5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
### Summary of the performance of symbol-table implementations

Order of growth of the frequency of operations.

<table>
<thead>
<tr>
<th>implementation</th>
<th>typical case</th>
<th>ordered operations</th>
<th>operations on keys</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search</td>
<td>insert</td>
<td>delete</td>
</tr>
<tr>
<td>red-black BST</td>
<td>(\log n)</td>
<td>(\log n)</td>
<td>(\log n)</td>
</tr>
<tr>
<td>hash table</td>
<td>(1^\dagger)</td>
<td>(1^\dagger)</td>
<td>(1^\dagger)</td>
</tr>
</tbody>
</table>

\[^\dagger\text{under uniform hashing assumption}\]

**Q.** Can we do better?

**A.** Yes, if we can avoid examining the entire key, as with string sorting.
String symbol table implementations cost summary

**Challenge.** Efficient performance for string keys.

<table>
<thead>
<tr>
<th>implementation</th>
<th>character accesses (typical case)</th>
<th>dedup</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search hit</td>
<td>search miss</td>
</tr>
<tr>
<td>red-black BST</td>
<td>$L + c \lg^2 n$</td>
<td>$c \lg^2 n$</td>
</tr>
<tr>
<td>hashing (linear probing)</td>
<td>$L$</td>
<td>$L$</td>
</tr>
</tbody>
</table>

$n = \text{number of string}$  
$L = \text{length of string}$  
$R = \text{radix}$

<table>
<thead>
<tr>
<th>file</th>
<th>size</th>
<th>words</th>
<th>distinct</th>
</tr>
</thead>
<tbody>
<tr>
<td>moby.txt</td>
<td>1.2 MB</td>
<td>210 K</td>
<td>32 K</td>
</tr>
<tr>
<td>actors.txt</td>
<td>82 MB</td>
<td>11.4 M</td>
<td>900 K</td>
</tr>
</tbody>
</table>
String symbol table basic API

**String symbol table.** Symbol table specialized to string keys.

```java
public class StringST<Value>

  StringST()
  put(String key, Value val)
  get(String key)
  delete(String key)

create an empty symbol table
put key–value pair into the symbol table
return value paired with given key
delete key and corresponding value
```

**Goal.** Faster than hashing, more flexible than BSTs.
5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
Tries
**Tries.** [from retrieval, but pronounced “try”]

- Store characters in nodes (not keys).
- Each node has $R$ children, one for each possible character. (for now, we do not draw null links)

<table>
<thead>
<tr>
<th>key</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>by</td>
<td>4</td>
</tr>
<tr>
<td>sea</td>
<td>6</td>
</tr>
<tr>
<td>sells</td>
<td>1</td>
</tr>
<tr>
<td>she</td>
<td>0</td>
</tr>
<tr>
<td>shells</td>
<td>3</td>
</tr>
<tr>
<td>shore</td>
<td>7</td>
</tr>
<tr>
<td>the</td>
<td>5</td>
</tr>
</tbody>
</table>

**Diagram:**
- Root node to trie for all keys that start with **s**.
- Link to trie for all keys that start with **she**.
- Value for **she** in node corresponding to last character in key.
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit:** node where search ends has a non-null value.
- **Search miss:** reach null link or node where search ends has null value.

`get("shells")`

```
return value in node corresponding to last character in key (return 3)
```
Search in a trie

Follow links corresponding to each character in the key.
- **Search hit:** node where search ends has a non-null value.
- **Search miss:** reach null link or node where search ends has null value.

get("she")

[Diagram of a trie tree with nodes labeled b, s, h, y, e, a, l, o, e, h, t, and l, with arrows indicating search path and a note indicating search may terminate at an intermediate node (return 0).]
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit**: node where search ends has a non-null value.
- **Search miss**: reach null link or node where search ends has null value.

get(“shell”)
Search in a trie

Follow links corresponding to each character in the key.

- **Search hit**: node where search ends has a non-null value.
- **Search miss**: reach null link or node where search ends has null value.

get("shelter")
**Insertion into a trie**

Follow links corresponding to each character in the key.
- Encounter a null link: create new node.
- Encounter the last character of the key: set value in that node.

```plaintext
put("shore", 7)
```

![Trie Diagram]

- Node labels indicate the character and value.
- The red path from the root to a leaf node represents the insertion path for "shore".
- The value associated with the leaf node is 7.
Trie construction demo

trie

trie
Trie representation: Java implementation

**Node.** A value, plus references to \( R \) nodes.

```java
private static class Node {
    private Object val; ← no generic array creation
    private Node[] next = new Node[R];
}
```

**Remark.** Neither keys nor characters are stored explicitly.
R-way trie: Java implementation

```java
public class TrieST<Value>
{
    private static final int R = 256;
    private Node root = new Node();

    private static class Node
    {
        /* see previous slide */
    }

    public void put(String key, Value val)
    {
        root = put(root, key, val, 0);
    }

    private Node put(Node x, String key, Value val, int d)
    {
        if (x == null) x = new Node();
        if (d == key.length()) { x.val = val; return x; }
        char c = key.charAt(d);
        x.next[c] = put(x.next[c], key, val, d+1);
        return x;
    }

    private Value get(String key)
    {
        /* similar, see book or booksite */
    }
}
```
What is order of growth of the running time (in the worst case) to insert a key of length $L$ into an $R$-way trie?

A. $L$
B. $R + L$
C. $n + L$
D. $RL$

$R = \text{alphabet size}$

$L = \text{length of key}$

$n = \text{number of keys}$
Trie performance

**Search hit.** Need to examine all $L$ characters for equality.

**Search miss.**
- Worst case: examine $L$ characters.
- Typical case: examine only a few characters before mismatch (sublinear).

**Space.** At least $R$ links per key.

**Bottom line.** Fast search hit and even faster search miss, but wastes space.
Deletion in an R-way trie

To delete a key–value pair:

- Find the node corresponding to key and set value to null.
- If node has null value and all null links, remove that node (and recur).

```python
delete("shells")
```
Deletion in an R-way trie

To delete a key–value pair:

- Find the node corresponding to key and set value to null.
- If node has null value and all null links, remove that node (and recur).

```
delete("shells")
```
String symbol table implementations cost summary

<table>
<thead>
<tr>
<th>implementation</th>
<th>character accesses (typical case)</th>
<th>dedup</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search hit</td>
<td>search miss</td>
</tr>
<tr>
<td>red–black BST</td>
<td>$L + c \lg^2 n$</td>
<td>$c \lg^2 n$</td>
</tr>
<tr>
<td>hashing (linear probing)</td>
<td>$L$</td>
<td>$L$</td>
</tr>
<tr>
<td>R–way trie</td>
<td>$L$</td>
<td>$\log_R n$</td>
</tr>
</tbody>
</table>

R–way trie.
- Method of choice for small $R$.
- Works well for medium $R$.
- Too much memory for large $R$.

Challenge. Use less memory, e.g., a 65,536-way trie for Unicode!
5.2 Tries

- R-way tries
- ternary search tries
- character-based operations
Ternary search tries

- Store characters and values in nodes (not keys).
- Each node has 3 children: smaller (left), equal (middle), larger (right).

Fast Algorithms for Sorting and Searching Strings

Jon L. Bentley*  Robert Sedgewick#

Abstract

We present theoretical algorithms for sorting and searching multikey data, and derive from them practical C implementations for applications in which keys are character strings. The sorting algorithm blends Quicksort and radix sort, it is competitive with the best known C sort codes. The searching algorithm blends tries and binary search trees, it is faster than hashing and other commonly used search methods. The basic ideas behind the algo-

that is competitive with the most efficient string sorting programs known. The second program is a symbol table implementation that is faster than hashing, which is commonly regarded as the fastest symbol table implementation. The symbol table implementation is much more space-efficient than multiway trees, and supports more advanced searches.

In many application programs, sorts use a Quicksort implementation based on an abstract compare operation,
Ternary search tries

- Store characters and values in nodes (not keys).
- Each node has 3 children: smaller (left), equal (middle), larger (right).

abstract trie

TST

link to TST for all keys that start with a character less than s

link to TST for all keys that start with s

each node has 3 links
Which value is associated with the key CAC?

A. 3
B. 4
C. 5
D. null
Search hit in a TST

get("sea")

return value in node corresponding to last character in key
Search miss in a TST

get("shelter")

no link to t (return null)
Search in a TST

Follow links corresponding to each character in the key.
- If less, take left link; if greater, take right link.
- If equal, take the middle link and move to the next key character.

Search hit. Node where search ends has a non-null value.
Search miss. Reach a null link or node where search ends has null value.
Ternary search trie construction demo

ternary search trie

![Ternary Trie Diagram](image-url)
In which subtrie would the key CCC be inserted?

A. 
B. 
C. 
D. 
E.
26-way trie vs. TST

**26-way trie.** 26 null links in each leaf.

![26-way trie diagram](image)

26–way trie (1035 null links, not shown)

**TST.** 3 null links in each leaf.

![TST diagram](image)

TST (155 null links)
TST representation in Java

A TST node is five fields:

- A value.
- A character.
- A reference to a left TST.
- A reference to a middle TST.
- A reference to a right TST.

```java
private class Node {
    private Value val;
    private char c;
    private Node left, mid, right;
}
```
TST: Java implementation

```java
public class TST<Value> {
    private Node root;
    private class Node {
        /* see previous slide */
    }

    public Value get(String key) {
        return get(root, key, 0);
    }

    private Value get(Node x, String key, int d) {
        if (x == null) return null;
        char c = key.charAt(d);
        if (c < x.c) return get(x.left, key, d);
        else if (c > x.c) return get(x.right, key, d);
        else if (d < key.length() - 1) return get(x.mid, key, d+1);
        else return x.val;
    }

    public void put(String Key, Value val) {
        /* similar, see book or booksite */
    }
}
```
## String symbol table implementation cost summary

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Character accesses (typical case)</th>
<th>Dedup</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search hit</td>
<td>search miss</td>
</tr>
<tr>
<td>red-black BST</td>
<td>$L + c \log^2 n$</td>
<td>$c \log^2 n$</td>
</tr>
<tr>
<td>hashing (linear probing)</td>
<td>$L$</td>
<td>$L$</td>
</tr>
<tr>
<td>R-way trie</td>
<td>$L$</td>
<td>$\log_R n$</td>
</tr>
<tr>
<td>TST</td>
<td>$L + \ln n$</td>
<td>$\ln n$</td>
</tr>
</tbody>
</table>

**Remark.** Can build balanced TSTs via rotations to achieve $L + \log n$ worst-case guarantees.

**Bottom line.** TST is as fast as hashing (for string keys) and space efficient.
TST with $R^2$ branching at root

Hybrid of R-way trie and TST.
- Do $R^2$-way branching at root.
- Each of $R^2$ root nodes points to a TST.

Q. What about one- and two-letter words?
## String symbol table implementation cost summary

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Character accesses (typical case)</th>
<th>Dedup</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>search hit</td>
<td></td>
</tr>
<tr>
<td></td>
<td>search miss</td>
<td></td>
</tr>
<tr>
<td></td>
<td>insert</td>
<td></td>
</tr>
<tr>
<td></td>
<td>space (references)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>moby.txt</td>
<td></td>
</tr>
<tr>
<td></td>
<td>actors.txt</td>
<td></td>
</tr>
<tr>
<td>red-black BST</td>
<td>(L + c \log^2 n)</td>
<td>1.4</td>
</tr>
<tr>
<td></td>
<td>(c \log^2 n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c \log^2 n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4n)</td>
<td>97.4</td>
</tr>
<tr>
<td>hashing (linear probing)</td>
<td>(L)</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>(L)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(L)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4n) to (16n)</td>
<td></td>
</tr>
<tr>
<td>R-way trie</td>
<td>(L)</td>
<td>1.12</td>
</tr>
<tr>
<td></td>
<td>(\log_R n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(R + L)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((R+1)n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>out of memory</td>
<td></td>
</tr>
<tr>
<td>TST</td>
<td>(L + \ln n)</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>(\ln n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(L + \ln n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4n)</td>
<td>38.7</td>
</tr>
<tr>
<td>TST with (R^2)</td>
<td>(L + \ln n)</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>(\ln n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(L + \ln n)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4n + R^2)</td>
<td>32.7</td>
</tr>
</tbody>
</table>

**Bottom line.** Faster than hashing for our benchmark client.
TST vs. hashing

Hashing.
- Need to examine entire key.
- Search hits and misses cost about the same.
- Performance relies on hash function.
- Does not support ordered symbol table operations.

TSTs.
- Works only for string (or digital) keys.
- Search miss may involve only a few characters.
- Supports ordered symbol table operations (plus extras!).

Bottom line. TSTs are:
- Faster than hashing (especially for search misses).
- More flexible than red–black BSTs.  
  
  [stay tuned]
5.2 TRIES

- R-way tries
- ternary search tries
- character-based operations
Character-based operations. The string symbol table API supports several useful character-based operations.

<table>
<thead>
<tr>
<th>key</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>by</td>
<td>4</td>
</tr>
<tr>
<td>sea</td>
<td>6</td>
</tr>
<tr>
<td>sells</td>
<td>1</td>
</tr>
<tr>
<td>she</td>
<td>0</td>
</tr>
<tr>
<td>shells</td>
<td>3</td>
</tr>
<tr>
<td>shore</td>
<td>7</td>
</tr>
<tr>
<td>the</td>
<td>5</td>
</tr>
</tbody>
</table>

Prefix match. Keys with prefix sh: she, shells, and shore.

Longest prefix. Key that is the longest prefix of shellsort: shells.
String symbol table API

```java
public class StringST<Value>

    StringST() // create a symbol table with string keys
    void put(String key, Value val) // put key-value pair into the symbol table
    Value get(String key) // value paired with key
    void delete(String key) // delete key and corresponding value
    :

    Iterable<String> keys() // all keys
    Iterable<String> keysWithPrefix(String s) // keys having s as a prefix
    String longestPrefixOf(String s) // longest key that is a prefix of s
```

**Remark.** Can also add other ordered ST methods, e.g., floor() and rank().
Warmup: ordered iteration

To iterate through all keys in sorted order:

- Do inorder traversal of trie; add keys encountered to a queue.
- Maintain sequence of characters on path from root to node.

```plaintext
keys()
  key  q
    b
    by
    s
    sea
    sel
    sell
    sells
    sh
    she
    shell
    shells
    sho
    shor
    shore
    t
    th
    the
```

To iterate through all keys in sorted order:

- Do inorder traversal of trie; add keys encountered to a queue.
- Maintain sequence of characters on path from root to node.

```java
public Iterable<String> keys()
{
    Queue<String> queue = new Queue<String>();
    collect(root, "", queue);
    return queue;
}

private void collect(Node x, String prefix, Queue<String> queue)
{
    if (x == null) return;
    if (x.val != null) queue.enqueue(prefix);
    for (char c = 0; c < R; c++)
        collect(x.next[c], prefix + c, queue);
}
```

sequence of characters on path from root to x
or use StringBuilder
Prefix matches

Find all keys in a symbol table starting with a given prefix.

**Ex.** Autocomplete in a cell phone, search bar, text editor, or shell.
- User types characters one at a time.
- System reports all matching strings.
Prefix matches in an R-way trie

Find all keys in a symbol table starting with a given prefix.

```java
keysWithPrefix("sh");
```
Longest prefix

Find longest key in symbol table that is a prefix of query string.

Ex 1. To send packet toward destination IP address, router chooses IP address in routing table that is longest prefix match.

- "128"
- "128.112"
- "128.112.055"
- "128.112.055.15"
- "128.112.136"
- "128.112.155.11"
- "128.112.155.13"
- "128.222"
- "128.222.136"

represented as 32-bit binary number for IPv4 (instead of string)

\[\text{longestPrefixOf}("128.112.136.11") = "128.112.136"
\]
\[\text{longestPrefixOf}("128.112.100.16") = "128.112"
\]
\[\text{longestPrefixOf}("128.166.123.45") = "128"
\]

Note. Not the same as floor: \[\text{floor}("128.112.100.16") = "128.112.055.15"
\]
Longest prefix in an R-way trie

Find longest key in symbol table that is a prefix of query string.
- Search for query string.
- Keep track of longest key encountered.

Possibilities for longestPrefixOf()
T9 texting (predictive texting)

**Goal.** Type text messages on a phone keypad.

**Multi-tap input.** Enter a letter by repeatedly pressing a key.

**Ex.** good: 4 6 6 6 6 6 6 3

“a much faster and more fun way to enter text”

**T9 text input (on 4 billion handsets).**

- Find all words that correspond to given sequence of numbers.
  4663: good, home, gone, hoof. textonyms
- Press * to select next option.
- Press 0 to see all completion options.
- System adapts to user’s tendencies.

http://www.t9.com
Q. How to implement T9 texting on a mobile phone?
Patricia trie

**Patricia trie.** [Practical Algorithm to Retrieve Information Coded in Alphanumeric]
- Remove one-way branching.
- Each node represents a sequence of characters.
- Implementation: one step beyond this course.

**Applications.**
- Database search.
- P2P network search.
- IP routing tables: find longest prefix match.
- Compressed quad-tree for $n$-body simulation.
- Efficiently storing and querying XML documents.

**Also known as:** crit-bit tree, radix tree.
Suffix tree

**Suffix tree.**

- Patricia trie of suffixes of a string.
- Linear-time construction: well beyond scope of this course.

**suffix tree for BANANAS**

```
  BANANAS
   /   \\  \\  \\
  A     NA S
  /     /  \\  \\
 NA S NAS S
 / /  \\
NAS S
```

**Applications.**

- Linear-time: longest repeated substring, longest common substring, longest palindromic substring, substring search, tandem repeats, ....
- Computational biology databases (BLAST, FASTA).
String symbol tables summary

A success story in algorithm design and analysis.

**Red–black BST.**
- Performance guarantee: \( \log n \) key compares.
- Supports ordered symbol table API.

**Hash tables.**
- Performance guarantee: constant number of probes.
- Requires good hash function for key type.

**Tries.** R-way, TST.
- Performance guarantee: \( \log n \) characters accessed.
- Supports character-based operations.