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Definition of a Group

**Definition 2.7 (Group).** Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a group if the following hold:

1. **Closure of $\mathcal{G}$ under $\otimes$:** $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. **Associativity:** $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. **Neutral element:** $\exists e \in \mathcal{G} \forall x \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. **Inverse element:** $\forall x \in \mathcal{G} \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$, where $e$ is the neutral element. We often write $x^{-1}$ to denote the inverse element of $x$.

If additionally $\forall x, y \in \mathcal{G} : x \otimes y = y \otimes x$, then $G = (\mathcal{G}, \otimes)$ is an Abelian group (commutative).
Example: Vectors in $\mathbb{R}^n$ under addition

1. **Closure**: $\vec{a}, \vec{b} \in \mathbb{R}^n \Rightarrow \vec{a} + \vec{b} \in \mathbb{R}^n$

2. **Associativity**: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

3. **Neutral element**: $\vec{a} + \vec{0} = \vec{a}$

4. **Inverse element**: $\vec{a} + -\vec{a} = \vec{0}$

5. (abelian) **Commutativity**: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
Definition 2.9 (Vector Space). A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

- **“Inner Operation”** $+: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$
- **“Outer Operation”** $\cdot: \mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}$

where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity:
   1. $\forall \lambda \in \mathbb{R}, x, y \in \mathcal{V}: \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$
   2. $\forall \lambda, \psi \in \mathbb{R}, x \in \mathcal{V}: (\lambda + \psi) \cdot x = \lambda \cdot x + \psi \cdot x$
3. Associativity (outer operation): $\forall \lambda, \psi \in \mathbb{R}, x \in \mathcal{V}: \lambda \cdot (\psi \cdot x) = (\lambda \psi) \cdot x$
4. Neutral element with respect to the outer operation: $\forall x \in \mathcal{V}: 1 \cdot x = x$
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Row-Echelon Form

Definition

A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix.\(^a\)
- Looking at nonzero rows only, the pivot\(^b\) is always strictly to the right of the pivot of the row above it.

\(^a\)Correspondingly, all rows that contain at least one nonzero element are on top of rows that contain only zeros.

\(^b\)the first nonzero value from the left, also called the leading coefficient.
Row-Echelon Form

Examples

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
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Reduced Row-Echelon Form

Definition
A matrix is in reduced row-echelon form if
- It is in row-echelon form
- Every pivot\(^a\) is 1
- The pivot is the only nonzero entry in its column.

\(^a\)The first nonzero value from the left in each row
Reduced Row-Echelon Form

**Examples**

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 9 \\
0 & 0 & 0 & 1 & -4 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 1 & 2 & 12 \\
\end{bmatrix}
\]

In general, row-echelon form and reduced row-echelon form make it easier for us to determine a particular solution and the general solution.
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Elementary Transformations

Given a matrix \( A \), there are three elementary operations one can perform on \( A \) to transform \( A \) into reduced row-echelon form without changing the solution set of \( Ax = b \).

- Addition of two rows
- Multiplication of a row with a constant \( \lambda \in \mathbb{R} \), where \( \lambda \neq 0 \)
- Exchange two rows of a matrix
- Exchange two columns of a matrix
Given a matrix $A$, there are three elementary operations one can perform on $A$ to transform $A$ into reduced row-echelon form without changing the solution set of $Ax = b$.

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Gaussian elimination is an algorithm that performs elementary transformations to bring a system of linear equations into reduced row-echelon form.
The above system of equations can be represented by this augmented matrix:

\[
\begin{bmatrix}
1 & 1 & -1 \\
7 & 1 & -1 & 2 \\
3 & 2 & 1 \\
9 & 2 & 1 & 3
\end{bmatrix}
\]

We will perform Gaussian Elimination on this system of equations (Open Colab Notebook)
Gaussian Elimination

\[
\begin{align*}
  x_1 + x_2 - x_3 &= 7 \\
  x_1 - x_2 + 2x_3 &= 3 \\
  2x_1 + x_2 + x_3 &= 9 
\end{align*}
\]

The above system of equations can be represented by this augmented matrix:

\[
\begin{bmatrix}
  1 & 1 & -1 & 7 \\
  1 & -1 & 2 & 3 \\
  2 & 1 & 1 & 9
\end{bmatrix}
\]

We will perform Gaussian Elimination on this system of equations (Open Colab Notebook)
Invert Matrix via Gaussian Elimination

\[ A = \begin{bmatrix}
1 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 0 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \]
Invert Matrix via Gaussian Elimination

Perform Gaussian Elimination on the following Augmented Matrix:

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & \mid & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & \mid & 0 & 1 & 0 & 0 \\
1 & 2 & 0 & 1 & \mid & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & \mid & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Invert Matrix via Gaussian Elimination

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 & 2 \\
0 & 1 & 0 & 0 & \cdots & 1 & -1 & 2 & -2 \\
0 & 0 & 1 & 0 & \cdots & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 1 & \cdots & -1 & 0 & -1 & 2 \\
\end{bmatrix}
\]
Invert Matrix via Gaussian Elimination

\[ A^{-1} = \begin{bmatrix}
  -1 & 2 & -2 & 2 \\
  1 & -1 & 2 & -2 \\
  1 & -1 & 1 & -1 \\
  -1 & 0 & -1 & 2 \\
\end{bmatrix} \]
Each elementary operation on $A$ can be written as left multiplying $A$ by a matrix. Transforming $A$ to the identity matrix can be written as: $E_1E_2\cdots E_nA = I$. This implies that $E_1E_2\cdots E_nAA^{-1} = IA^{-1} = A^{-1}$, which implies that $E_1E_2\cdots E_nI = A^{-1}$. This means that applying the sequence of elementary operations that transformed $A$ to the identity matrix on $I$ will transform $I$ to $A^{-1}$. 