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4.2 Directed Graphs

- *introduction*
- *digraph API*
- *depth-first search*
- *breadth-first search*
- *topological sort*

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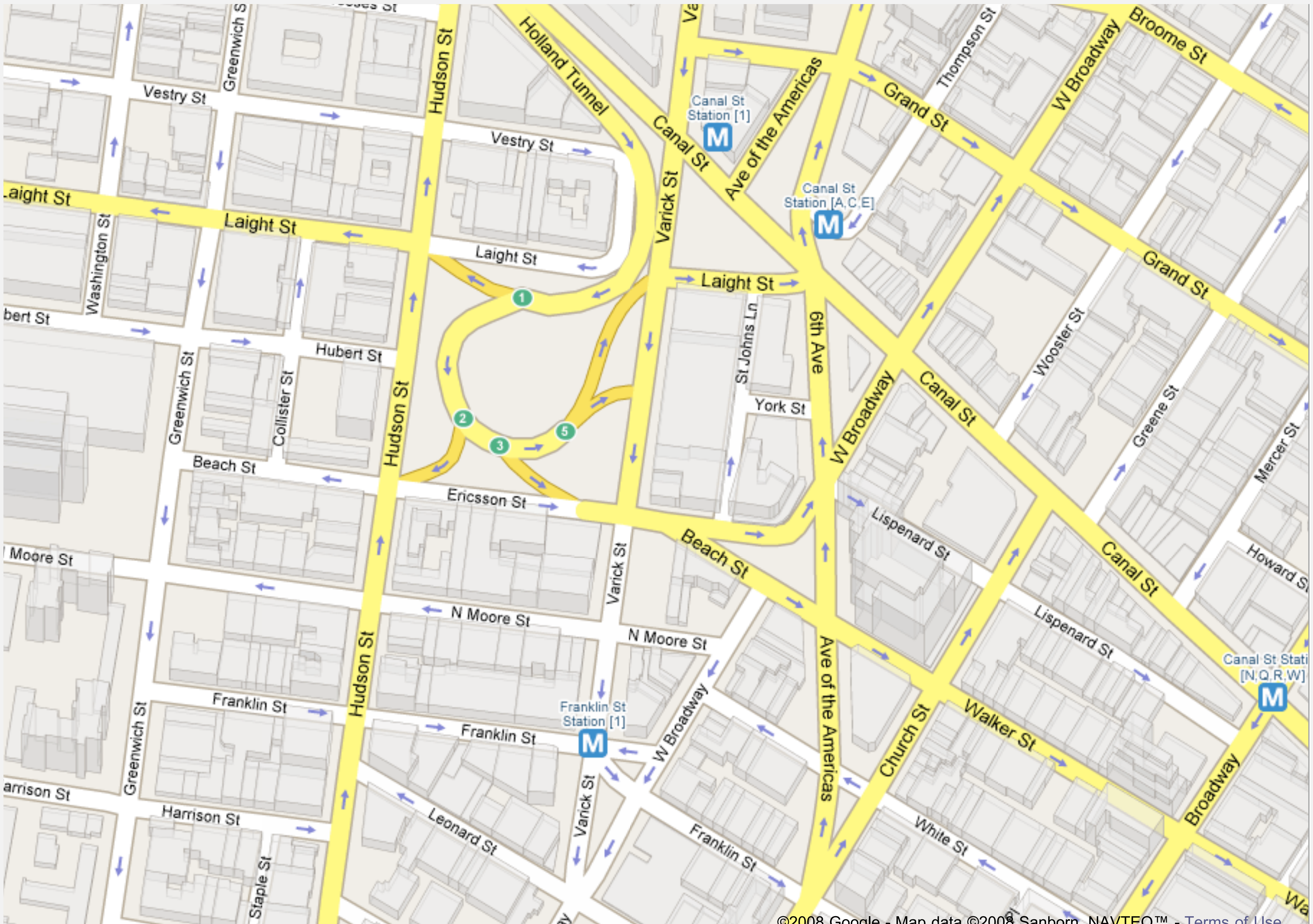
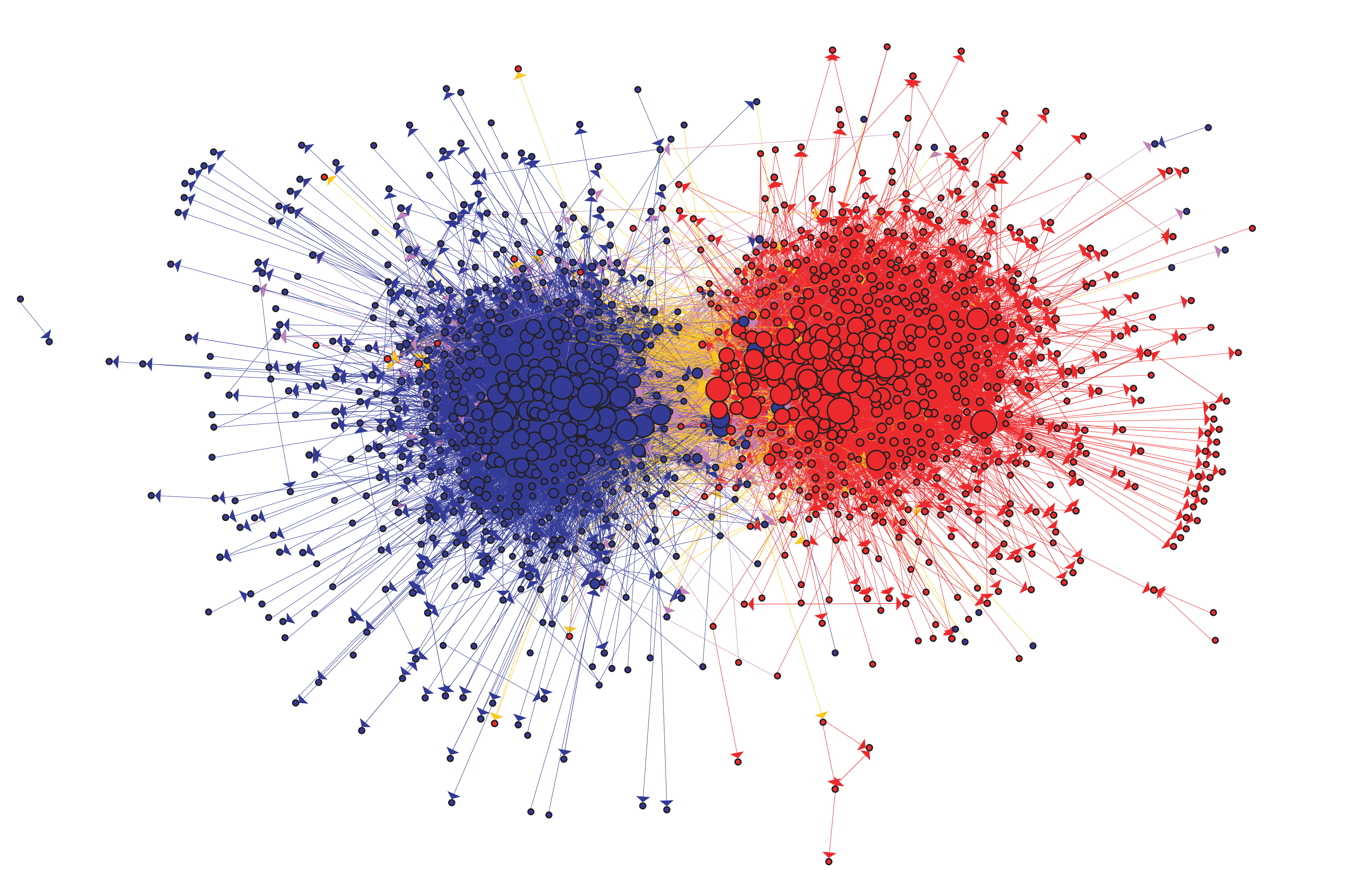
Road networks

Vertex = intersection; edge = one-way street.



Political blogosphere links

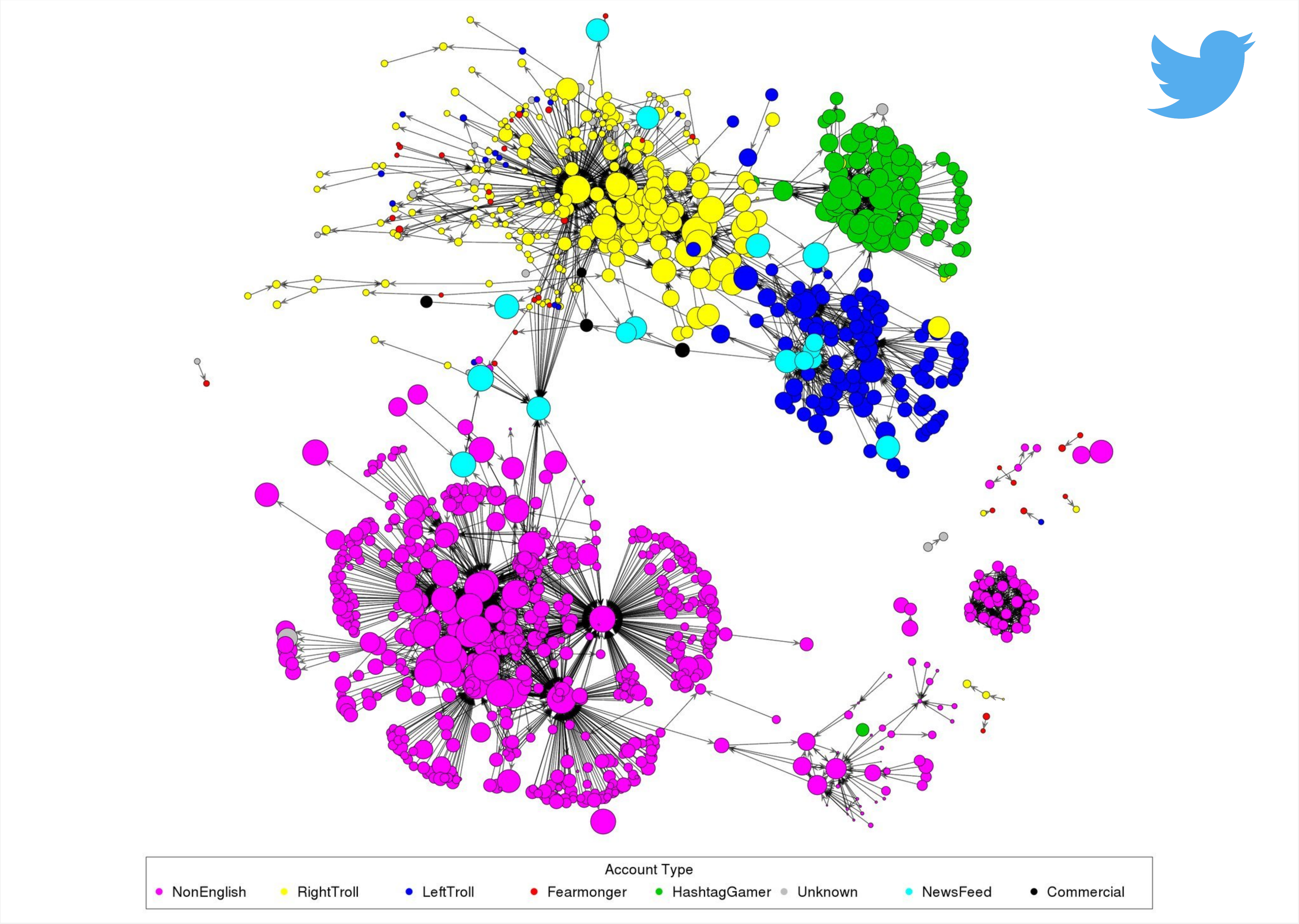
Vertex = political blog; edge = link.



The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

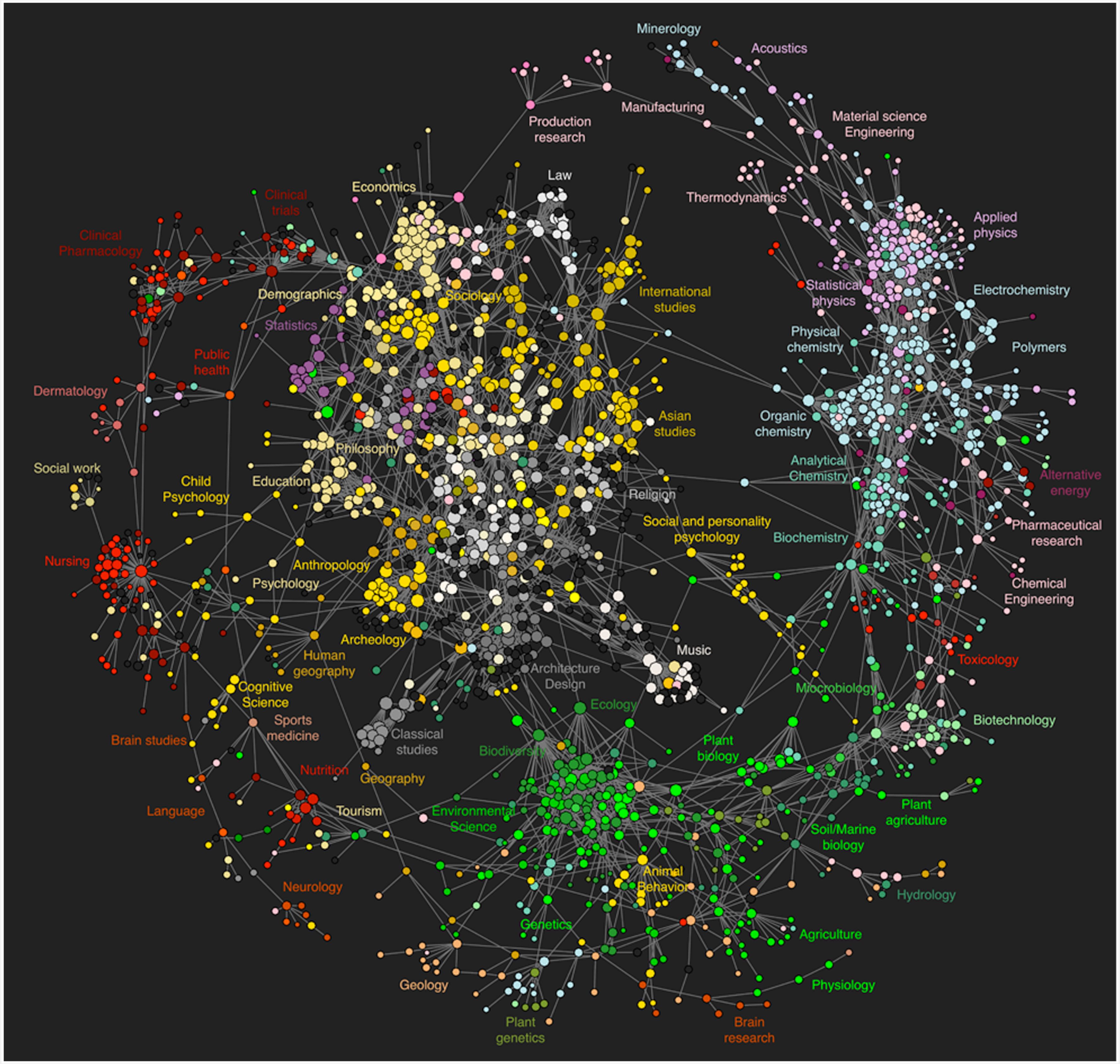
Russian troll network

Vertex = Russian troll; edge = Twitter mention.



Russian Troll-to-Russian Troll Twitter Mention Network, fivethirtyeight.com

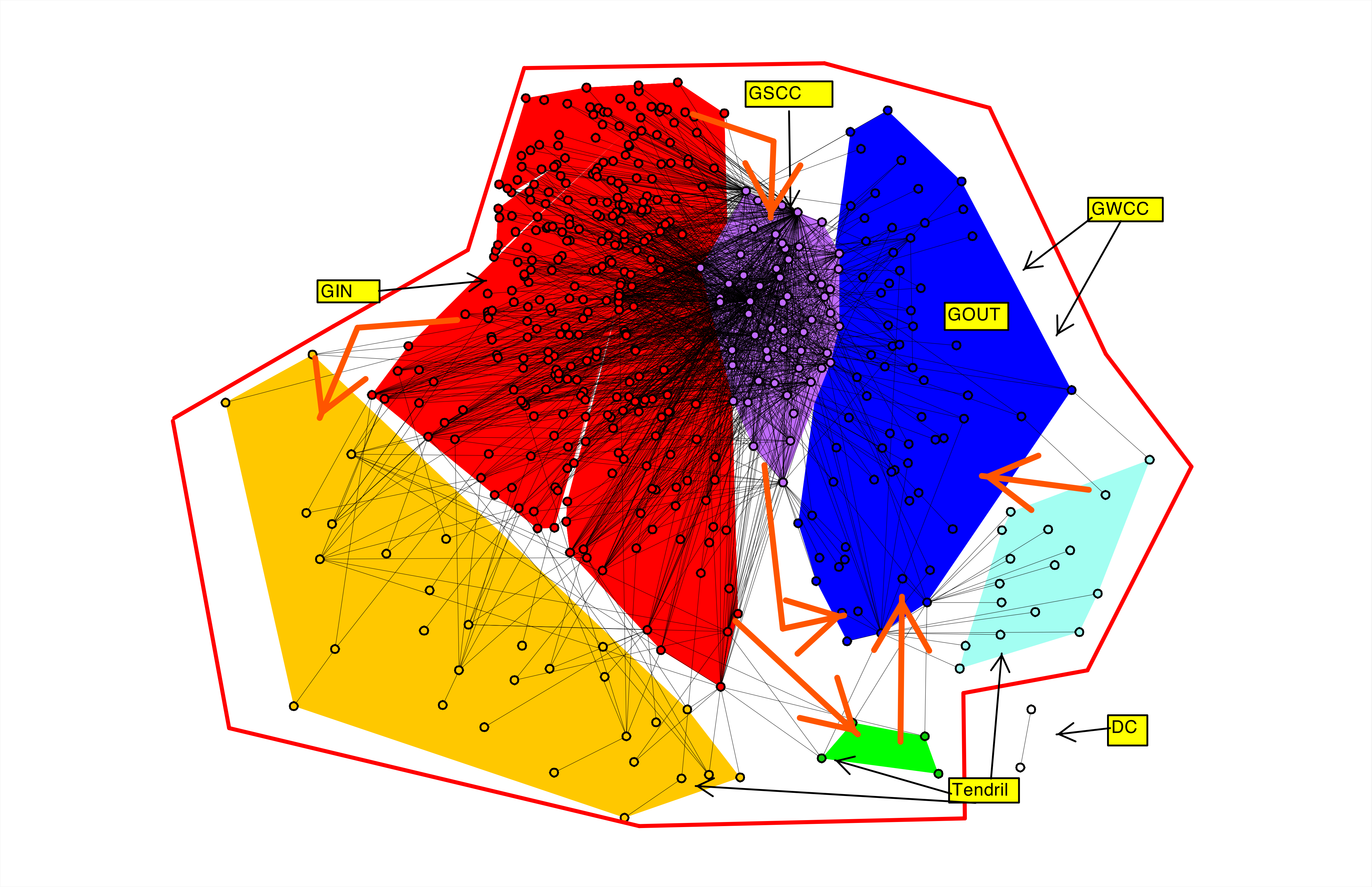
Science clickstreams



http://www.plosone.org/article/info:doi/10.1371/journal.pone.0004803

Overnight interbank loans

Vertex = bank; edge = overnight loan.



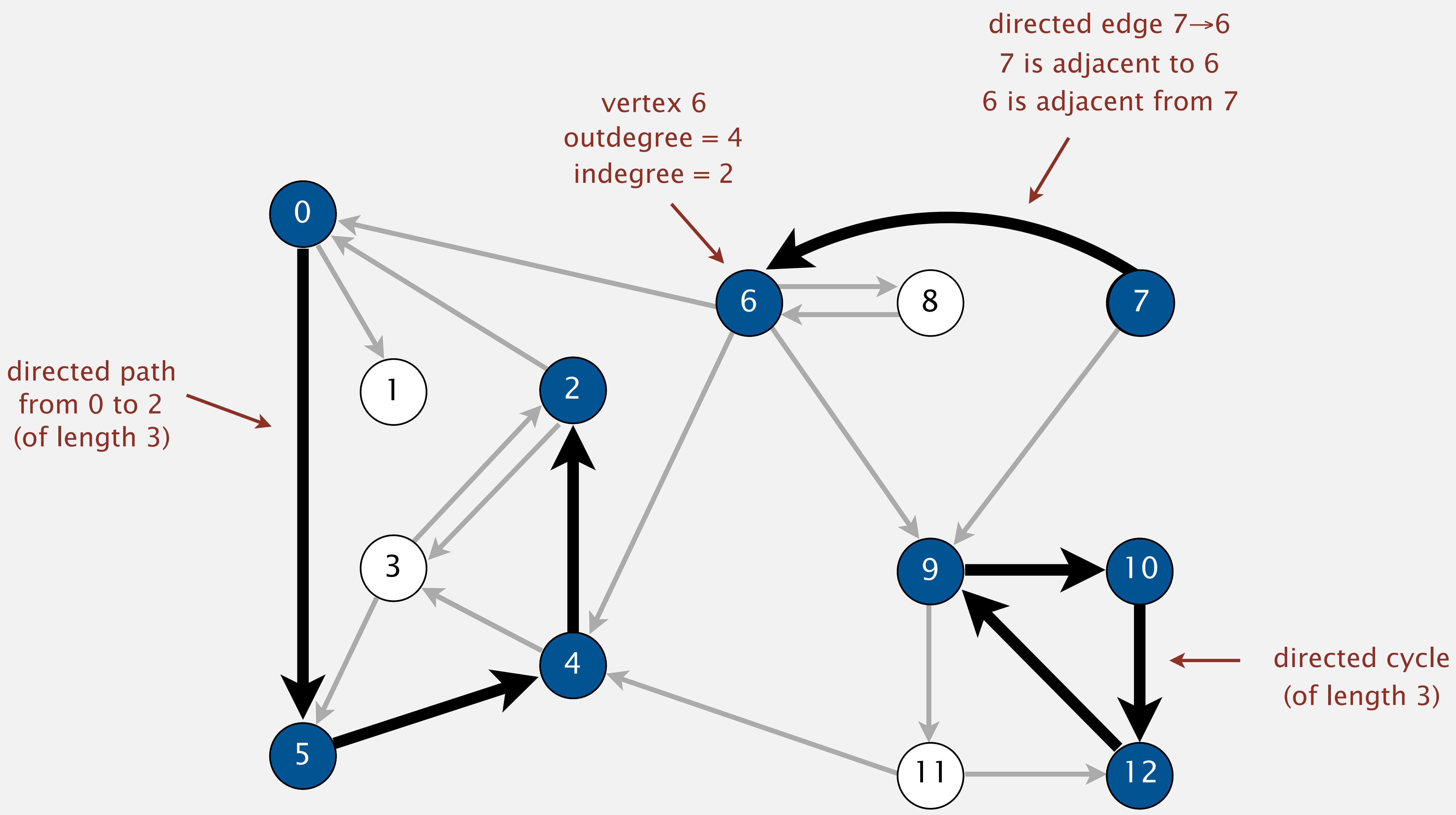
The Topology of the Federal Funds Market, Bech and Atalay, 2008

Digraph applications

digraph	vertex	directed edge
transportation	street intersection	one-way street
web	web page	hyperlink
food web	species	predator-prey relationship
WordNet	synset	hypernym
scheduling	task	precedence constraint
financial	bank	transaction
cell phone	person	placed call
infectious disease	person	infection
game	board position	legal move
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

Directed graph terminology

Digraph. Set of vertices connected pairwise by directed edges.



Some digraph problems

problem	description
s→t path	*Is there a path from s to t ?*
shortest s→t path	*What is the shortest path from s to t ?*
directed cycle	*Is there a directed cycle in the graph ?*
topological sort	*Can the digraph be drawn so that all edges point upwards?*
strong connectivity	*Is there a directed path between every pairs of vertices ?*
transitive closure	*For which vertices v and w is there a directed path from v to w ?*
PageRank	*What is the importance of a web page ?*

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Digraph API

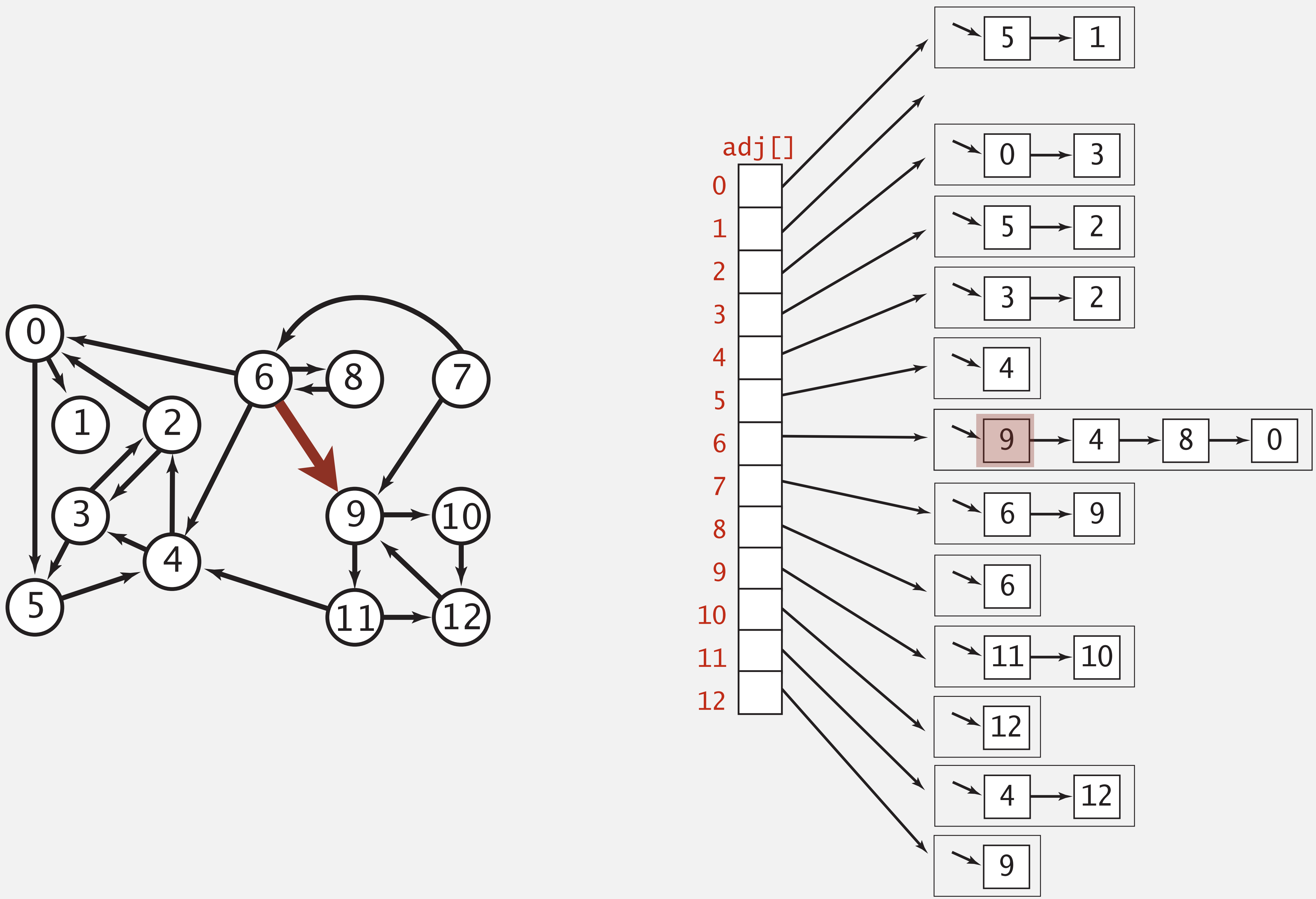
Almost identical to `Graph` API.

	`public class Digraph`	
	`Digraph(int V)`	*create an empty digraph with V vertices*
`void`	`addEdge(int v, int w)`	*add a directed edge v→w*
`Iterable<Integer>`	`adj(int v)`	*vertices adjacent from v*
`int`	`V()`	*number of vertices*
	⋮	⋮

Note. Full `Digraph` API includes additional methods, such as `reverse()`.

Digraph representation: adjacency lists

Maintain vertex-indexed array of lists.



Directed graphs: quiz 1

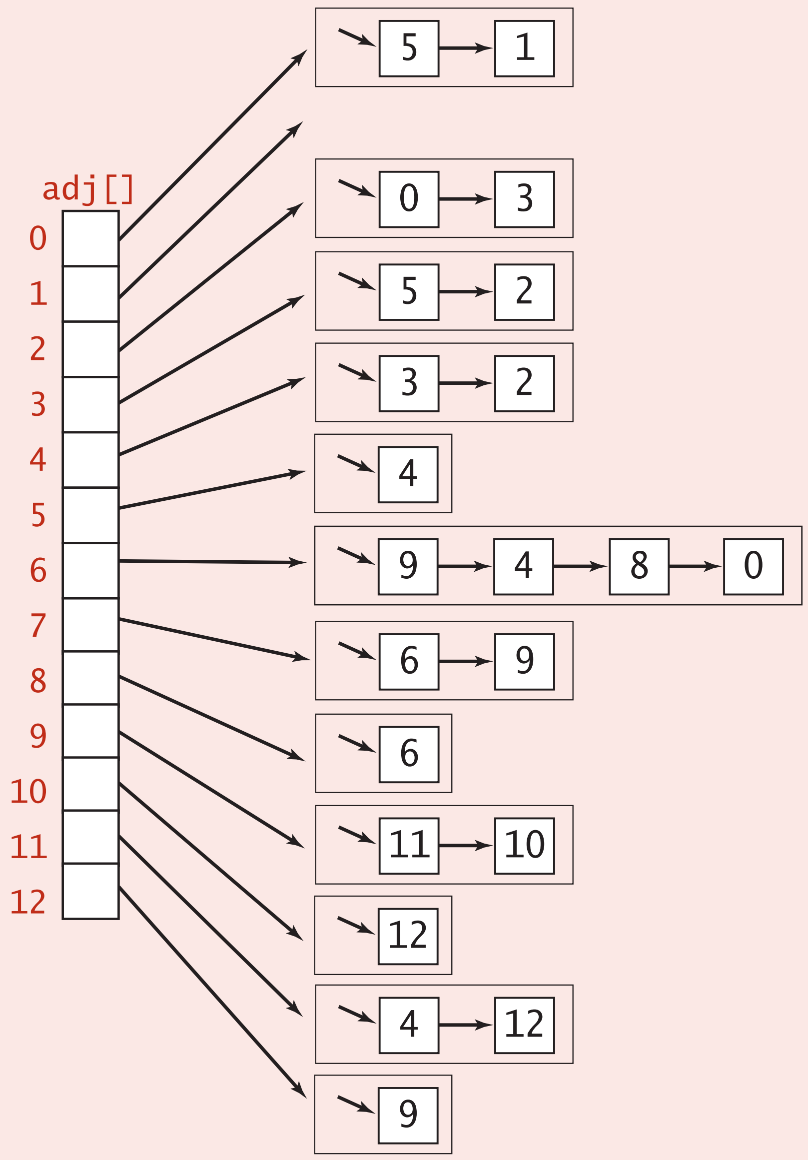
Which is the order of growth of the running time for removing an edge v→w from a digraph using the adjacency-lists representation, where V is the number of vertices and E is the number of edges?

A. 1

B. $outdegree(v)$

C. $indegree(w)$

D. $outdegree(v) + indegree(w)$



Which is the order of growth of the running time of the following code fragment if the digraph uses the adjacency-lists representation, where V is the number of vertices and E is the number of edges?

A. V

B. $E + V$

C. V^2

D. $V E$

```
for (int v = 0; v < G.V(); v++)
   for (int w : G.adj(v))
      StdOut.println(v + "->" + w);
```

prints each edge exactly once

Digraph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent from v.
- Real-world graphs tend to be sparse (not dense).

sparse: proportional to V edges

dense: proportional to V^2 edges

representation	space	insert edge from v to w	edge from v to w?	iterate over vertices adjacent from v?
list of edges	E	1	E	E
adjacency matrix	V^2	1 †	1	V
adjacency lists	$E + V$	1	$outdegree(v)$	$outdegree(v)$

† disallows parallel edges

Adjacency-lists graph representation (review): Java implementation

```
public class Graph
{
   private final int V;
   private Bag<Integer>[] adj;

   public Graph(int V)
   {
     this.V = V;
     adj = (Bag<Integer>[]) new Bag[V];
     for (int v = 0; v < V; v++)
        adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)
   {
     adj[v].add(w);
     adj[w].add(v);
   }

   public Iterable<Integer> adj(int v)
   {  return adj[v];   }
}
```

adjacency lists

create empty graph with V vertices

add edge v–w

iterator for vertices adjacent to v

Adjacency-lists digraph representation: Java implementation

```
public class Digraph
{
   private final int V;
   private Bag<Integer>[] adj;

   public Digraph(int V)
   {
      this.V = V;
      adj = (Bag<Integer>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Integer>();
   }

   public void addEdge(int v, int w)
   {
      adj[v].add(w);

   }

   public Iterable<Integer> adj(int v)
   {  return adj[v];  }
}
```

- adjacency lists
- create empty digraph with V vertices
- add edge v→w
- iterator for vertices adjacent from v

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4.2 Directed Graphs

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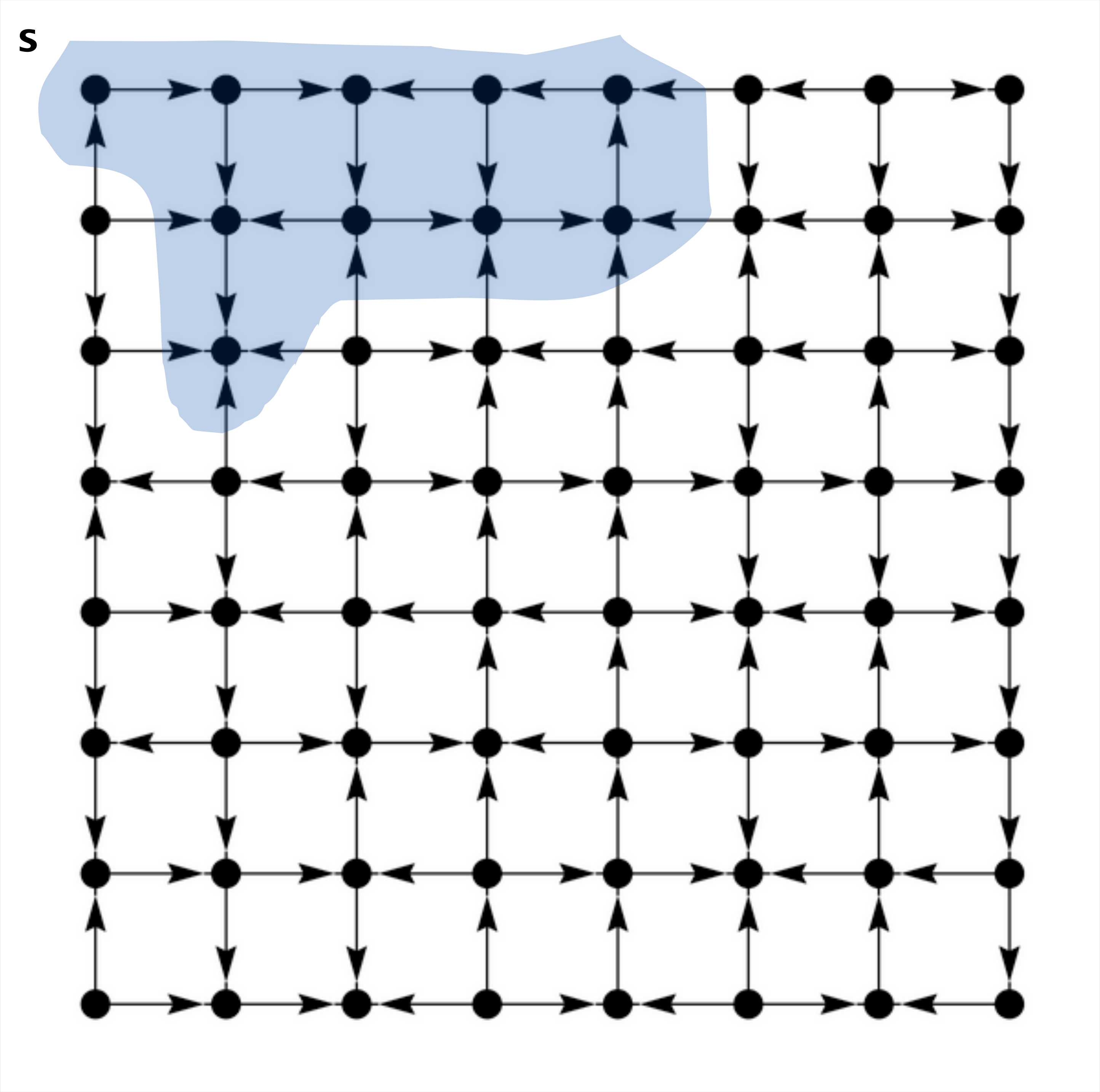
- breadth-first search
- topological sort

Reachability

Problem. Find all vertices reachable from s along a directed path.



Depth-first search in digraphs

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- DFS is a digraph algorithm.

DFS (to visit a vertex v)

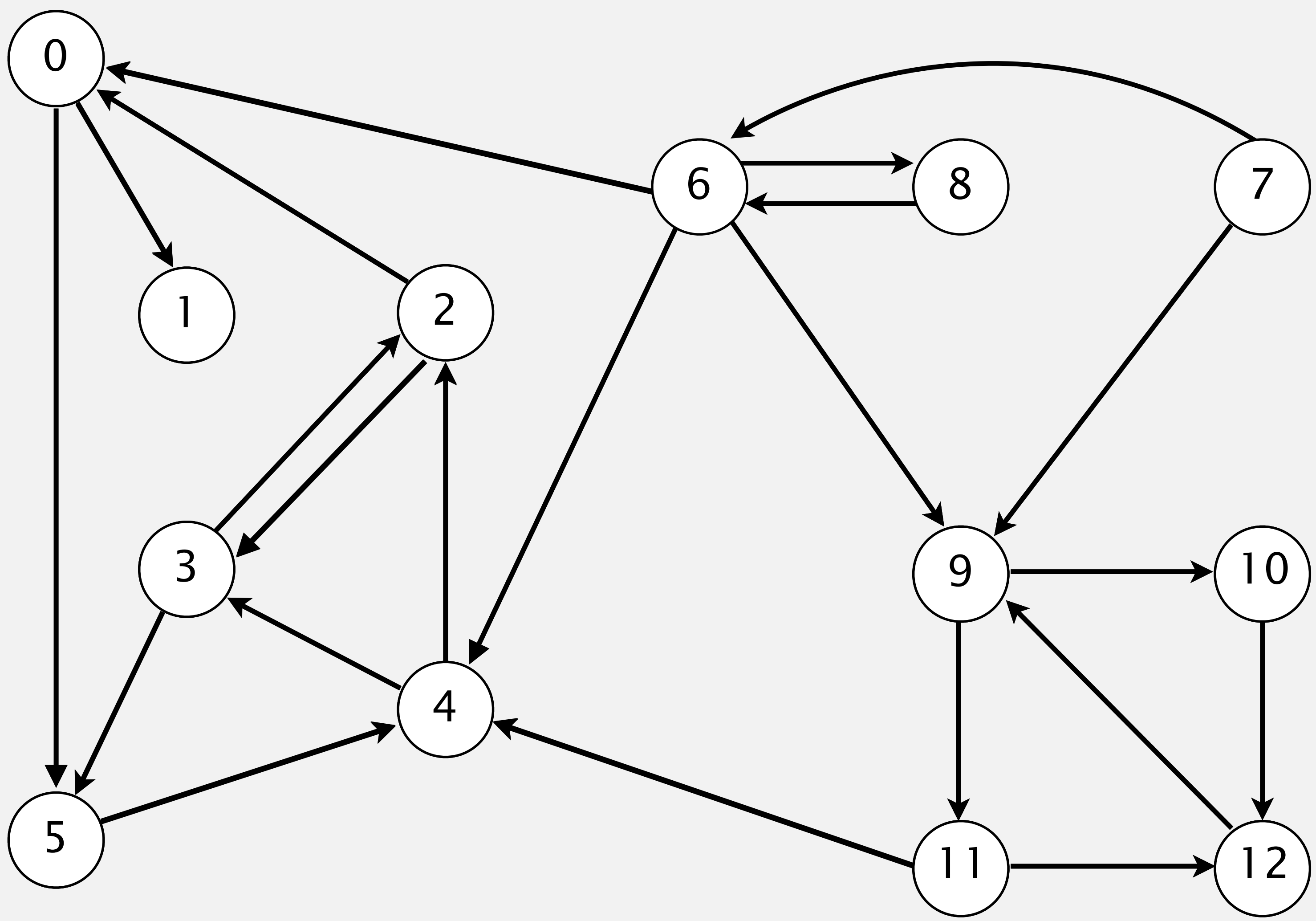
Mark vertex v.

Recursively visit all unmarked vertices w adjacent from v.

Depth-first search demo

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices adjacent from v.



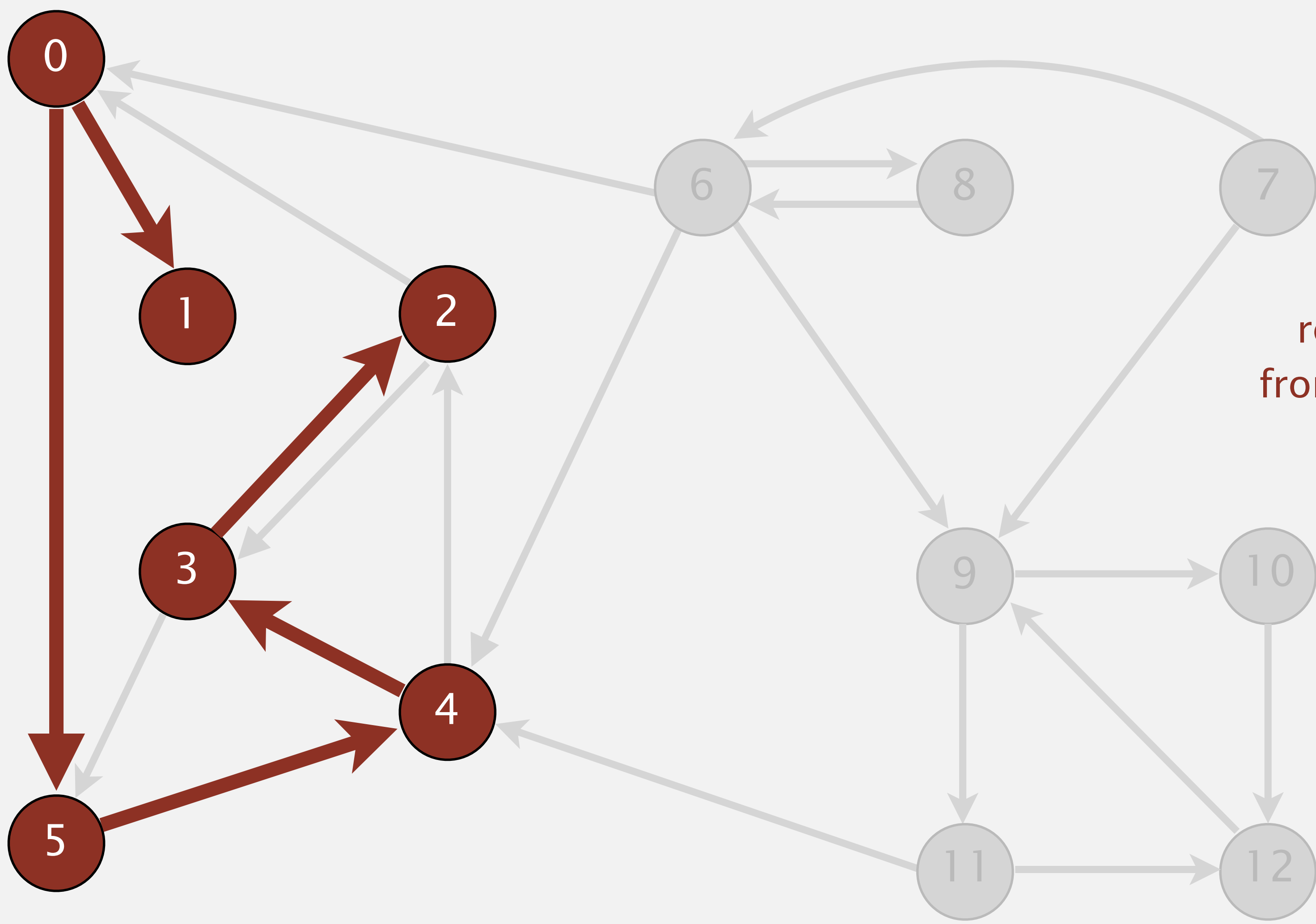
a directed graph

4→2
2→3
3→2
6→0
0→1
2→0
11→12
12→9
9→10
9→11
8→9
10→12
11→4
4→3
3→5
6→8
8→6
5→4
0→5
6→4
6→9
7→6

Depth-first search demo

To visit a vertex v :

- Mark vertex v as visited.
- Recursively visit all unmarked vertices adjacent from v.



reachable from 0

reachable from vertex 0 → (rows 0–5)

v	marked[]	edgeTo[]
0	T	–
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	–
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–
12	F	–

Depth-first search (in undirected graphs)

Recall code for undirected graphs.

```
public class DepthFirstSearch
{
   private boolean[] marked;

   public DepthFirstSearch(Graph G, int s)
   {
     marked = new boolean[G.V()];
     dfs(G, s);
   }

   private void dfs(Graph G, int v)
   {
     marked[v] = true;
     for (int w : G.adj(v))
        if (!marked[w])
           dfs(G, w);
   }

   public boolean visited(int v)
   {  return marked[v];  }
}
```

- `private boolean[] marked;` — true if connected to s
- constructor — constructor marks vertices connected to s
- `dfs` — recursive DFS does the work
- `visited` — is vertex v is connected to s ?

Depth-first search (in directed graphs)

Code for directed graphs identical to undirected one.

```
public class DirectedDFS
{
   private boolean[] marked;

   public DirectedDFS(Digraph G, int s)
   {
     marked = new boolean[G.V()];
     dfs(G, s);
   }

   private void dfs(Digraph G, int v)
   {
     marked[v] = true;
     for (int w : G.adj(v))
        if (!marked[w])
           dfs(G, w);
   }

   public boolean visited(int v)
   {  return marked[v];  }
}
```

- private boolean[] marked; ← true if connected to s
- constructor ← constructor marks vertices connected to s
- dfs() ← recursive DFS does the work
- visited() ← is vertex v reachable from s ?

Reachability application: program control-flow analysis

Every program is a digraph.

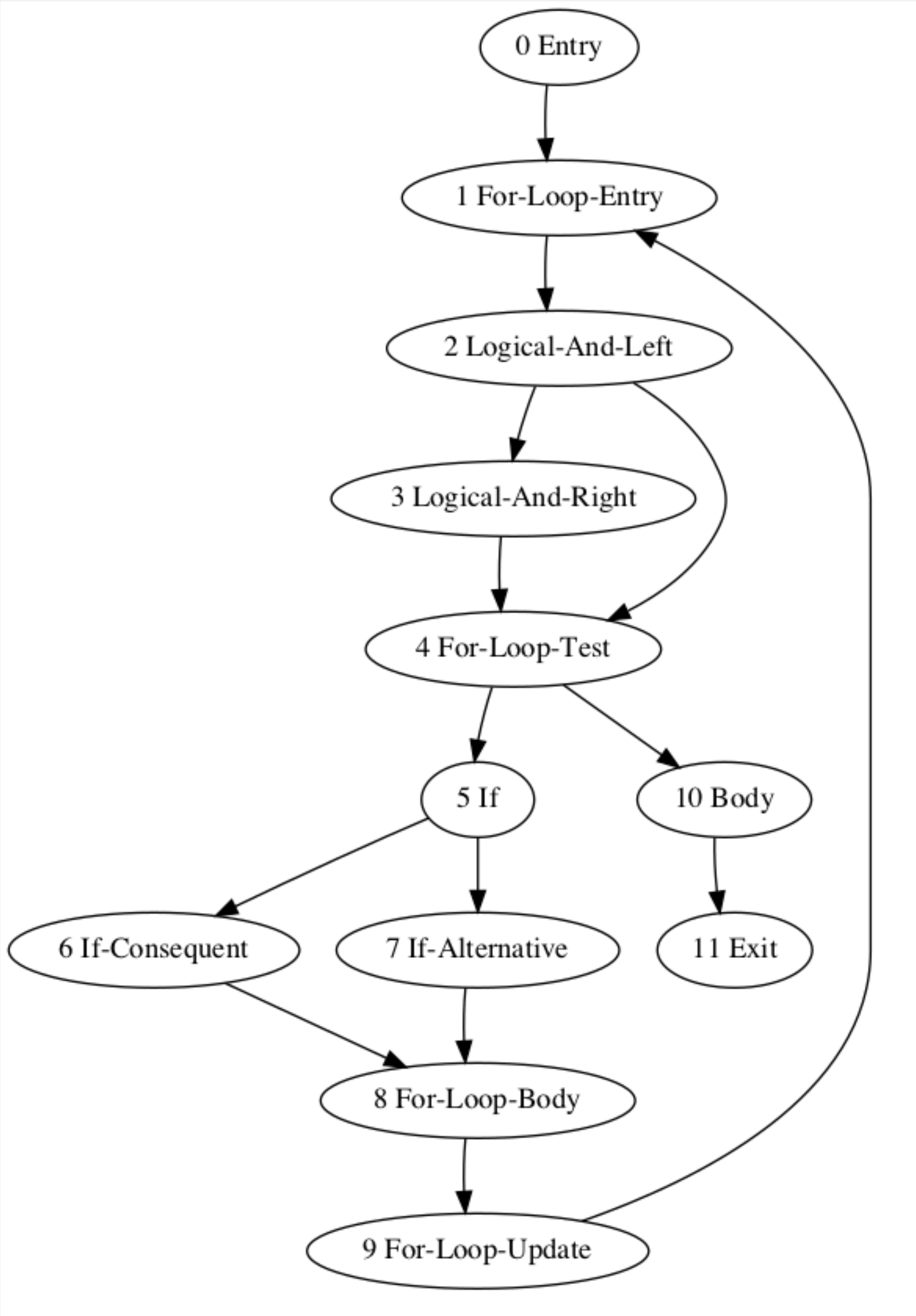
- Vertex = basic block of instructions (straight-line program).
- Edge = jump.

Dead-code elimination.

Find (and remove) unreachable code.

Infinite-loop detection.

Determine whether exit is unreachable.



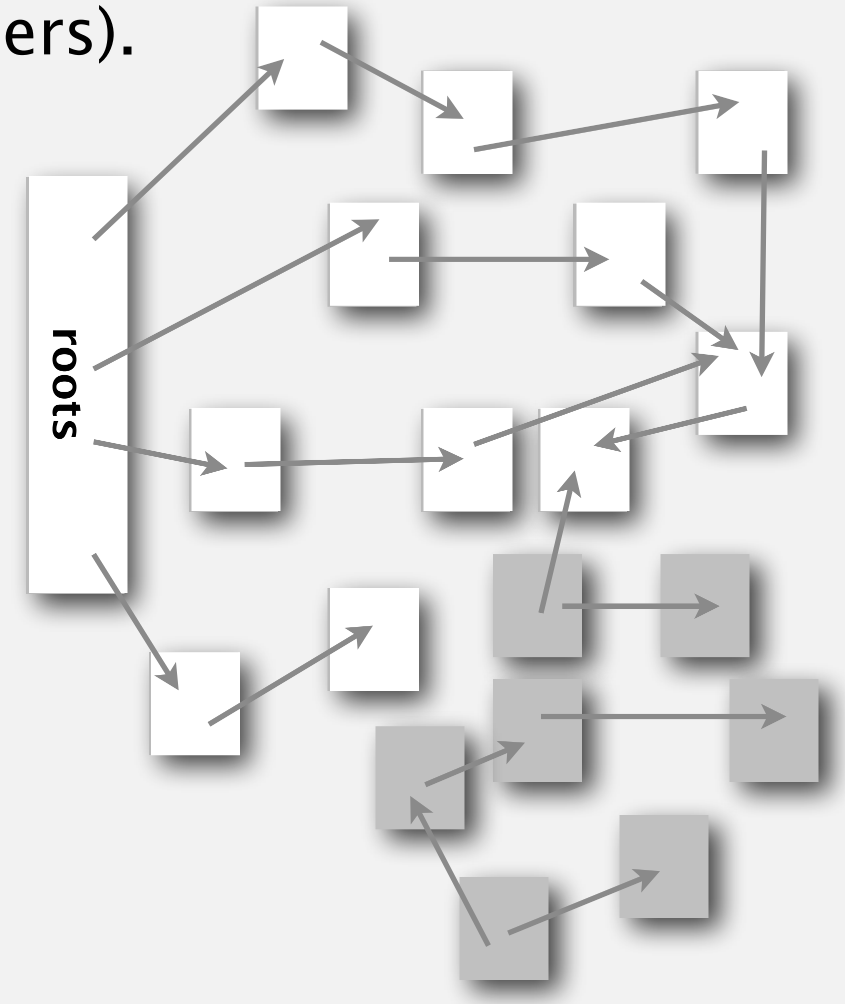
Reachability application: mark-sweep garbage collector

Every data structure is a digraph.

- Vertex = object.
- Edge = reference.

Roots. Objects known to be directly accessible by program (e.g., stack).

Reachable objects. Objects indirectly accessible by program (starting at a root and following a chain of pointers).

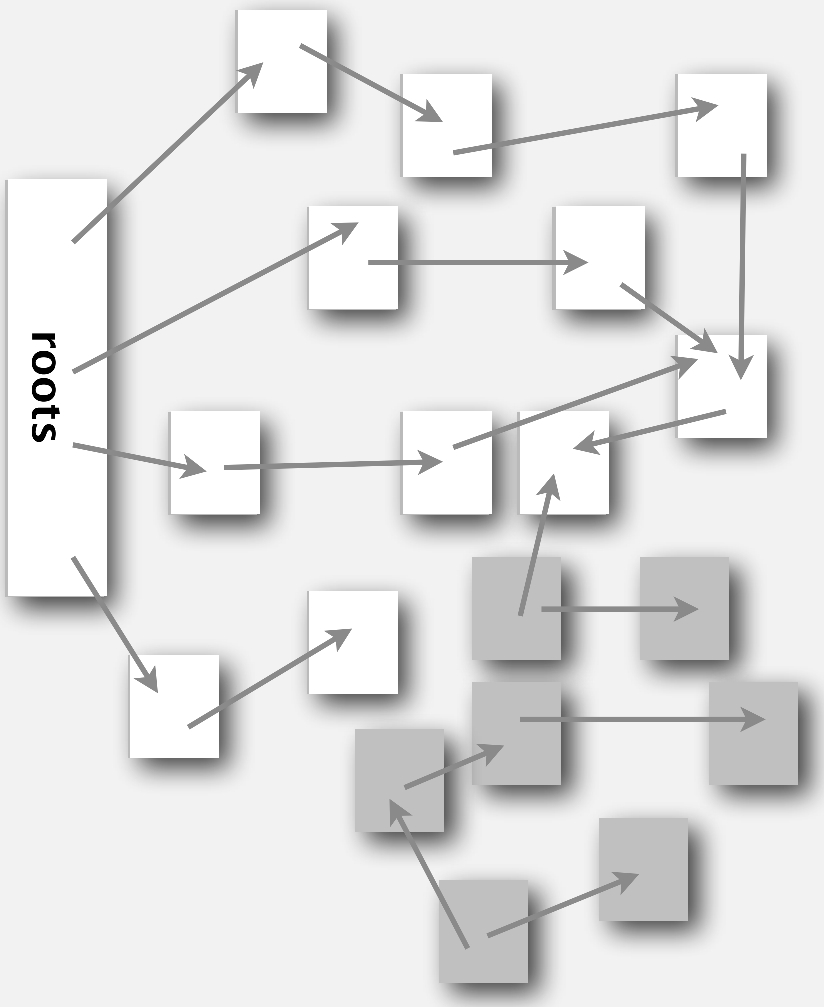


Reachability application: mark-sweep garbage collector

Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS stack).



Depth-first search in digraphs summary

DFS enables direct solution of simple digraph problems.

- ✓ Reachability.
- ✓ Path finding.
- Topological sort.
- Directed cycle detection.

Basis for solving difficult digraph problems.

- 2-satisfiability.
- Directed Euler path.
- Strongly connected components.

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an un-direct graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

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4.2 Directed Graphs

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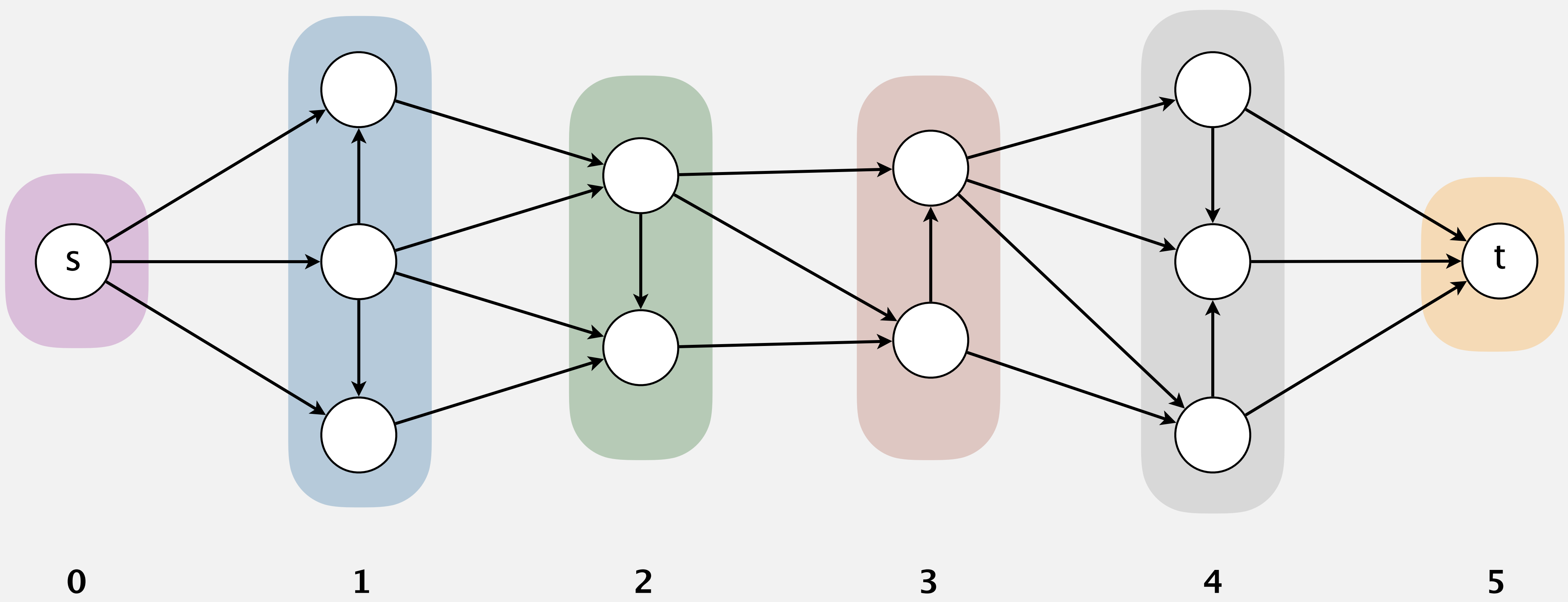
- depth-first search
- **breadth-first search**
- topological sort

Shortest directed paths

Problem. Find directed path from s to each vertex that uses fewest edges.



Breadth-first search in digraphs

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- BFS is a digraph algorithm.

BFS (from source vertex s)

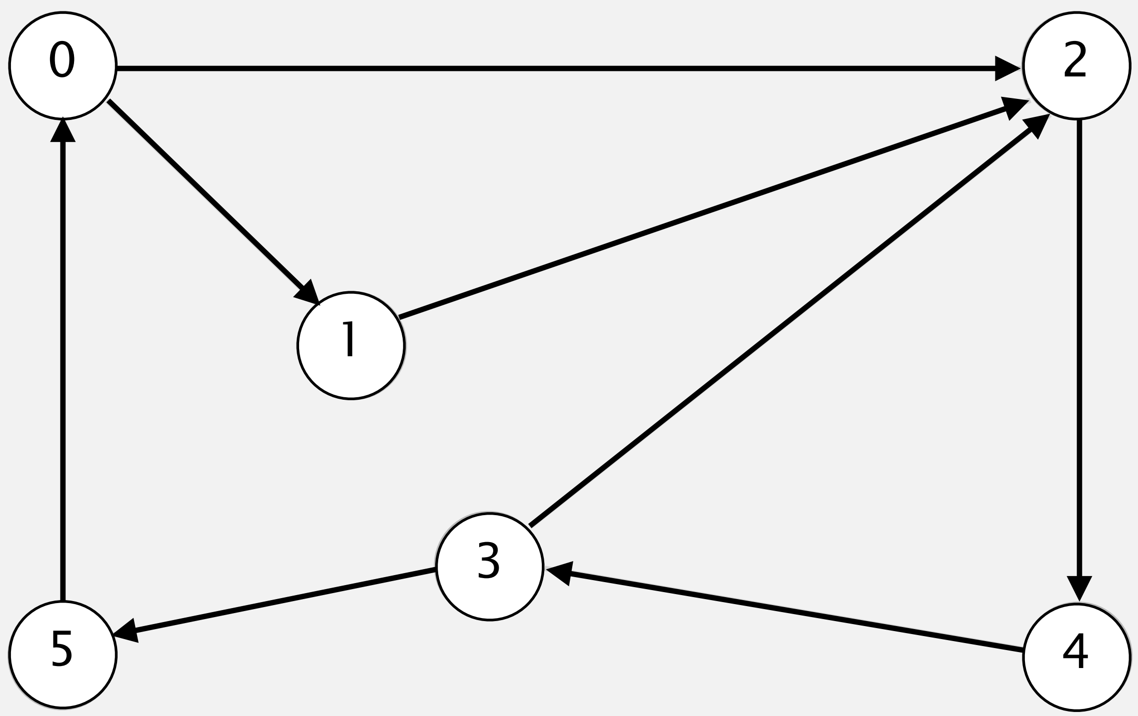
Put s onto a FIFO queue, and mark s as visited.
Repeat until the queue is empty:
- **remove the least recently added vertex v**
- **for each unmarked vertex adjacent from v: add to queue and mark as visited.**

Proposition. In worst case, BFS computes directed path with fewest edges from s to each vertex in time proportional to $E + V$.

Directed breadth-first search demo

Repeat until queue is empty:

- Remove vertex *v* from queue.
- Add to queue all unmarked vertices adjacent from *v* and mark them.



graph G

tinyDG2.txt

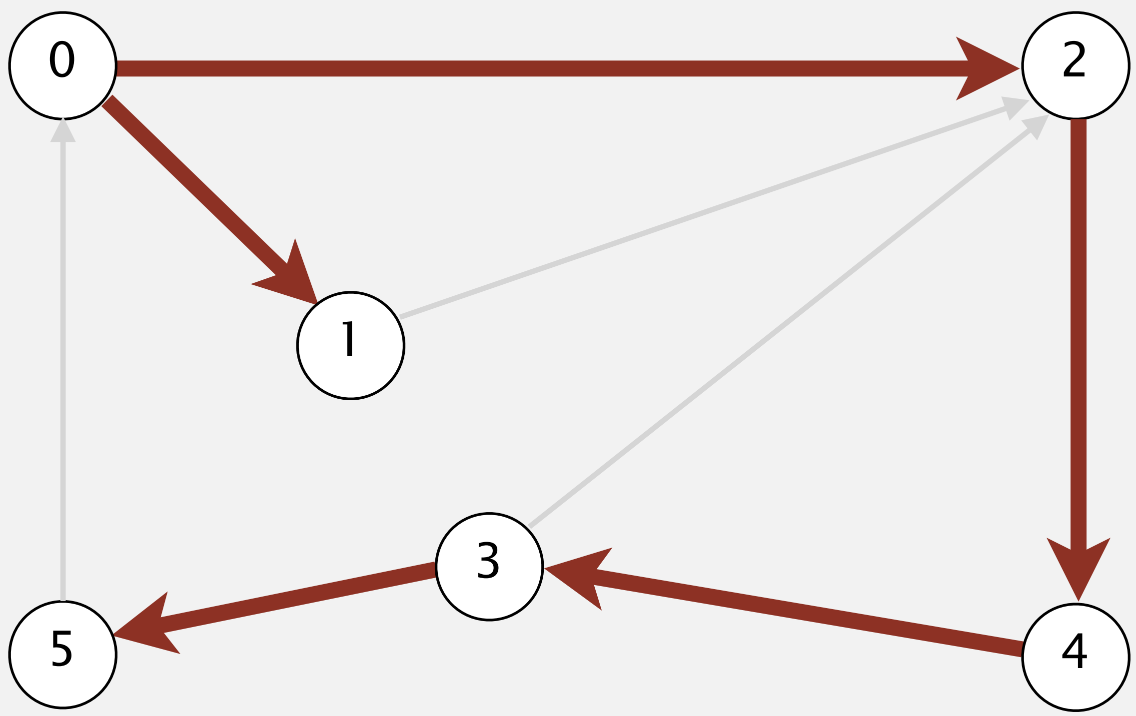
V → 6
E → 8

```
6
8
5 0
2 4
3 2
1 2
0 1
4 3
3 5
0 2
```

Directed breadth-first search demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent from v and mark them.



v	edgeTo[]	marked[]
0	–	T
1	0	T
2	0	T
3	4	T
4	2	T
5	3	T

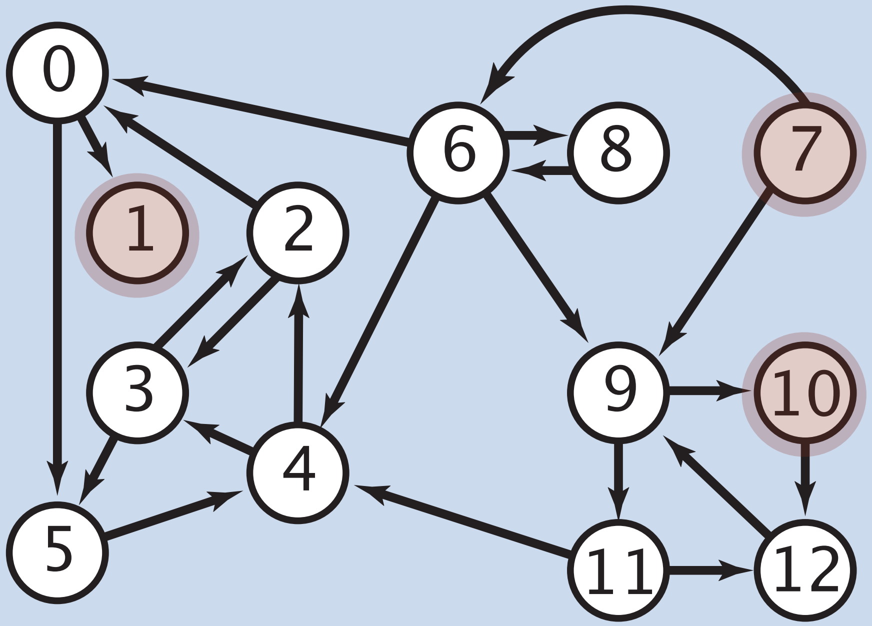
all done

Multiple-source shortest paths

Given a digraph and a set of source vertices, find shortest path from any vertex in the set to every other vertex.

Ex. $S = \{ 1, 7, 10 \}$.

- Shortest path to 4 is 7→6→4.
- Shortest path to 5 is 7→6→0→5.
- Shortest path to 12 is 10→12.

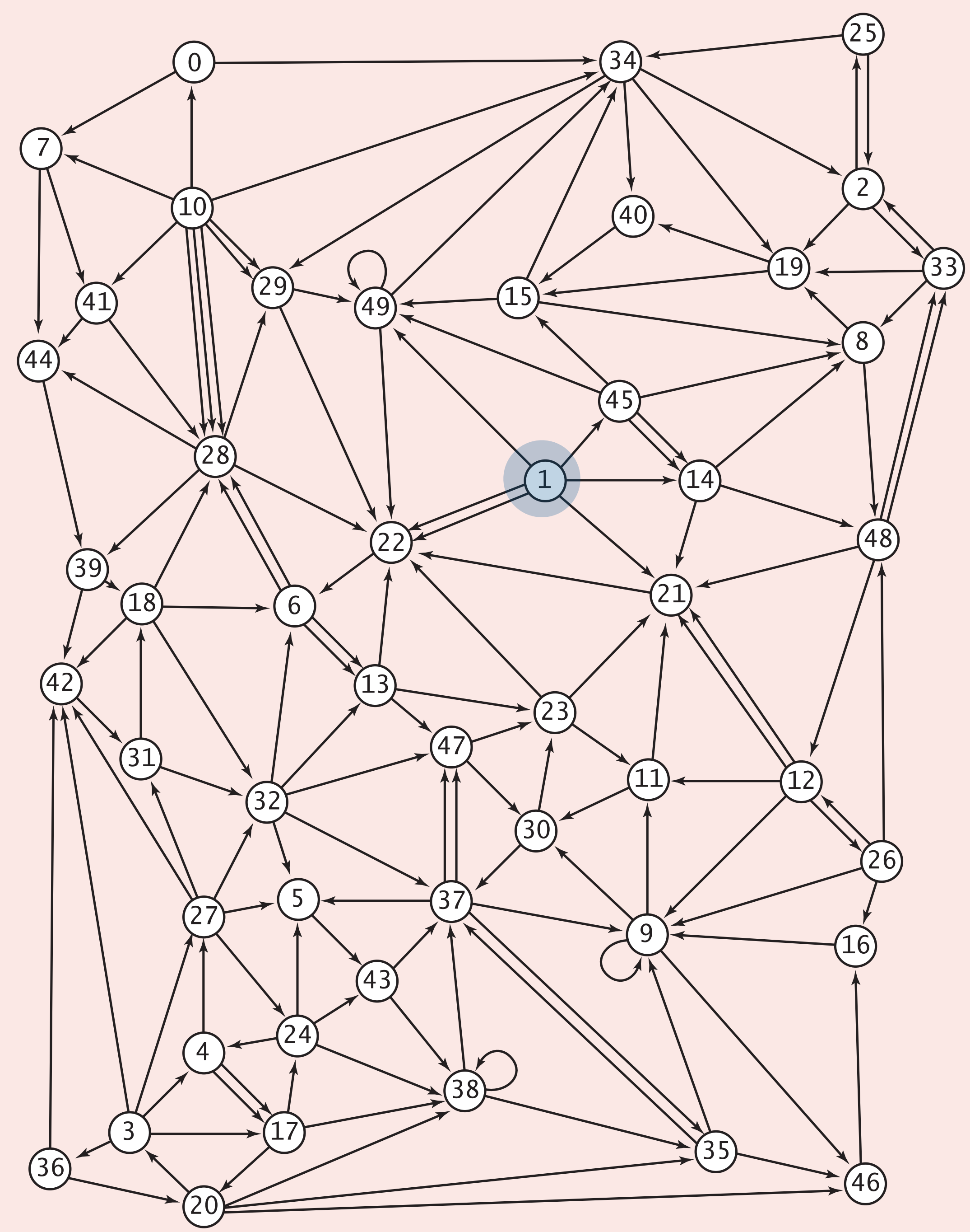


needed for Assignment 6

Q. How to implement multi-source shortest paths algorithm?

Suppose that you want to design a web crawler. Which graph-search algorithm should you use?

- **A.** depth-first search
- **B.** breadth-first search
- **C.** either A or B
- **D.** neither A nor B



Web crawler output

BFS crawl

```
http://www.princeton.edu
http://www.w3.org
http://ogp.me
http://giving.princeton.edu
http://www.princetonartmuseum.org
http://www.goprincetontigers.com
http://library.princeton.edu
http://helpdesk.princeton.edu
http://tigernet.princeton.edu
http://alumni.princeton.edu
http://gradschool.princeton.edu
http://vimeo.com
http://princetonusg.com
http://artmuseum.princeton.edu
http://jobs.princeton.edu
http://odoc.princeton.edu
http://blogs.princeton.edu
http://www.facebook.com
http://twitter.com
http://www.youtube.com
http://deimos.apple.com
http://qeprize.org
http://en.wikipedia.org
...
```

DFS crawl

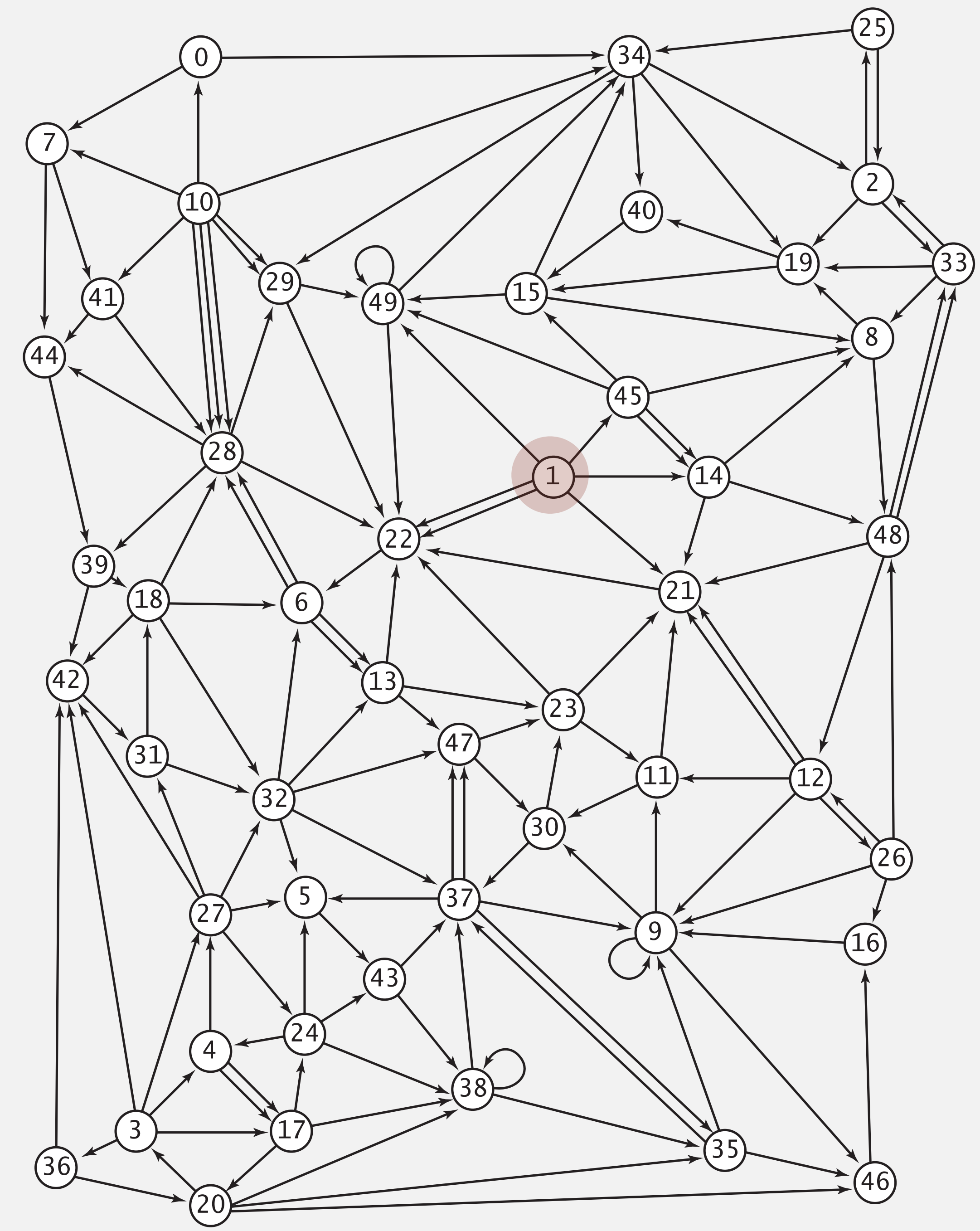
```
http://www.princeton.edu
http://deimos.apple.com
http://www.youtube.com
http://www.google.com
http://news.google.com
http://csi.gstatic.com
http://googlenewsblog.blogspot.com
http://labs.google.com
http://groups.google.com
http://img1.blogblog.com
http://feeds.feedburner.com
http:/buttons.googlesyndication.com
http://fusion.google.com
http://insidesearch.blogspot.com
http://agoogleaday.com
http://static.googleusercontent.com
http://searchresearch1.blogspot.com
http://feedburner.google.com
http://www.dot.ca.gov
http://www.TahoeRoads.com
http://www.LakeTahoeTransit.com
http://www.laketahoe.com
http://ethel.tahoeguide.com
...
```

Breadth-first search in digraphs application: web crawler

Goal. Crawl web, starting from some root web page, say `www.princeton.edu`.

Solution. [BFS with implicit digraph]

- Choose root web page as source s.
- Maintain a `Queue` of websites to explore.
- Maintain a `SET` of marked websites.
- Dequeue the next website and enqueue any unmarked websites to which it links.



Remark. Industrial-strength web crawlers use more sophisticated algorithms.

Bare-bones web crawler: Java implementation

```
Queue<String> queue = new Queue<String>();
SET<String> marked = new SET<String>();

String root = "http://www.princeton.edu";
queue.enqueue(root);
marked.add(root);

while (!queue.isEmpty())
{
   String v = queue.dequeue();
   StdOut.println(v);
   In in = new In(v);
   String input = in.readAll();

   String regexp = "http://(\\w+\\.)+(\\w+)";
   Pattern pattern = Pattern.compile(regexp);
   Matcher matcher = pattern.matcher(input);

   while (matcher.find())
   {
      String w = matcher.group();
      if (!marked.contains(w))
      {
          marked.add(w);
          q.enqueue(w);
      }
   }
}
```

- queue of websites to crawl
- set of marked websites
- start crawling from root website
- read in raw html from next website in queue
- use regular expression to find all URLs in website of form `http://xxx.yyy.zzz` [crude pattern misses relative URLs]
- if unmarked, mark and enqueue

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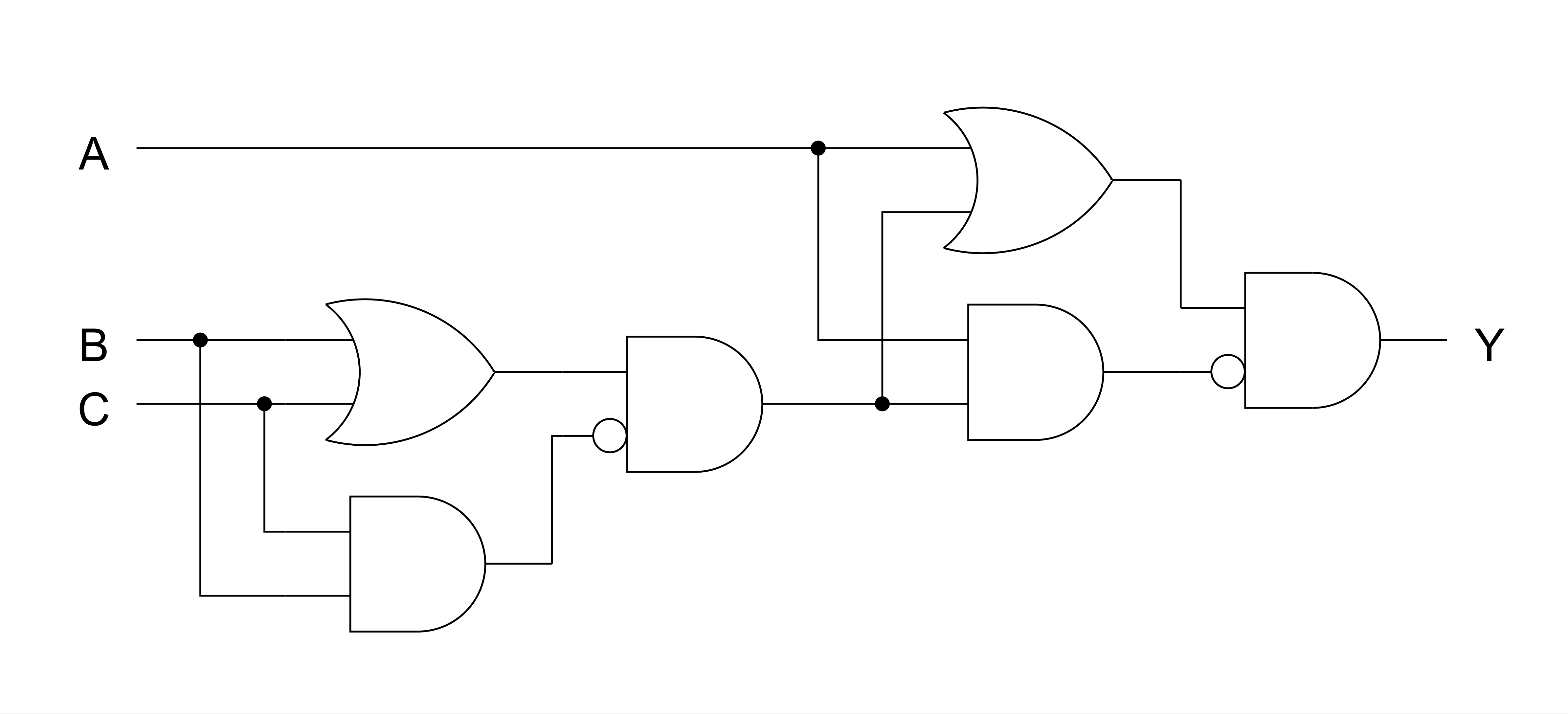
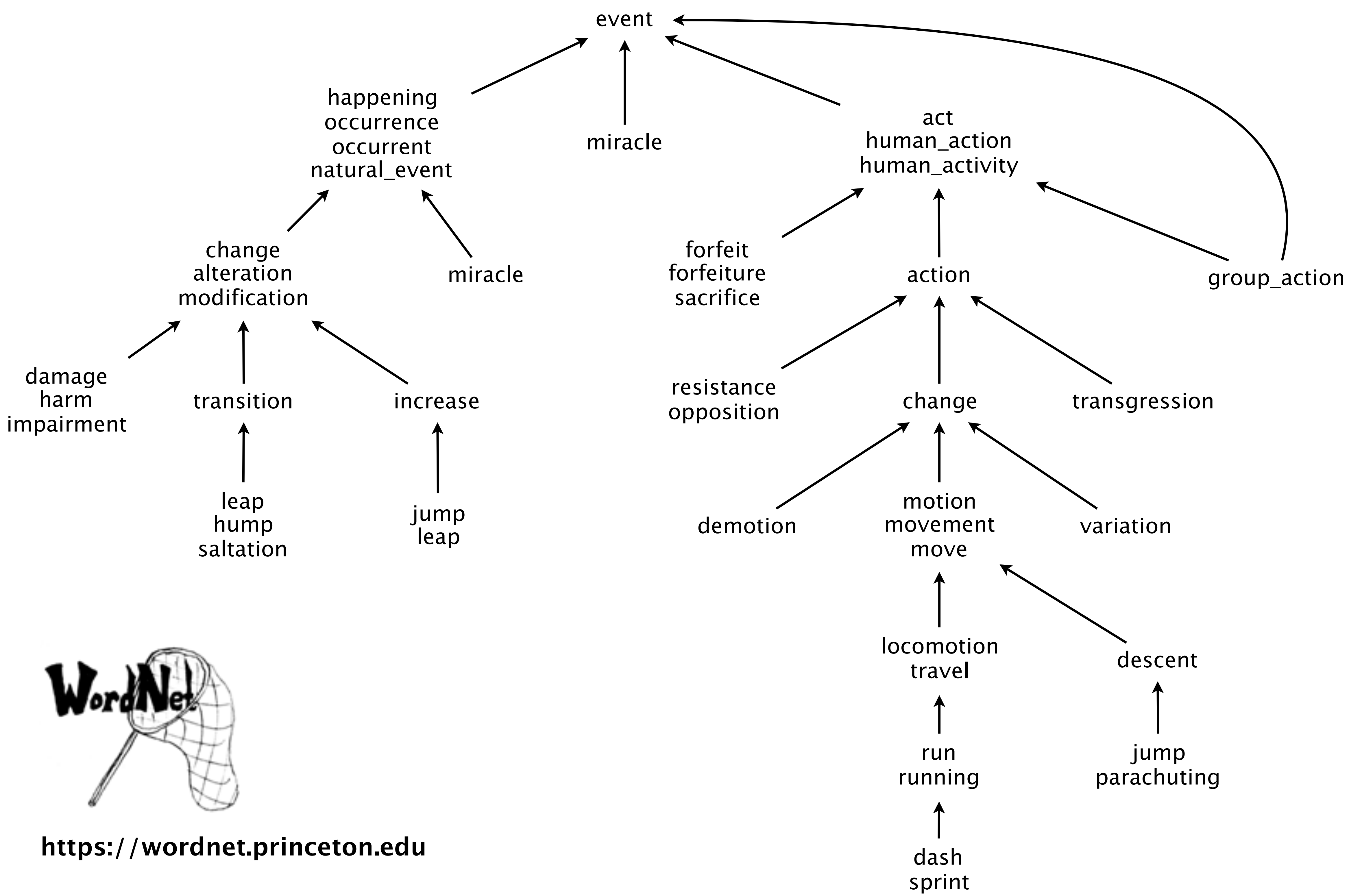
Combinational circuit

Vertex = logical gate; edge = wire.



WordNet digraph

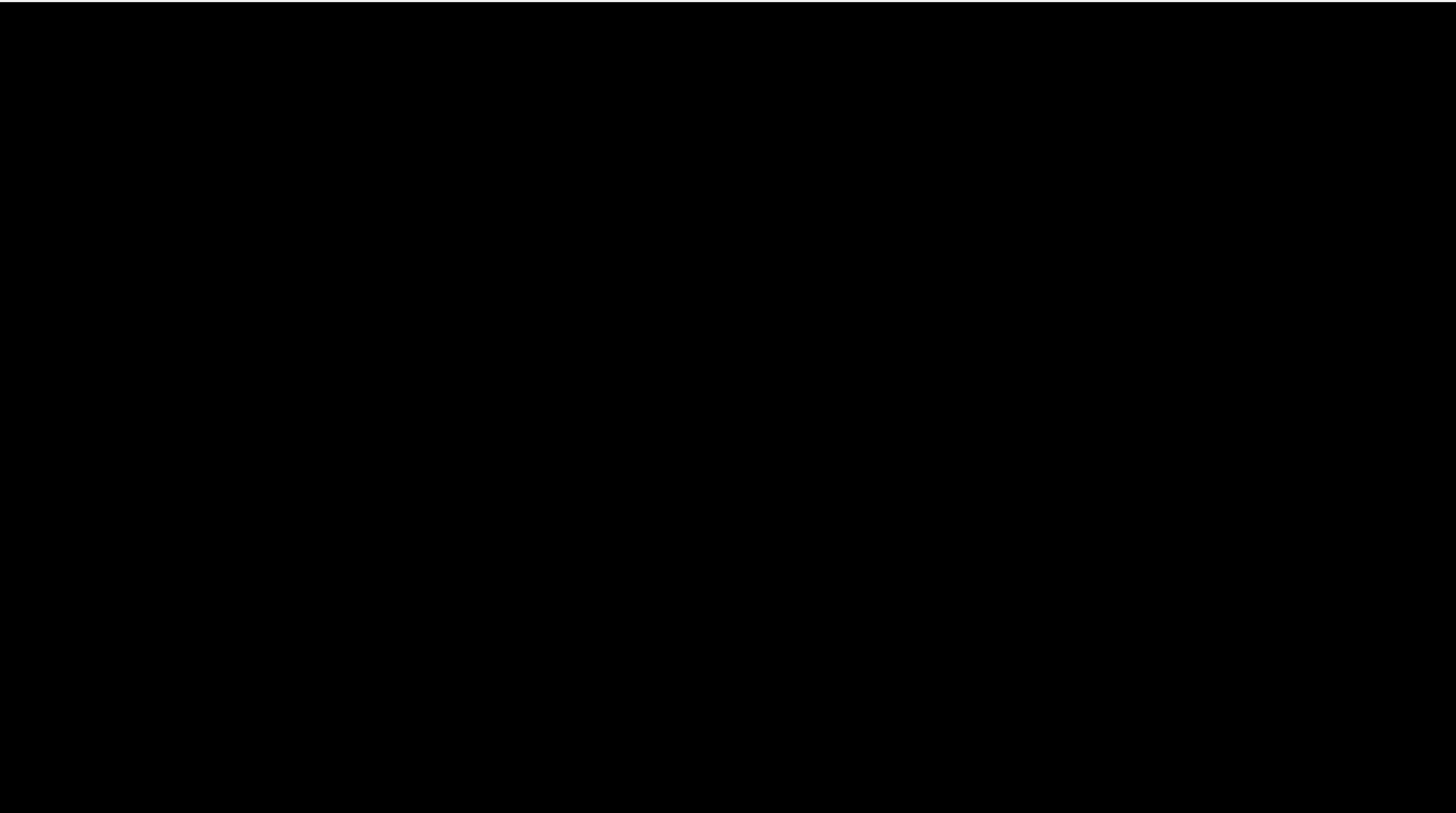
Vertex = synset; edge = hypernym relationship.



https://wordnet.princeton.edu

Git digraph

Vertex = revision of repository; edge = revision relationship.



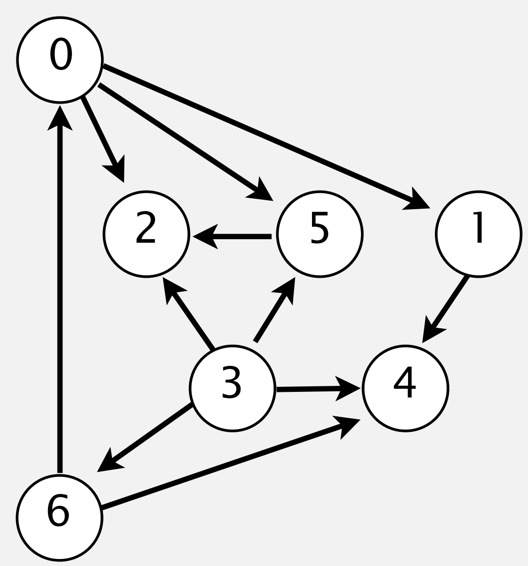
Precedence scheduling

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

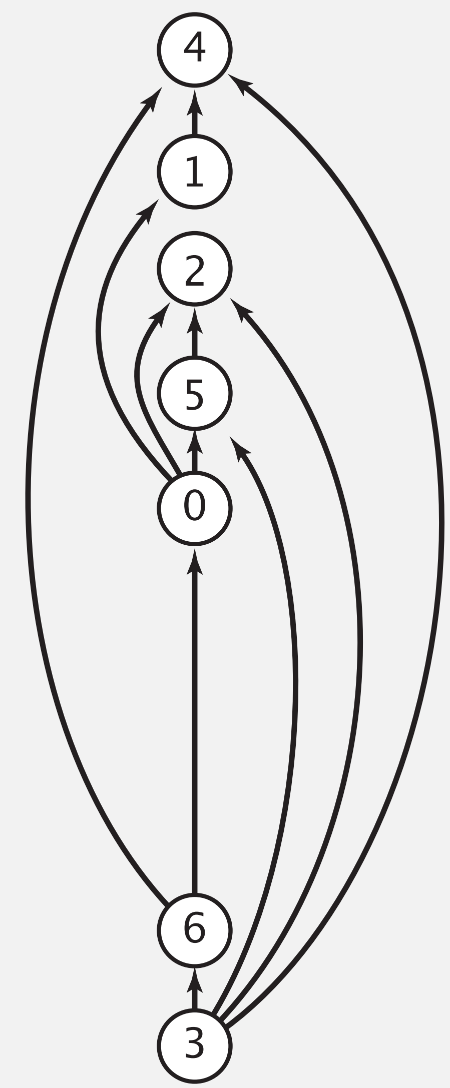
Digraph model. vertex = task; edge = precedence constraint.

0. Algorithms
1. Complexity Theory
2. Machine Learning
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Discrete Math

tasks



precedence constraint graph



feasible schedule

Topological sort

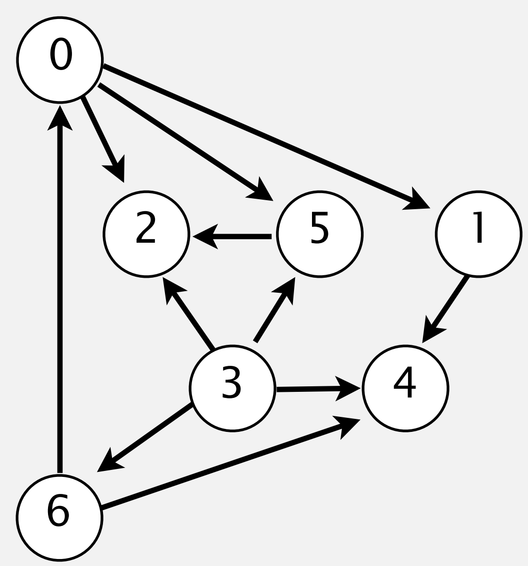
DAG. Directed acyclic graph.

Topological sort. Redraw DAG so all edges point upwards.

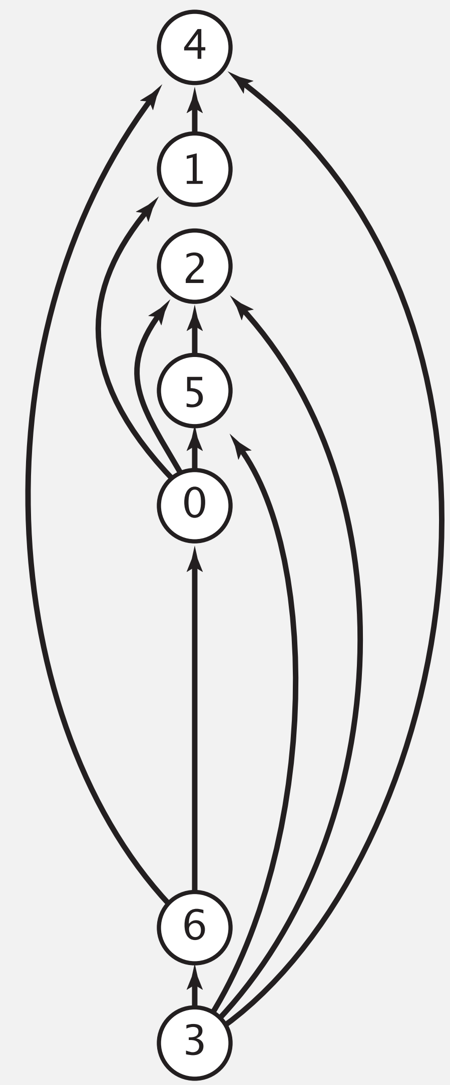
edges in DAG define a "partial order" for vertices

0→5	0→2
0→1	3→6
3→5	3→4
5→2	6→4
6→0	3→2
1→4	

directed edges



DAG

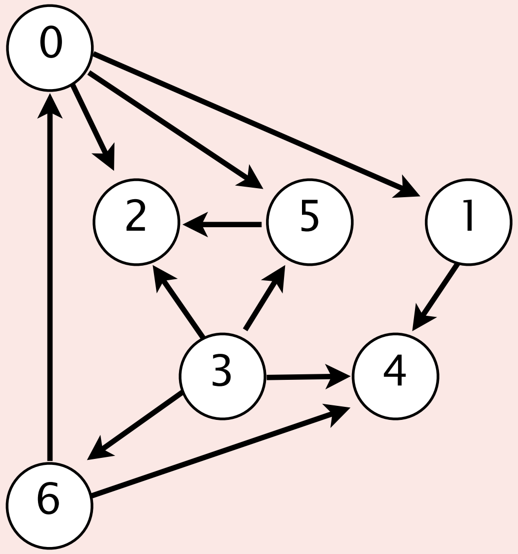


topological order

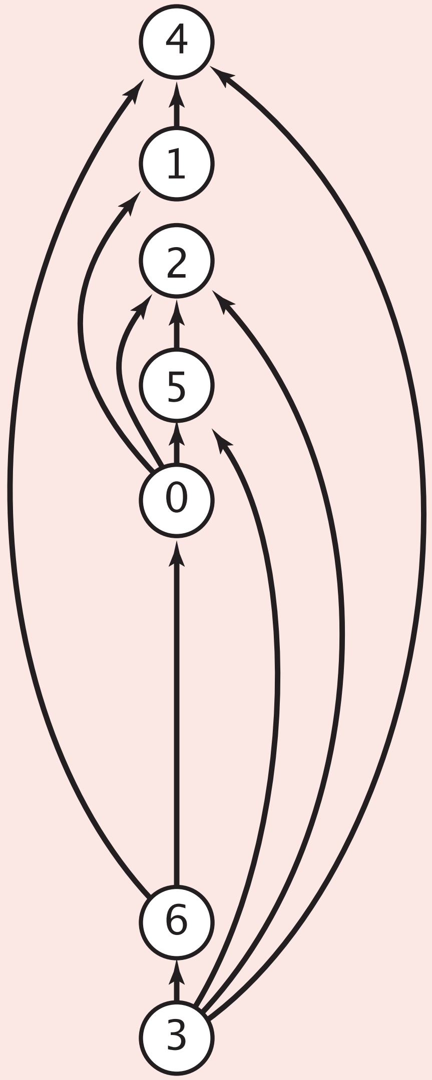
Directed graphs: quiz 4

Suppose that you want to topologically sort the vertices in a DAG.
Which graph-search algorithm should you use?

- **A.** depth-first search
- **B.** breadth-first search
- **C.** either A or B
- **D.** neither A nor B



DAG

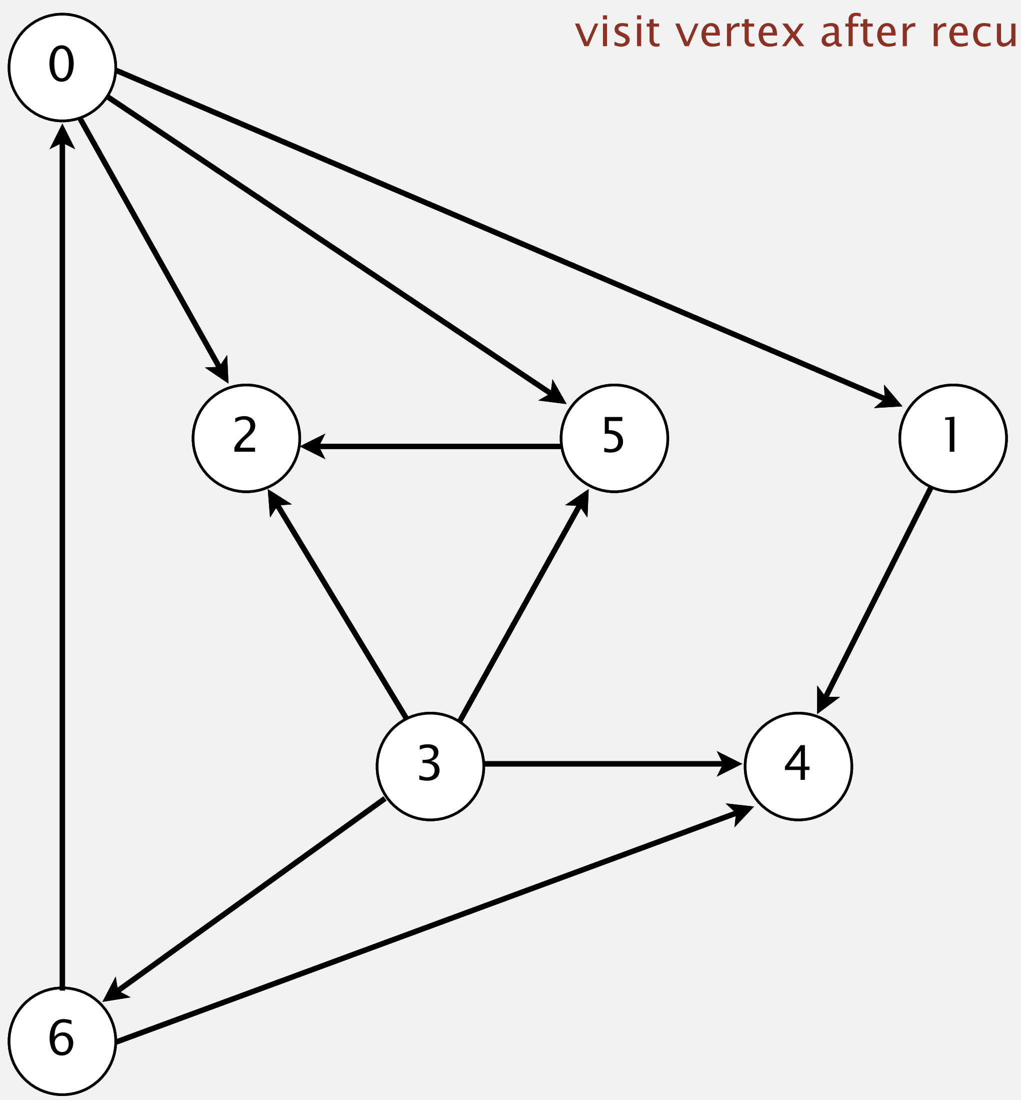


topological order

Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.

visit vertex after recursive calls



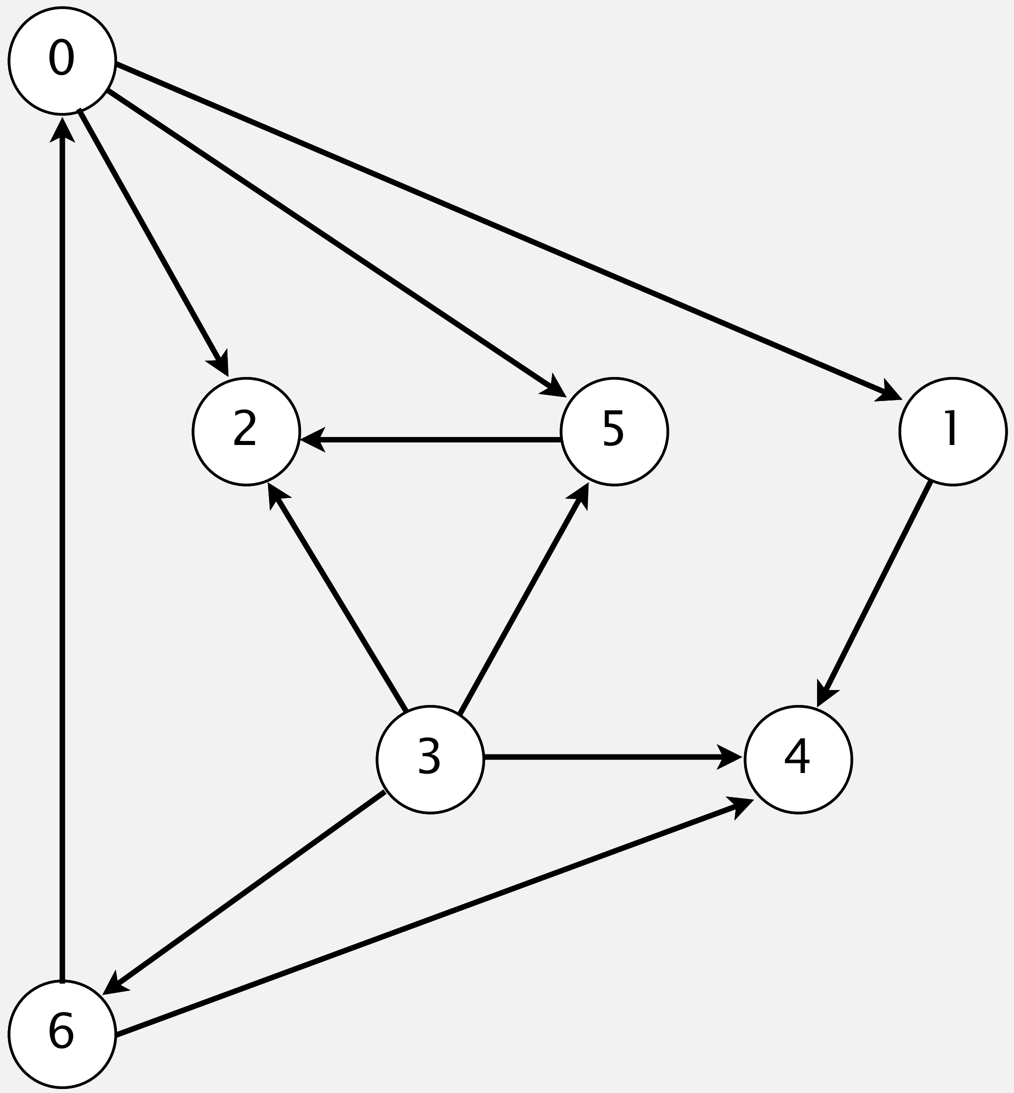
a directed acyclic graph

tinyDAG7.txt

```
7
11
 0  5
 0  2
 0  1
 3  6
 3  5
 3  4
 5  2
 6  4
 6  0
 3  2
```

Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.



DFS postorder

4 1 2 5 0 6 3

topological order
(reverse DFS postorder)

3 6 0 5 2 1 4

done

Depth-first search: reverse postorder

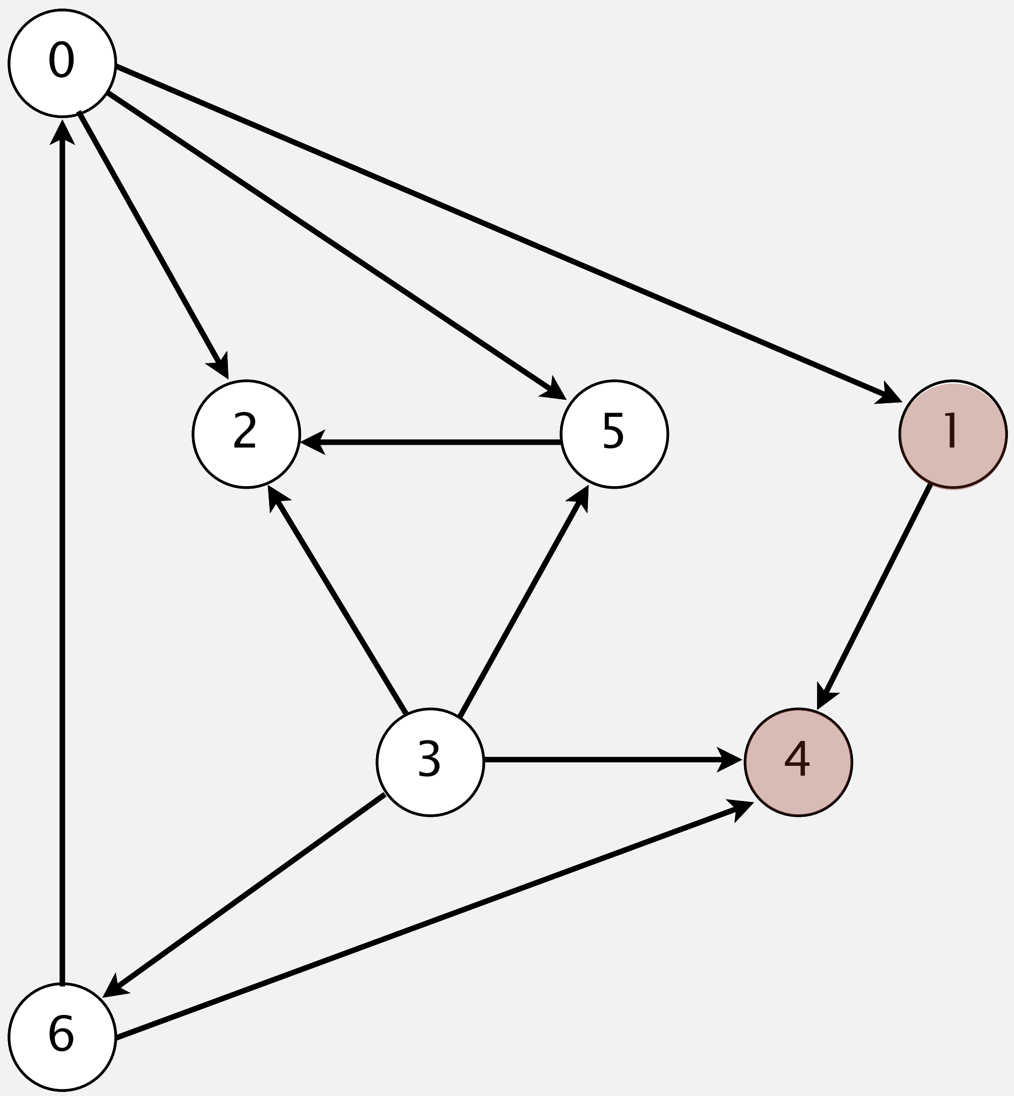
```
public class DepthFirstOrder
{
   private boolean[] marked;
   private Stack<Integer> reversePostorder;

   public DepthFirstOrder(Digraph G)
   {
      reversePostorder = new Stack<Integer>();
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) dfs(G, v);
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
      reversePostorder.push(v);
   }

   public Iterable<Integer> reversePostorder()
   {  return reversePostorder;  }

}
```

returns all vertices in "reverse DFS postorder"

Topological sort in a DAG: intuition

Why is the reverse DFS postorder a topological order?

- First vertex in DFS postorder (last in topological order) has outdegree 0.
- Second vertex in DFS postorder can point only to first vertex.
- ...



DFS postorder

4 1 2 5 0 6 3

topological order (reverse DFS postorder)

3 6 0 5 2 1 4

Topological sort in a DAG: correctness proof

Proposition. Reverse DFS postorder of a DAG is a topological order.

Pf. Consider any edge v→w. When `dfs(v)` is called:

- Case 1: `dfs(w)` has already been called and returned.
  - thus, w appears before v in DFS postorder
- Case 2: `dfs(w)` has not yet been called.
  - `dfs(w)` will get called directly or indirectly by `dfs(v)`
  - so, `dfs(w)` will return before `dfs(v)` returns
  - thus, w appears before v in DFS postorder
- Case 3: `dfs(w)` has already been called, but has not yet returned.
  - function-call stack contains directed path from w to v
  - edge v→w would complete a directed cycle
  - contradiction (it's a DAG)

```
dfs(0)
  dfs(1)
    dfs(4)
    4 done
  1 done
  dfs(2)
  2 done
  dfs(5)
    check 2
  5 done
0 done
check 1
check 2
dfs(3)
  check 2
  check 4
  check 5
  dfs(6)
    check 0
    check 4
  6 done
3 done
check 4
check 5
check 6
done
```

v = 3 → dfs(3)

case 1 (w = 2, 4, 5) → check 2, check 4, check 5

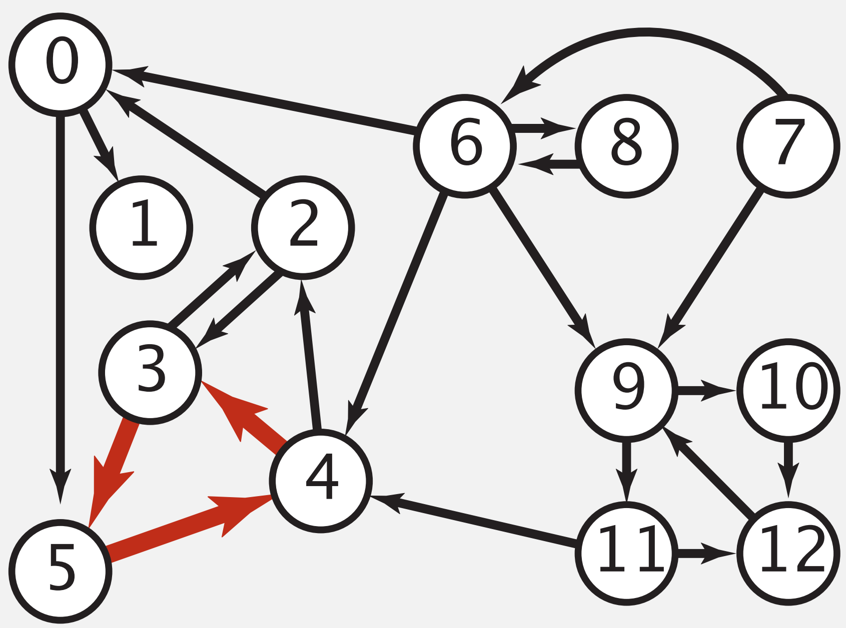
case 2 (w = 6) → dfs(6), 6 done

Directed cycle detection

Proposition. A digraph has a topological order iff no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



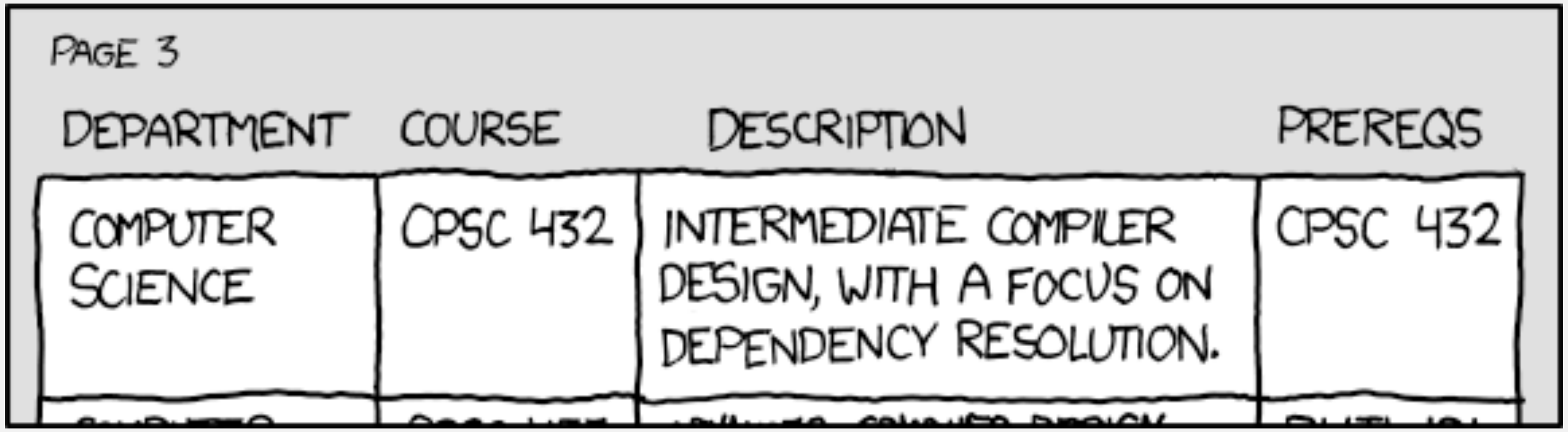
a digraph with a directed cycle

Goal. Given a digraph, find a directed cycle.

Solution. DFS. What else? See textbook.

Directed cycle detection application: precedence scheduling

Scheduling. Given a set of tasks to be completed with precedence constraints, in what order should we schedule the tasks?



http://xkcd.com/754

Remark. A directed cycle implies scheduling problem is infeasible.

Directed cycle detection application: cyclic inheritance

The Java compiler does directed cycle detection.

```
public class A extends B
{
   ...
}
```

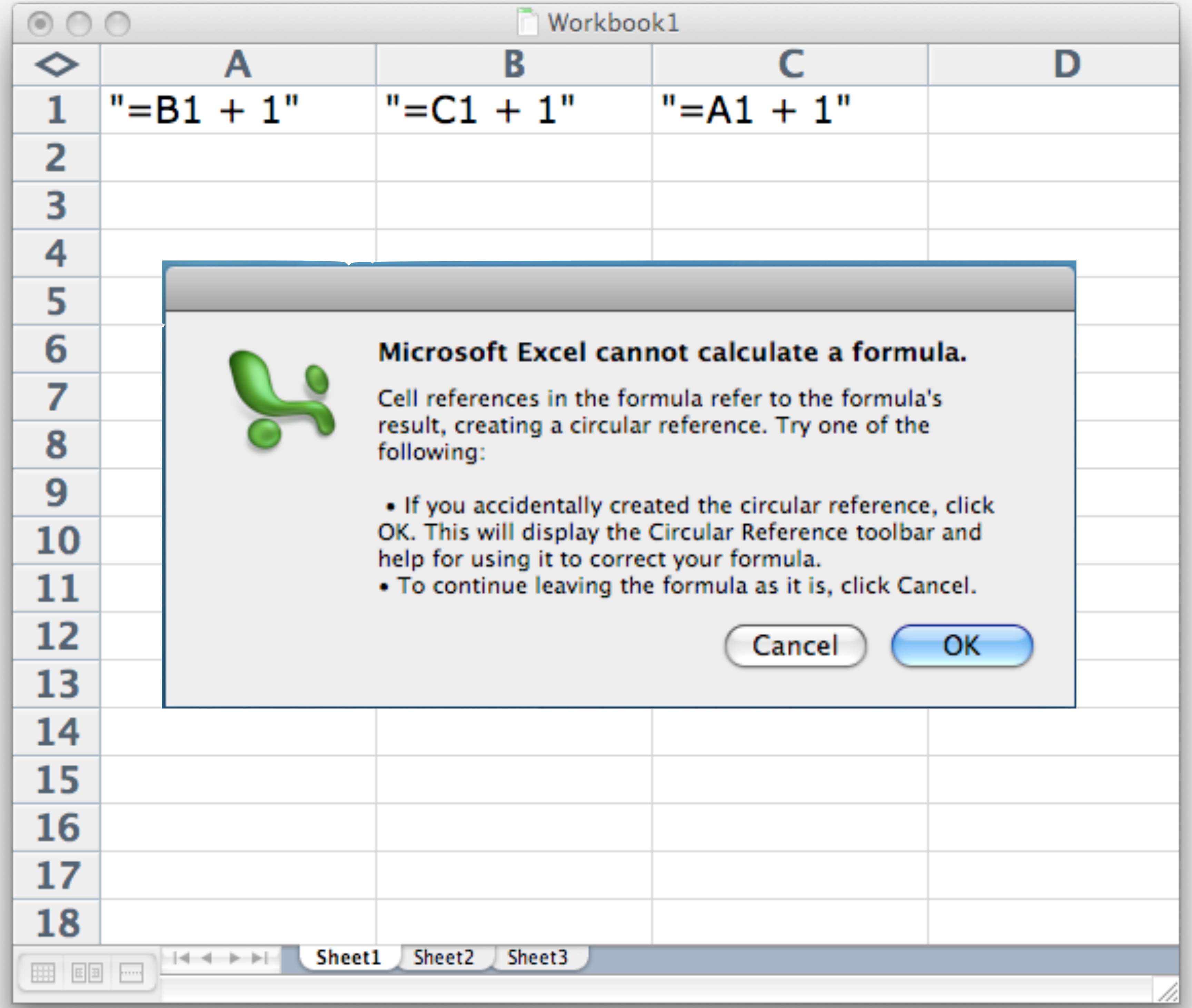
```
public class B extends C
{
   ...
}
```

```
public class C extends A
{
   ...
}
```

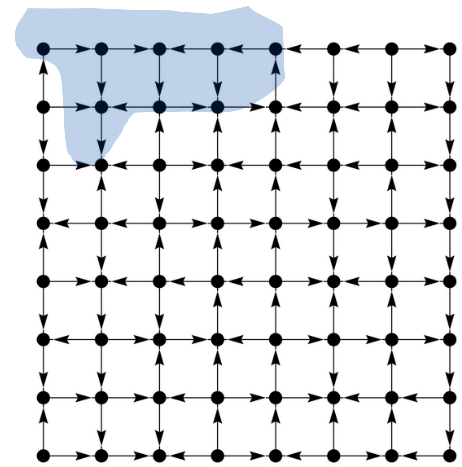
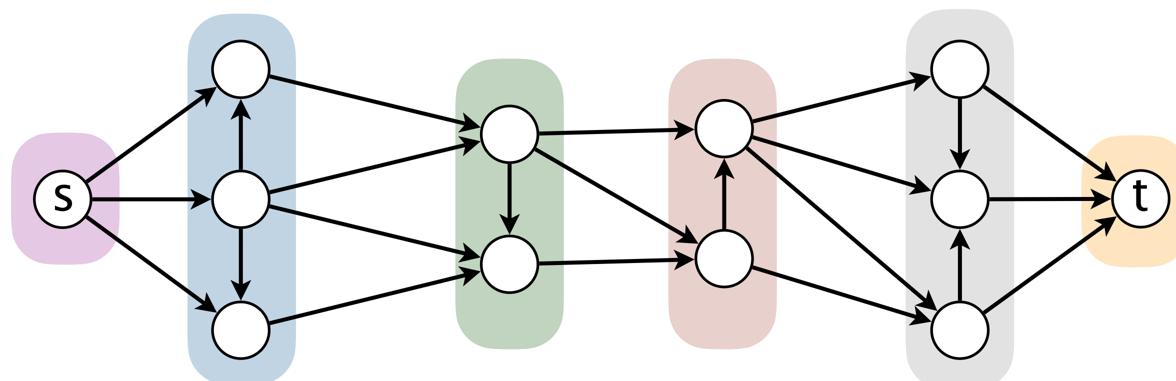
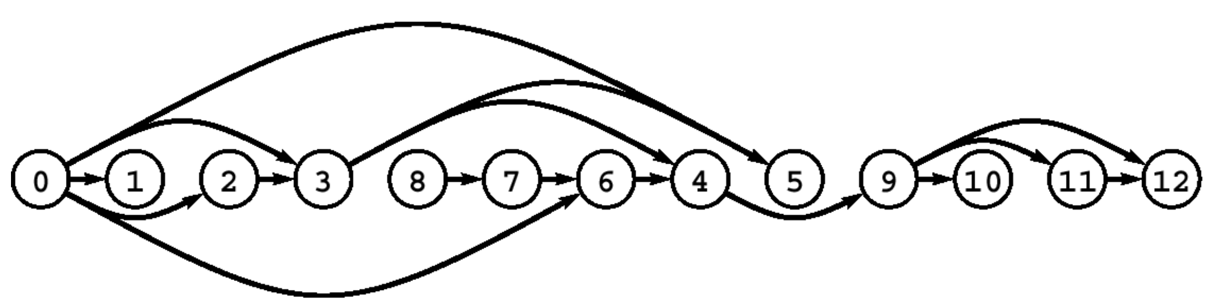
```
% javac A.java
A.java:1: cyclic inheritance
involving A
public class A extends B { }
       ^
1 error
```

Directed cycle detection application: spreadsheet recalculation

Microsoft Excel does directed cycle detection.



Digraph-processing summary: algorithms of the day

single-source reachability in a digraph		DFS/BFS
shortest path in a digraph		BFS
topological sort in a DAG		DFS