Linear Independence, Bases, and Rank

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Linear Combination

- Let $V$ be a vector space. $v \in V$ is a linear combination of vectors $x_1, \ldots, x_k \in V$ if
  \[ v = \lambda_1 x_1 + \ldots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i \in V \]

- Nontrivial linear combinations have at least one coefficient $\lambda_i \neq 0$
  - The 0-vector can be “trivially” represented as a linear combination $\sum_{i=1}^{k} 0 x_i$. 
• If there is at least one nontrivial linear combination of $x_1, \ldots, x_k \in \mathbb{V}$ such that $\sum_{i=1}^{k} \lambda_i x_i = 0$, then $x_1, \ldots, x_k$ are *linearly dependent*.

• Otherwise, when only the trivial solution exists, they are *linearly independent*. 
Linear (In)dependence

- Linearly independent vectors have no “redundancy”
  - If we remove any one of them, there will be certain vectors we can no longer represent via linear combinations.
- Equivalently, can’t express any $x_i$ as a linear combination of the others.
Example: Consider three vectors $a$, $b$, and $c$ where $c = a + b$.

These vectors are linearly dependent because $a + b - c = 0$. 
Use Gaussian Elimination to check linear (in-)dependence:

• Construct a matrix by stacking the vectors as columns
• Reduce to row echelon form
• If every column has a leading “1,” linearly independent
Checking Linear Independence

Example:

\[ x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}. \]

Transform the corresponding matrix to reduced row echelon form:

\[
\begin{bmatrix}
1 & 1 & -1 \\
2 & 1 & -2 \\
-3 & 0 & 1 \\
4 & 2 & 1 \\
\end{bmatrix} \quad \leadsto \quad \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Every column has a leading “1,” so the vectors are linearly independent.
The span of a set of vectors is the set of all their linear combinations. A set of vectors is a generating set for a vector space $\mathbb{V}$ if its span is $\mathbb{V}$. A basis is a minimal generating set.
Basis Example

In $\mathbb{R}^3$, the *canonical* or *standard* basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

Two other bases of $\mathbb{R}^3$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ -0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}.$$
Basis Non-Example

The set

\[ \mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \end{bmatrix} \right\} \]

is not a generating set (and so not a basis) of \( \mathbb{R}^4 \).
Remarks about Bases

• Every vector space possesses a basis, but there is no unique basis.
• All bases contain the same number of basis vectors.
• The dimension of $\mathbb{V}$ is the number of basis vectors of $\mathbb{V}$: intuitively, the dimension of a vector space can be thought of as the number of independent directions in this vector space.
• The dimension of a vector space is not necessarily the number of elements in a vector. For example,

$$\mathbb{V} = \text{span} \left( \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} \right)$$

is one-dimensional.
Finding a Basis

Use Gaussian Elimination to find a basis for the vector space spanned by $x_1 \ldots x_m$:

- Construct a matrix by stacking the vectors as columns
- Reduce to row echelon form
- Take every column with a leading “1”
Finding Basis: Example

Consider a subspace $U$ of $\mathbb{R}^5$ spanned by the vectors

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{pmatrix}, \quad x_4 = \begin{pmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{pmatrix}$$

To find which of the vectors are a basis for $U$...
Find Basis: Example

Write down matrix and reduce:

\[
\begin{bmatrix}
1 & 2 & 3 & -1 \\
2 & -1 & -4 & 8 \\
-1 & 1 & 3 & -5 \\
-1 & 2 & 5 & -6 \\
-1 & -2 & -3 & 1 \\
\end{bmatrix}
\sim \cdots \sim
\begin{bmatrix}
1 & 2 & 3 & -1 \\
0 & 1 & 2 & -2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Pivot columns (with leading ones) indicate linearly independent vectors, so \(x_1, x_2,\) and \(x_4\) form a basis for \(U.\)
• The rank of a matrix is the number of linearly independently rows (= the number of linearly independent columns)

• **Example:** The matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \]

has \( \text{rk}(A) = 2 \), because \( A \) has two linearly independent columns / rows.
Properties of Rank

- \( \text{rk}(A) = \text{rk}(A^\top) \)
- The columns of \( A \in \mathbb{R}^{m \times n} \) span a subspace \( U \subseteq \mathbb{R}^m \) with \( \dim(U) = \text{rk}(A) \)
- The rows of \( A \in \mathbb{R}^{m \times n} \) span a subspace \( W \subseteq \mathbb{R}^n \) with \( \dim(W) = \text{rk}(A) \).
- For all \( A \in \mathbb{R}^{n \times n} \), \( A \) is invertible iff \( \text{rk}(A) = n \)
Properties of Rank, cont.

• For $A \in \mathbb{R}^{m \times n}$, the subspace of solutions to $Ax = 0$ has dimension $n - \text{rk}(A)$. This is called the kernel or null space of $A$.

• A matrix $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions. In other words, the rank of a full rank matrix is $\text{rk}(A) = \min(m, n)$.

• A matrix is said to be rank deficient if it does not have full rank.

• A square matrix is singular if it does not have an inverse or, equivalently, is rank deficient.