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3.3 BALANCED SEARCH TREES

- *2–3 search trees*
- *red–black BSTs*
- *B-trees (see book or videos)*

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
goal	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

Challenge. Guarantee performance.

optimized for teaching and coding; introduced to the world in this course!

This lecture. 2–3 trees and left-leaning red–black BSTs.

co-invented by Bob Sedgewick

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3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

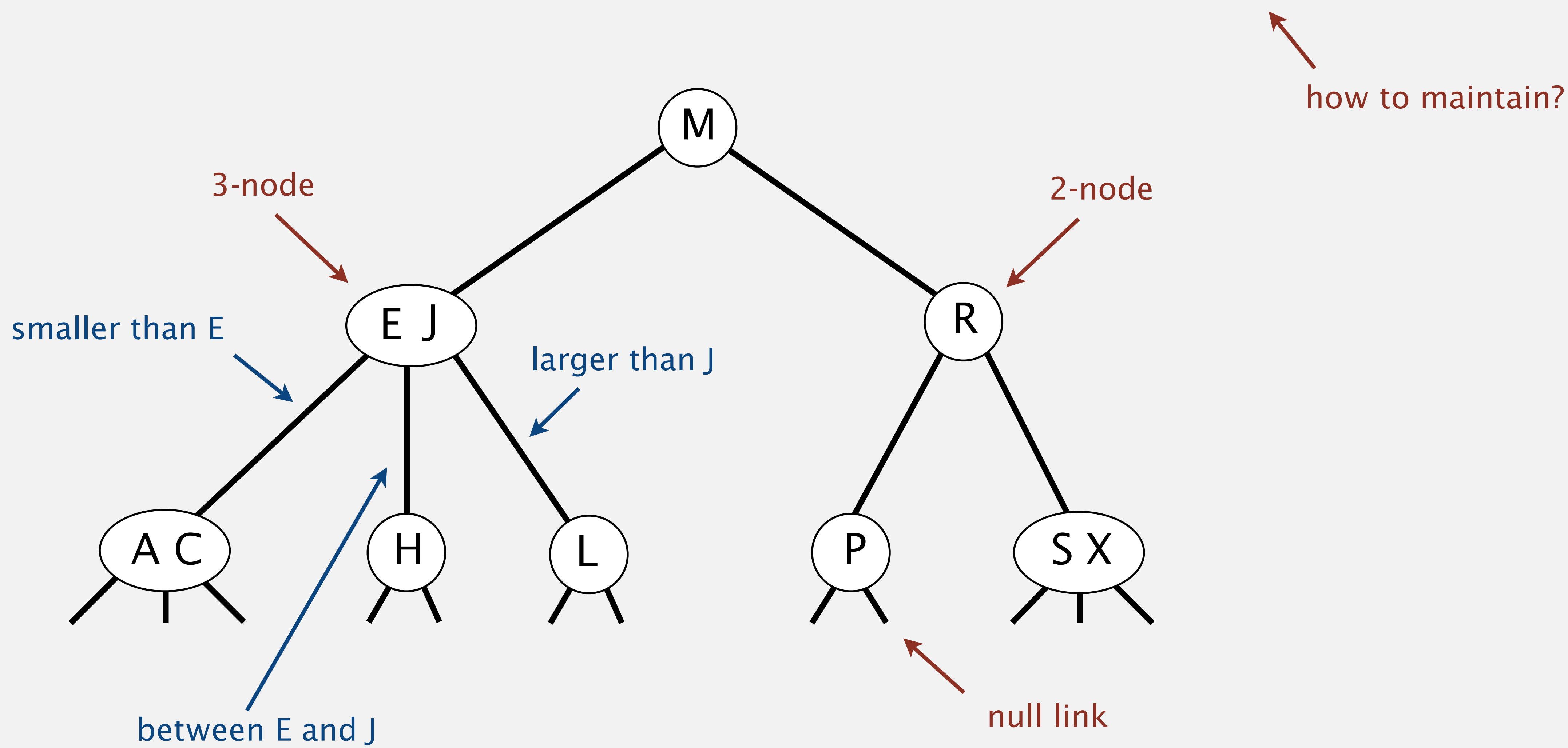
2–3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.

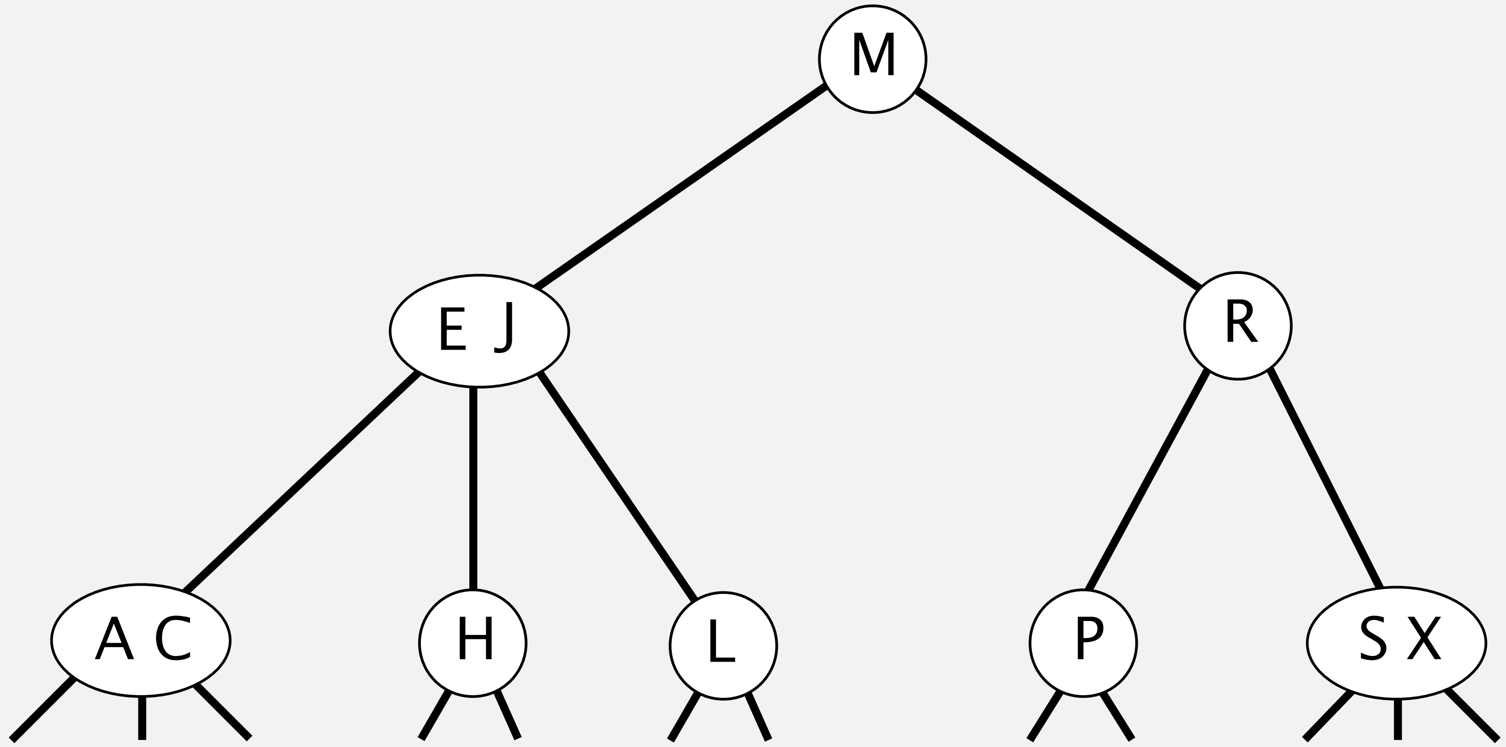


2-3 tree demo

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

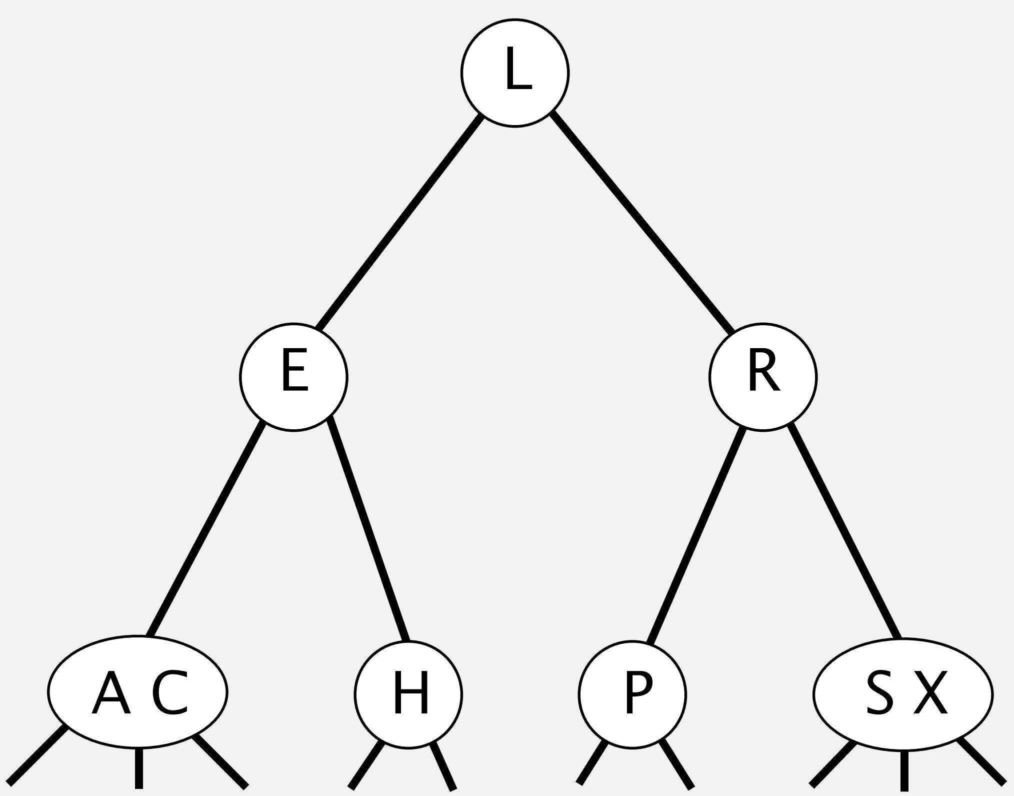


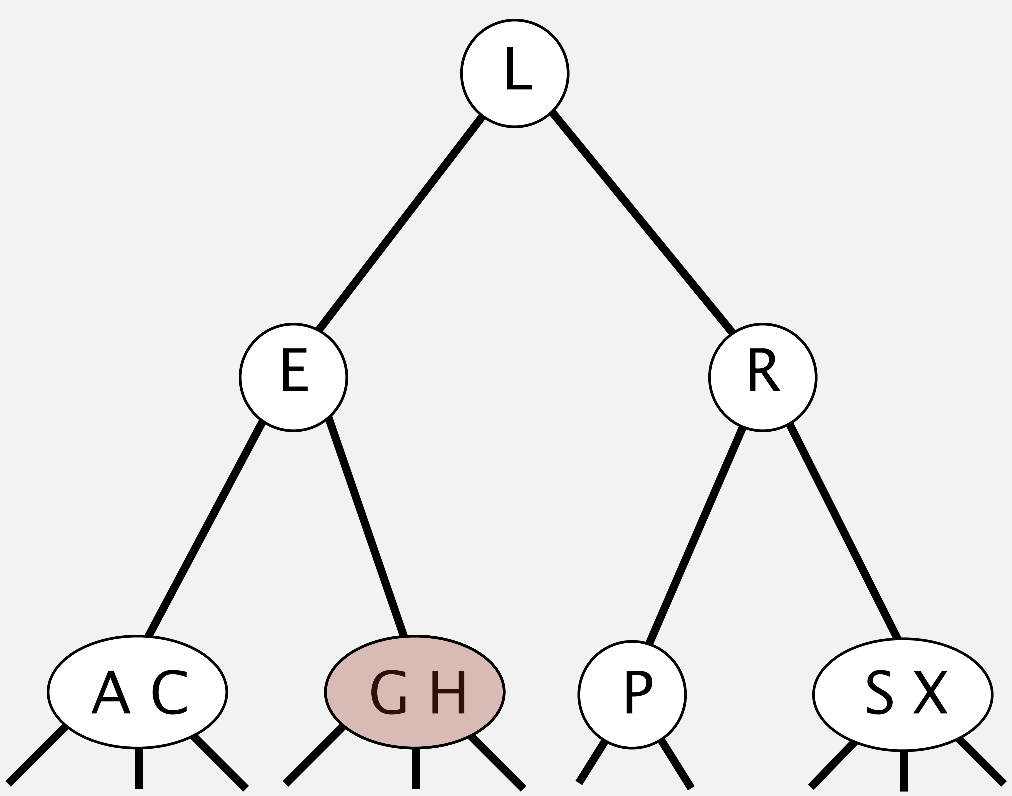
2–3 tree: insertion

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G



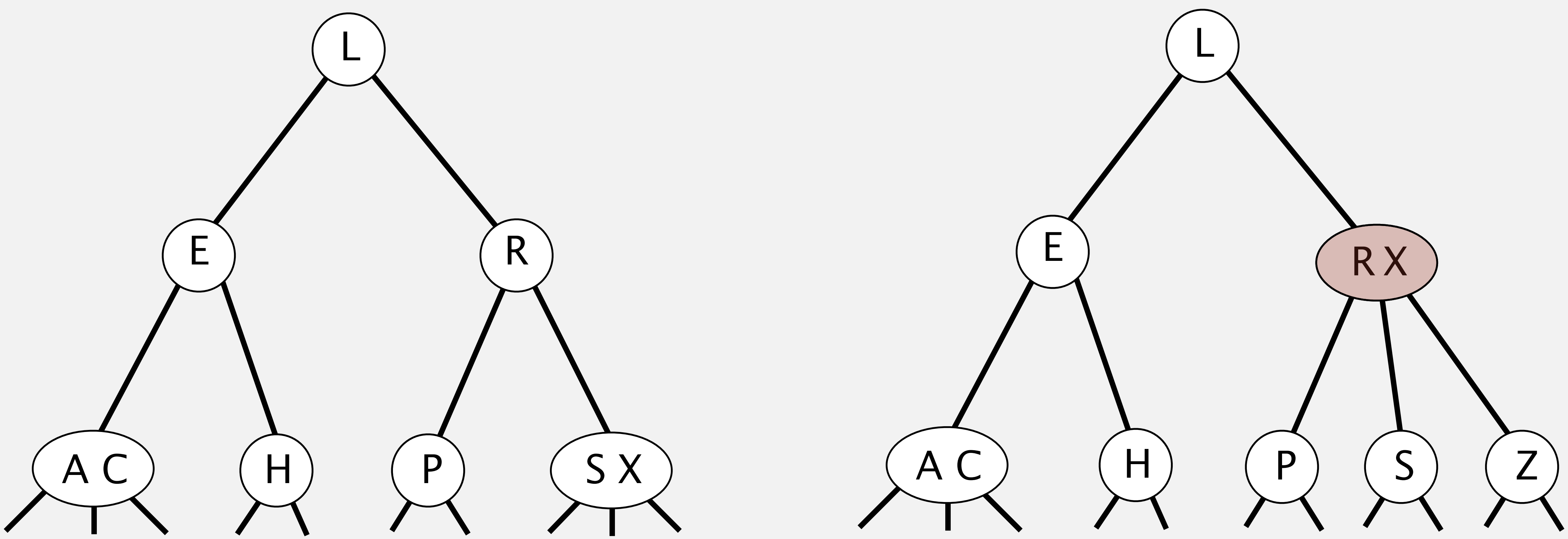


2–3 tree: insertion

Insertion into a 3-node at bottom.

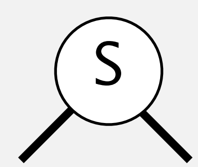
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert Z



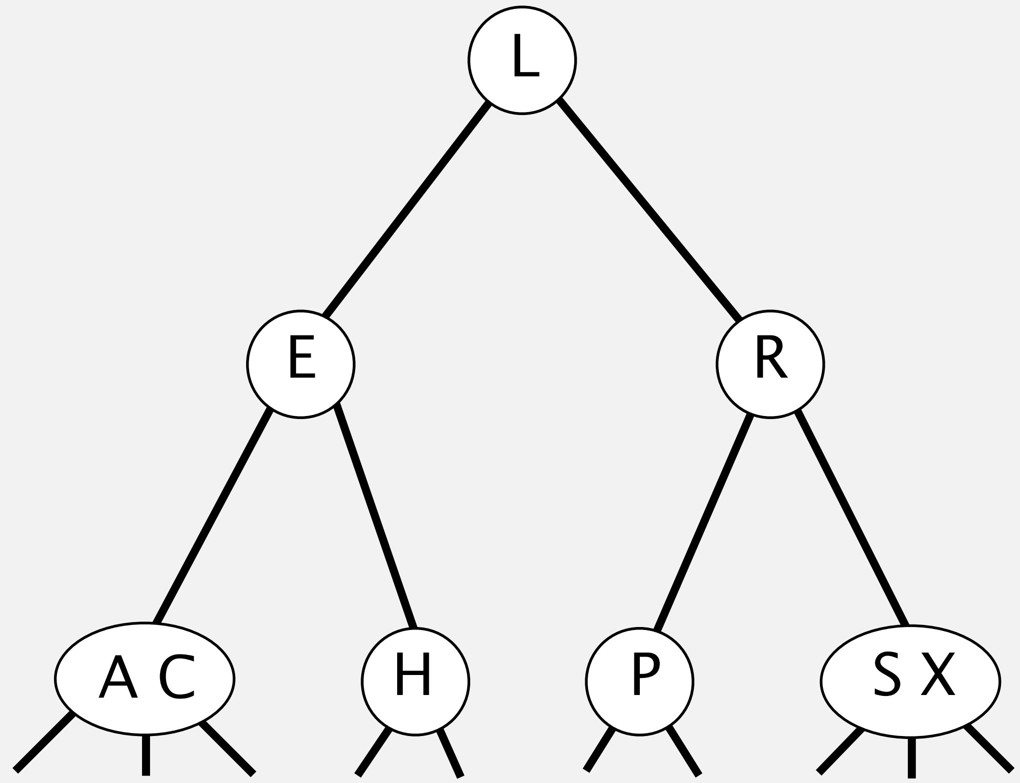
2-3 tree construction demo

insert S



2–3 tree construction demo

2–3 tree



Balanced search trees: quiz 1

What is the maximum height of a 2–3 tree with n keys?

A. $\sim \log_3 n$

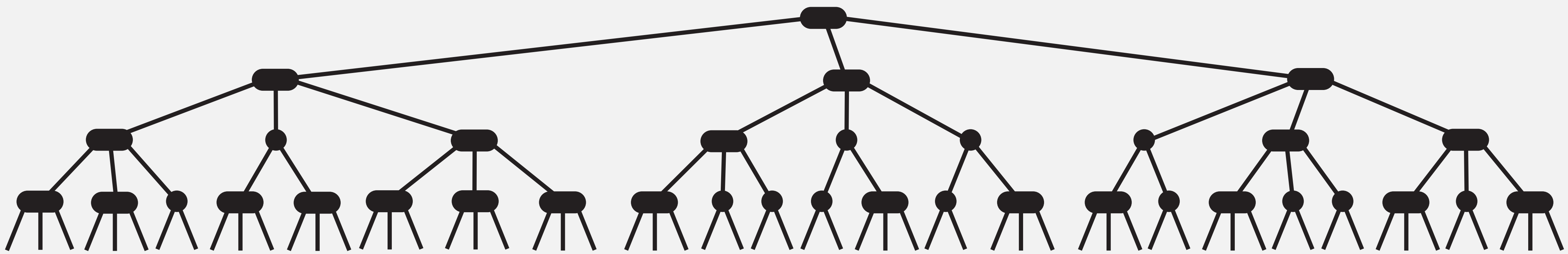
B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

D. $\sim n$

2–3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\log_2 n$. [all 2-nodes]
- Best case: $\log_3 n \approx 0.631 \log_2 n$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

but hidden constant c is large
(depends upon implementation)

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

fantasy code

```
public void put(Key key, Value val)
{
   Node x = root;
   while (x.getTheCorrectChild(key) != null)
   {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it, but there's a better way.

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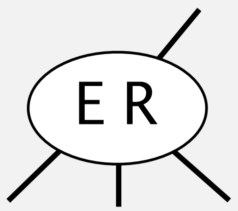
3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

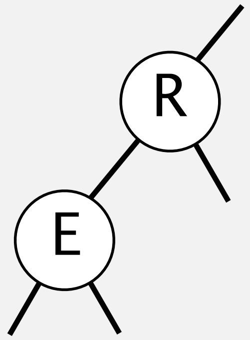
How to implement 2–3 trees with binary trees?

Challenge. How to represent a 3 node?



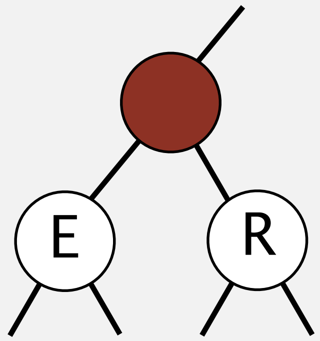
Approach 1. Regular BST.

- No way to tell a 3-node from two 2-nodes.
- Can't (uniquely) map from BST back to 2–3 tree.



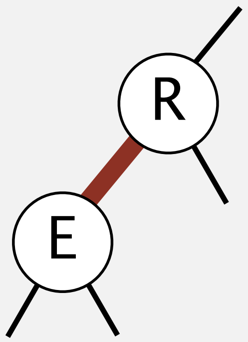
Approach 2. Regular BST with red "glue" nodes.

- Wastes space for extra node.
- Messy code.



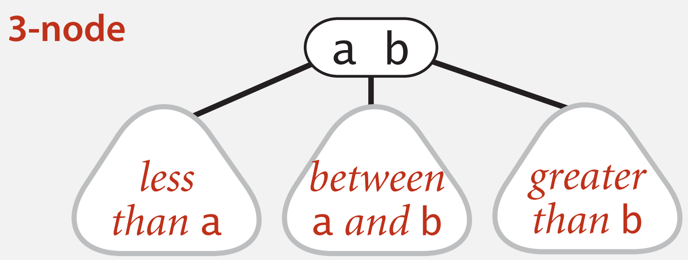
Approach 3. Regular BST with red "glue" links.

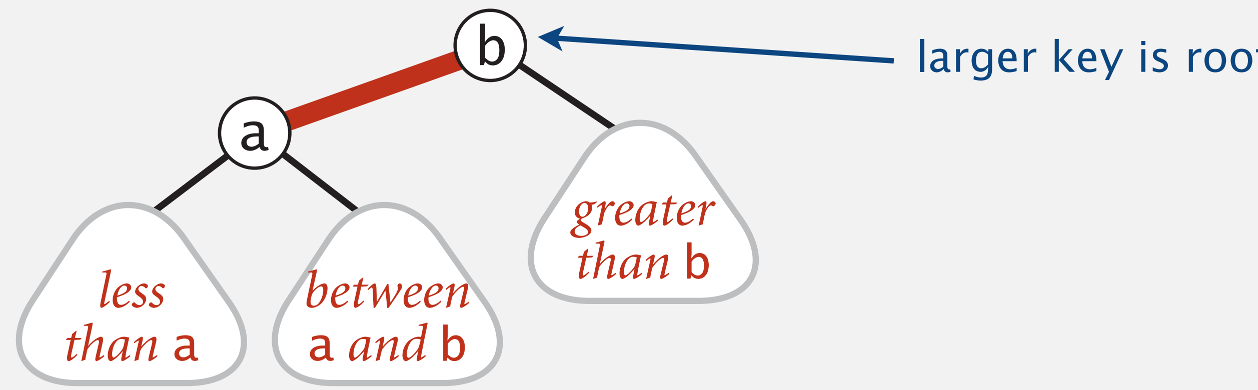
- Widely used in practice.
- Arbitrary restriction: red links lean left.

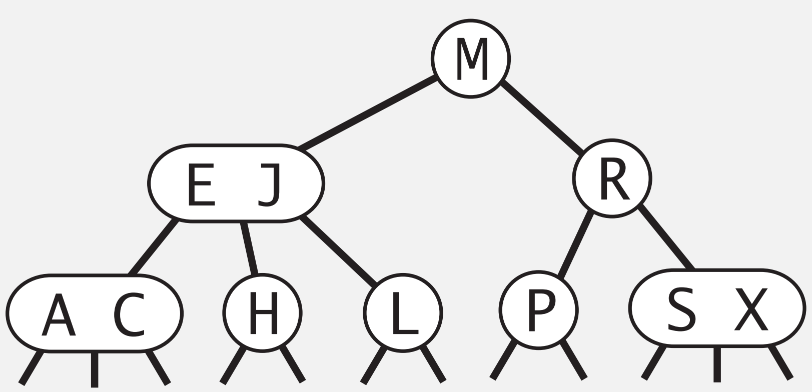


Left-leaning red–black BSTs (Guibas–Sedgewick 1979 and Sedgewick 2007)

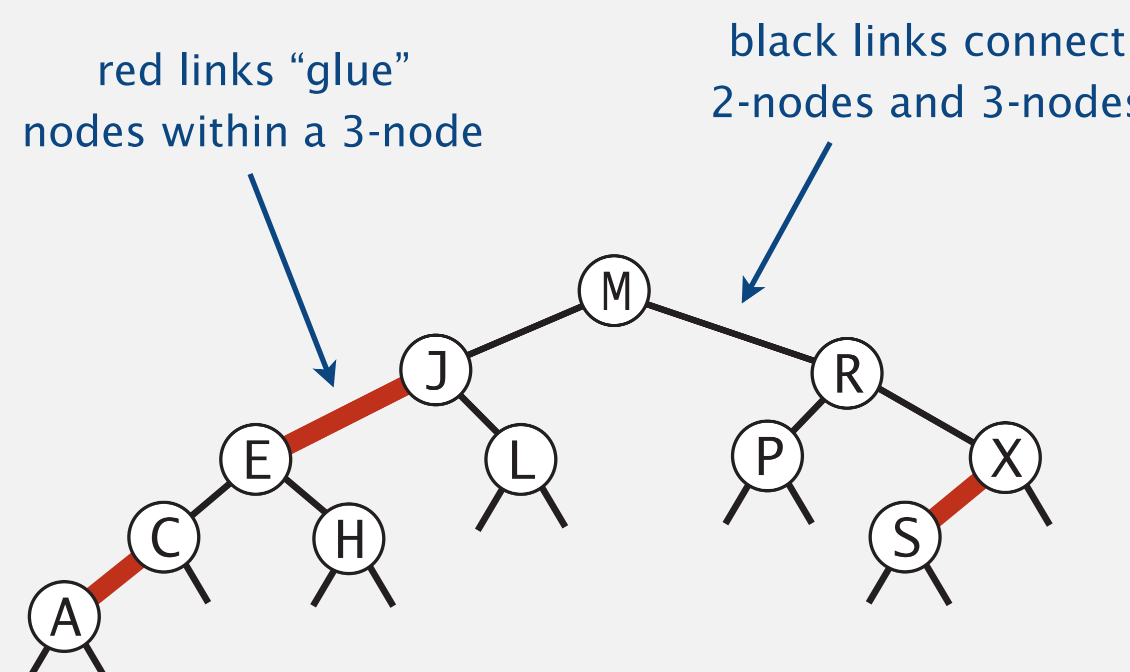
1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.







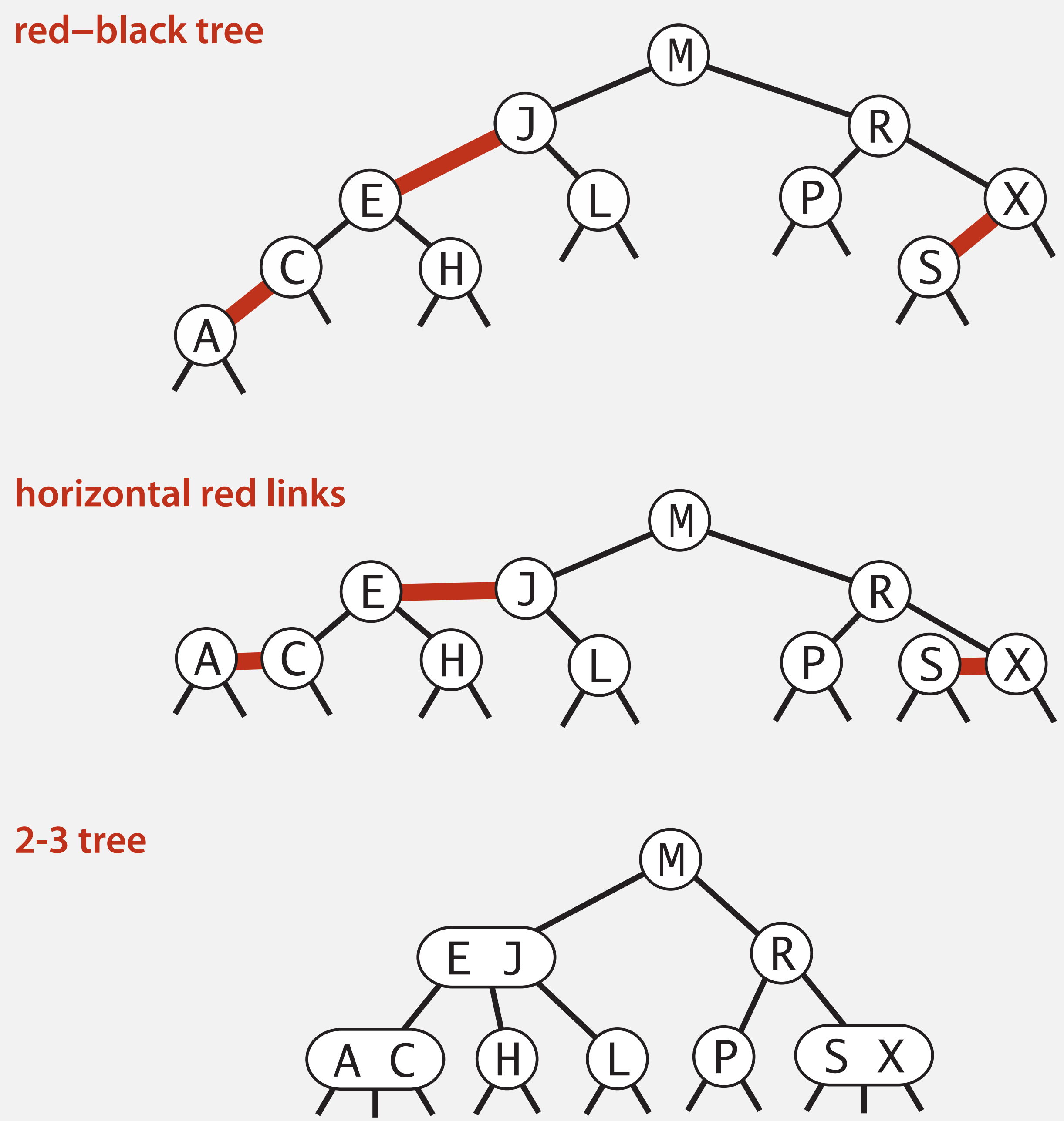
2–3 tree



corresponding red–black BST

Left-leaning red–black BSTs: 1–1 correspondence with 2–3 trees

Key property. 1–1 correspondence between 2–3 trees and LLRB trees.



An equivalent definition of LLRB trees (without reference to 2–3 trees)

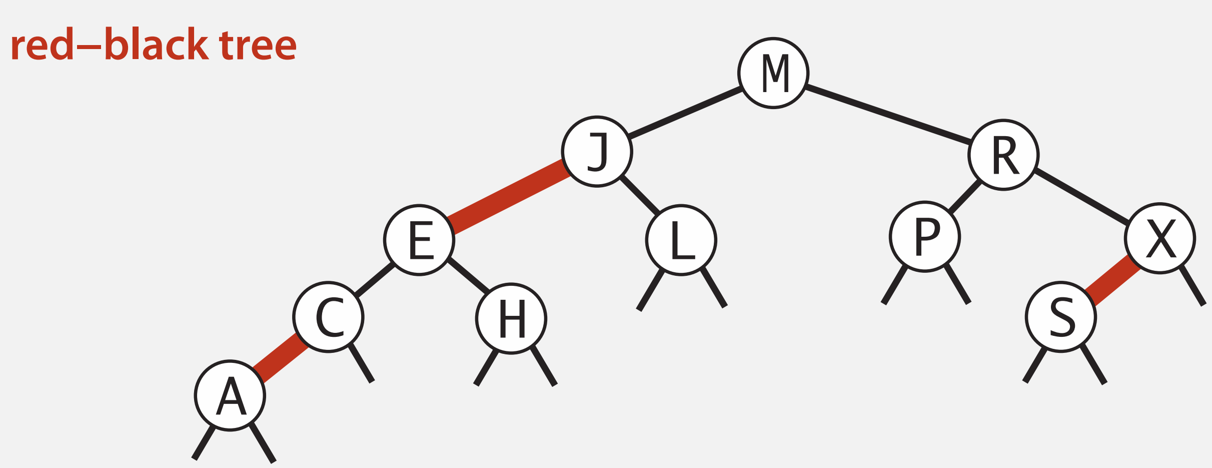
symmetric order

A BST such that:

- No node has two red links connected to it.
- Red links lean left.
- Every path from root to null link has the same number of black links.

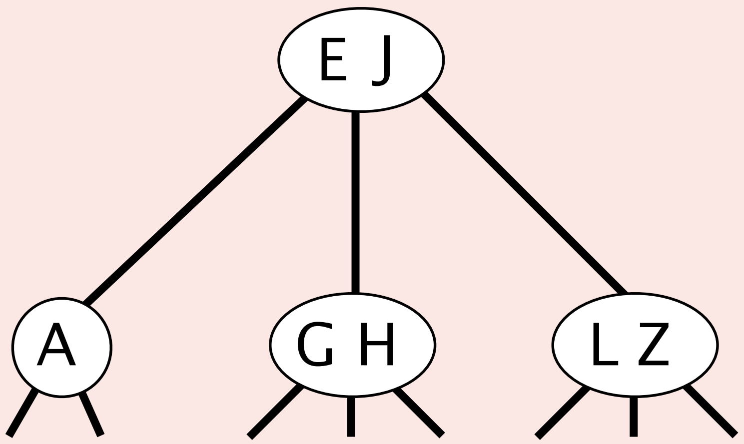
color invariants

"perfect black balance"

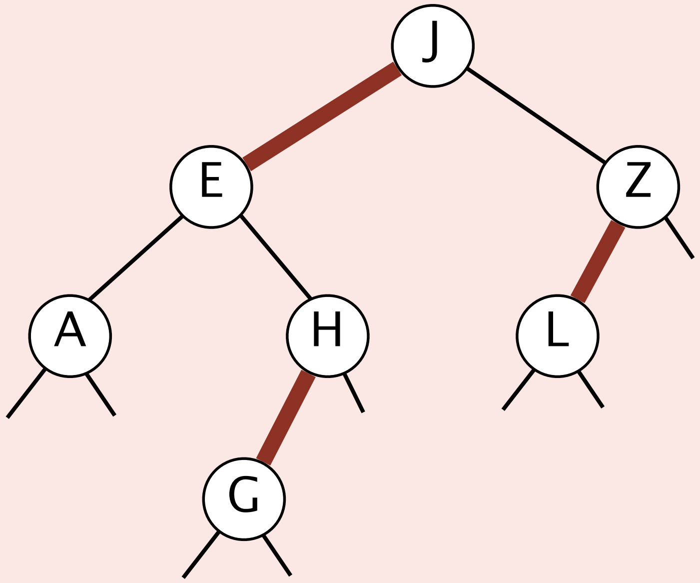


Balanced search trees: quiz 2

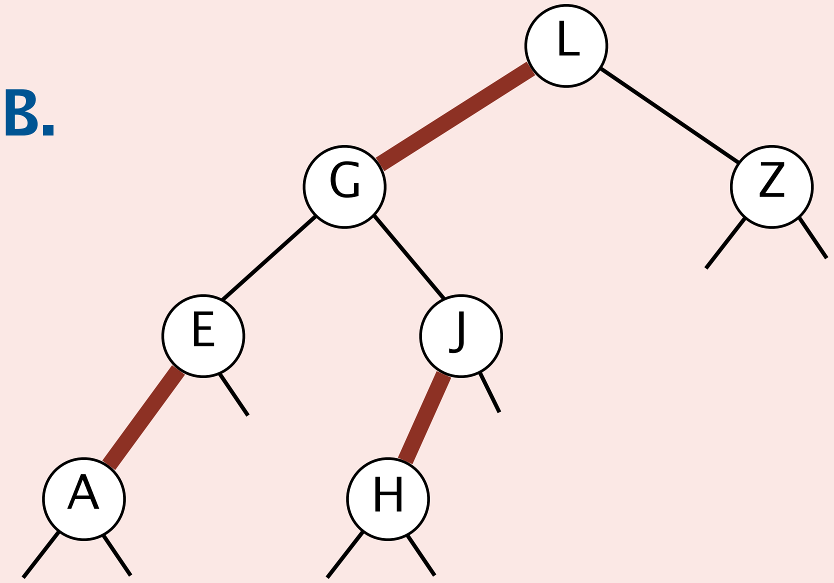
Which LLRB tree corresponds to the following 2–3 tree?



A.



B.



C. Both A and B.

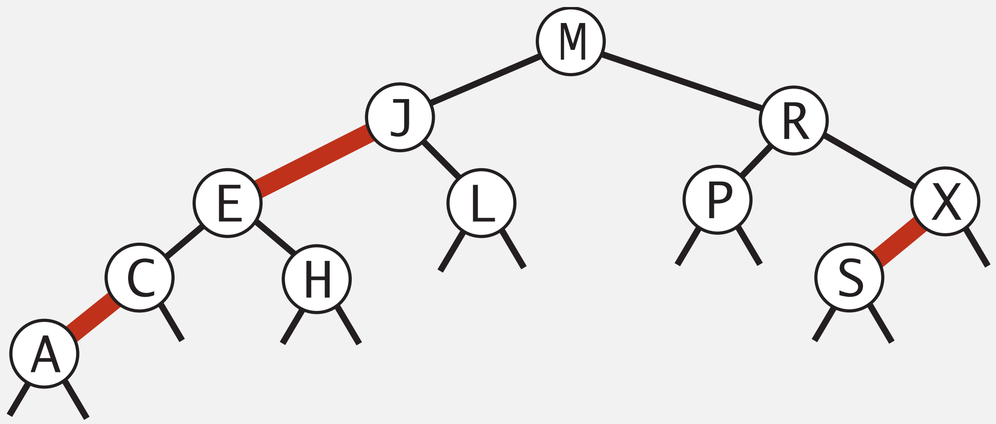
D. Neither A nor B.

Search implementation for red-black BSTs

Observation. Search is the same as for BST (ignore color).

but runs faster
(because of better balance)

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



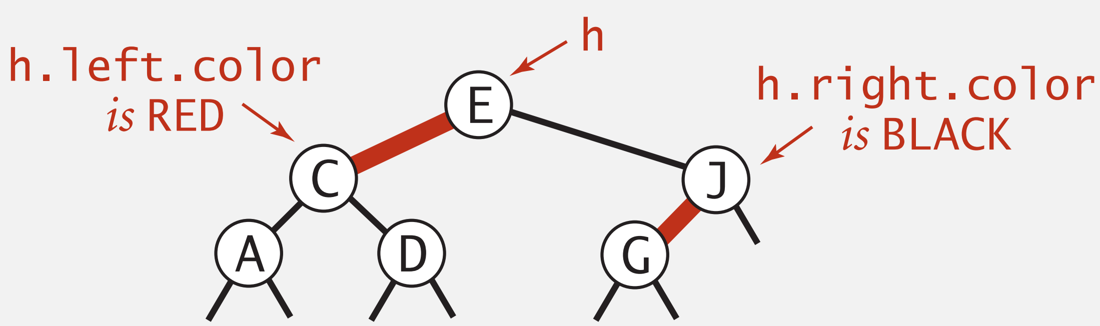
Remark. Many other ops (floor, iteration, rank, selection) are also identical.

Red-black BST representation

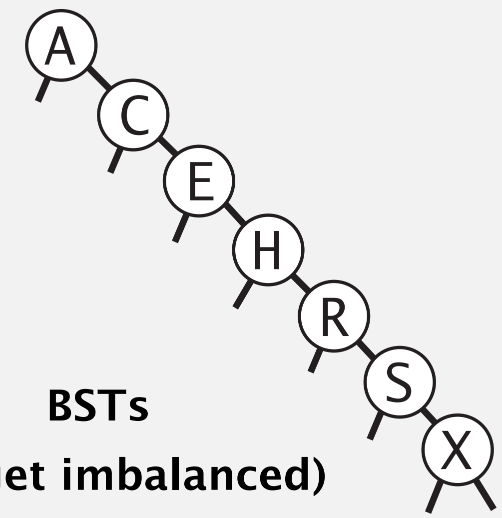
Each node is pointed to by precisely one link (from its parent) $\Rightarrow$ can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;    // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

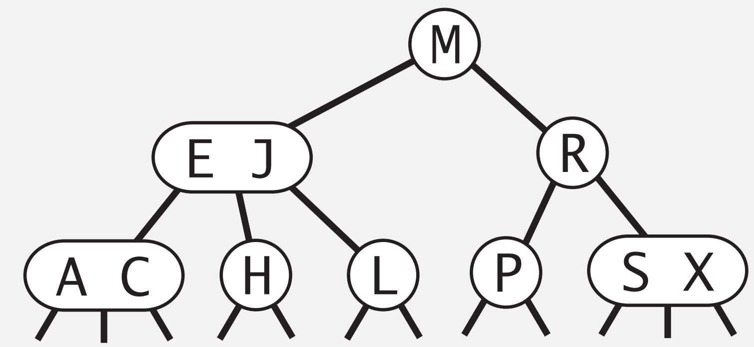
null links are black



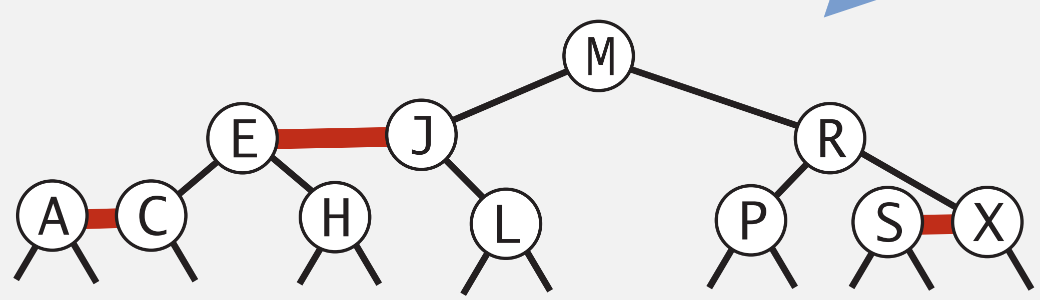
Review: the road to LLRB trees



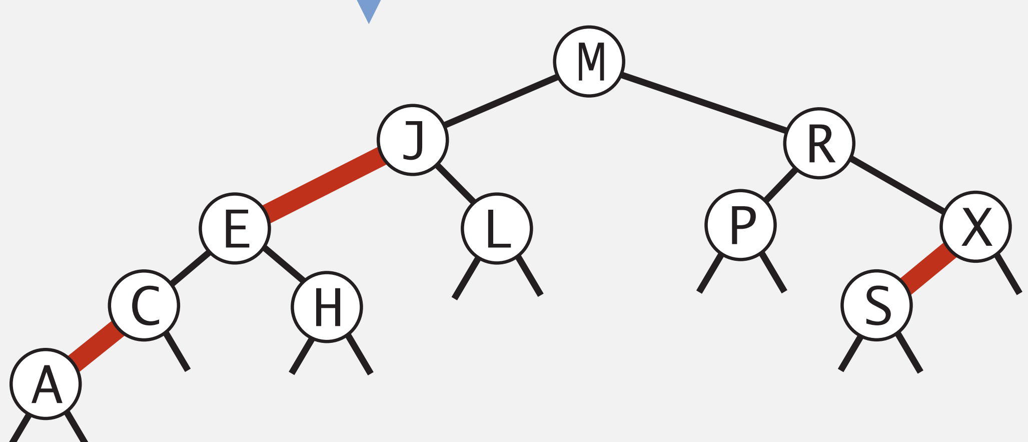
BSTs
(can get imbalanced)



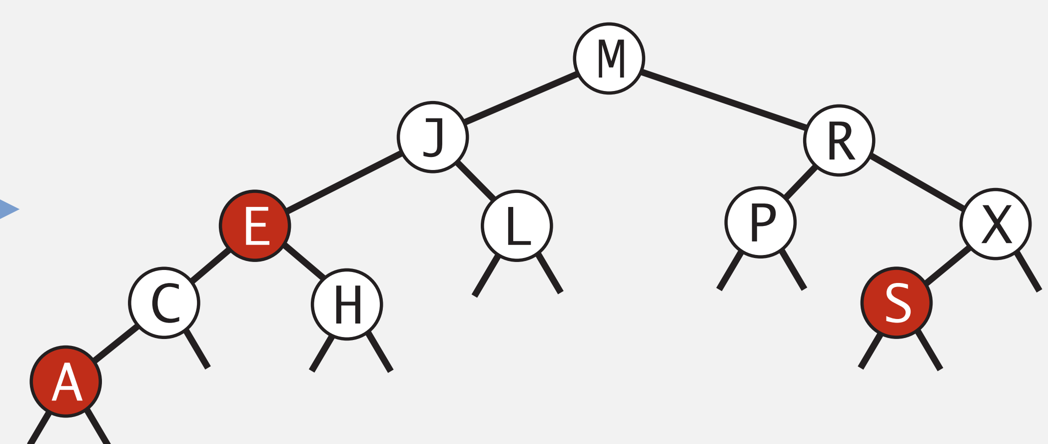
2-3 trees
(balanced but awkward to implement)



3-nodes "glued" together with red links



how we draw LLRB trees
(color in links)



how we implement LLRB trees
(color in nodes)

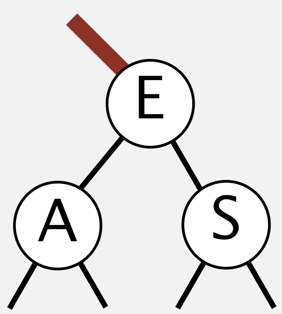
Insertion into a LLRB tree: overview

Basic strategy. Maintain 1–1 correspondence with 2–3 trees.

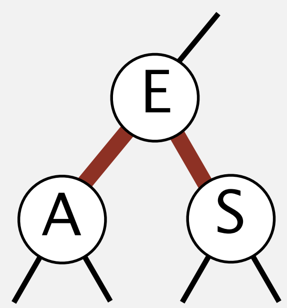
During internal operations, maintain:

- Symmetric order.
- Perfect black balance.
- [but not necessarily color invariants]

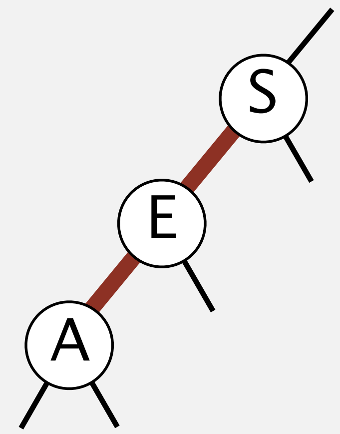
Example violations of color invariants:



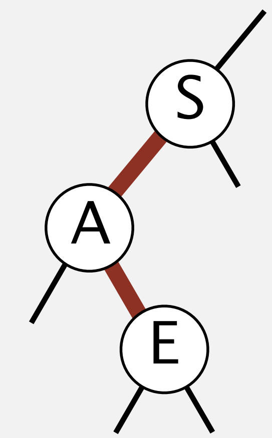
right-leaning red link



two red children (a temporary 4-node)



left-left red (a temporary 4-node)



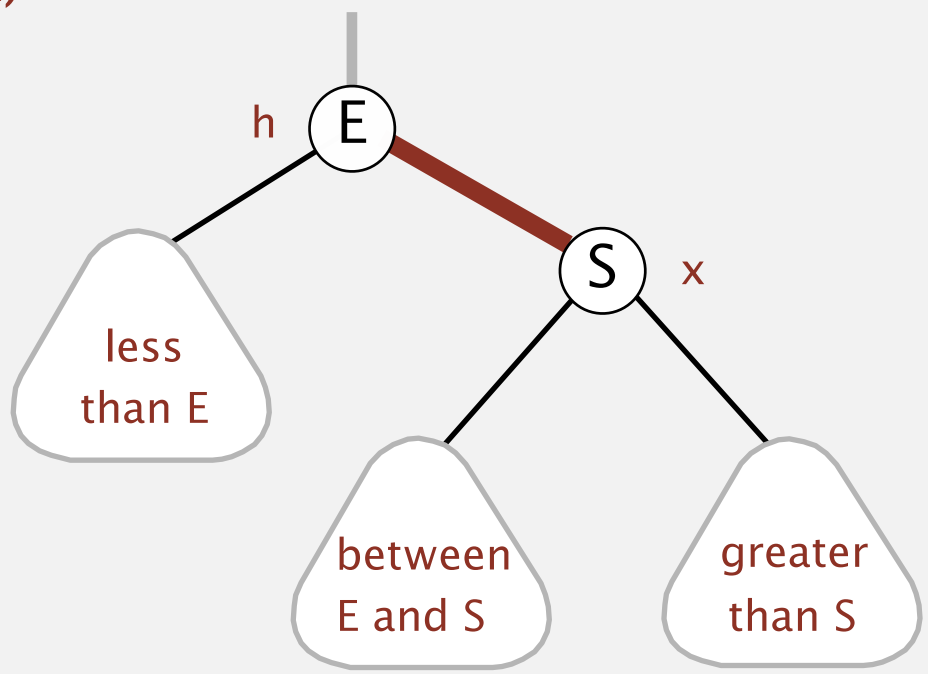
left-right red (a temporary 4-node)

To restore color invariants: perform rotations and color flips.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left
(before)

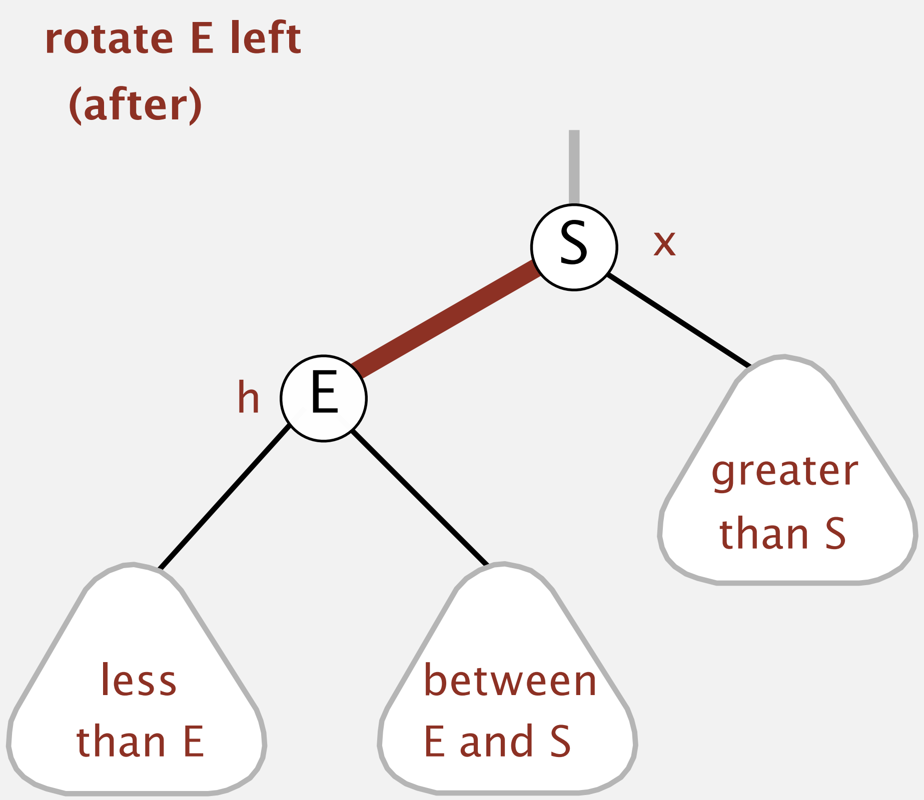



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.




```
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{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```
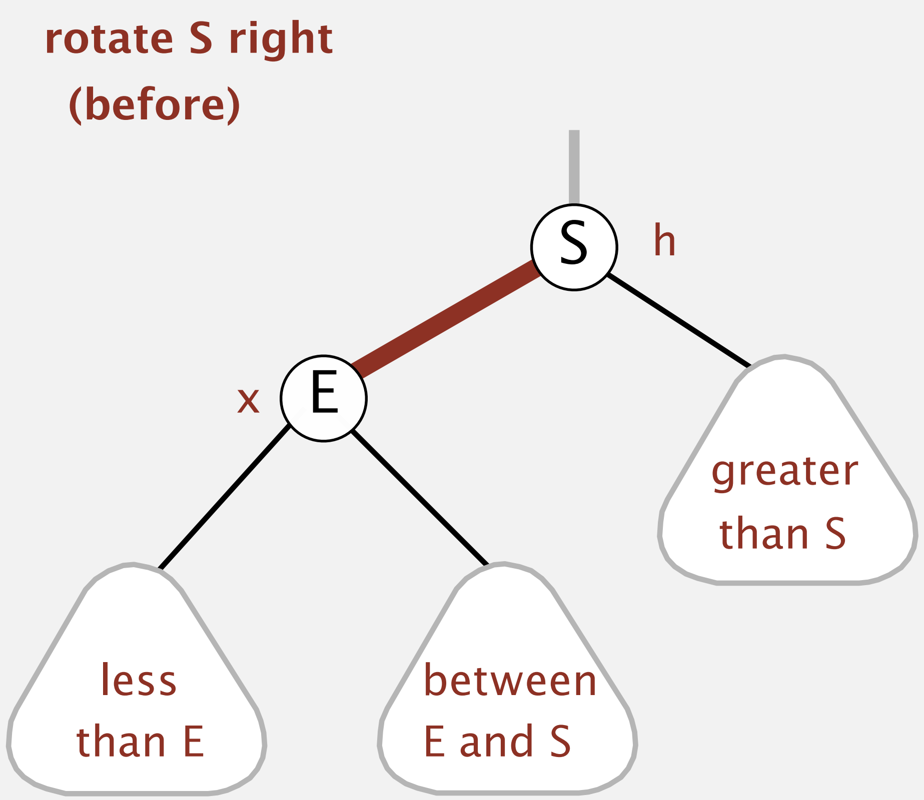

Invariants. Maintains symmetric order and perfect black balance.

Elementary red–black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

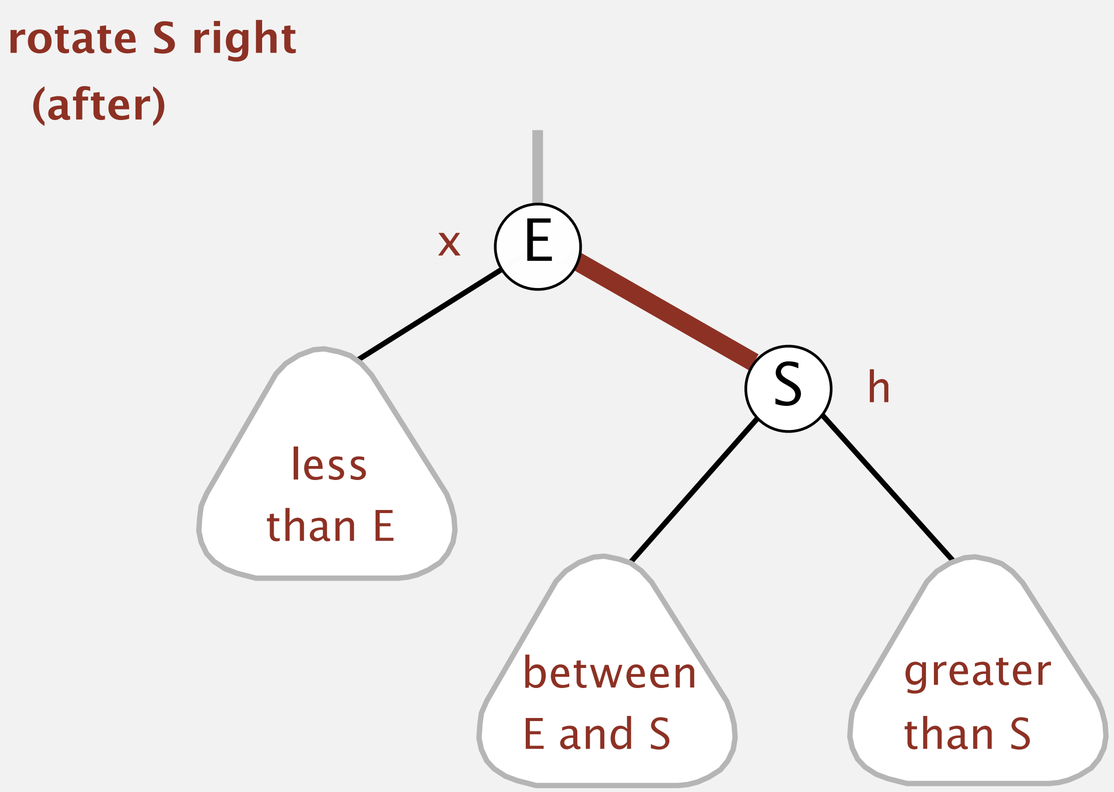



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

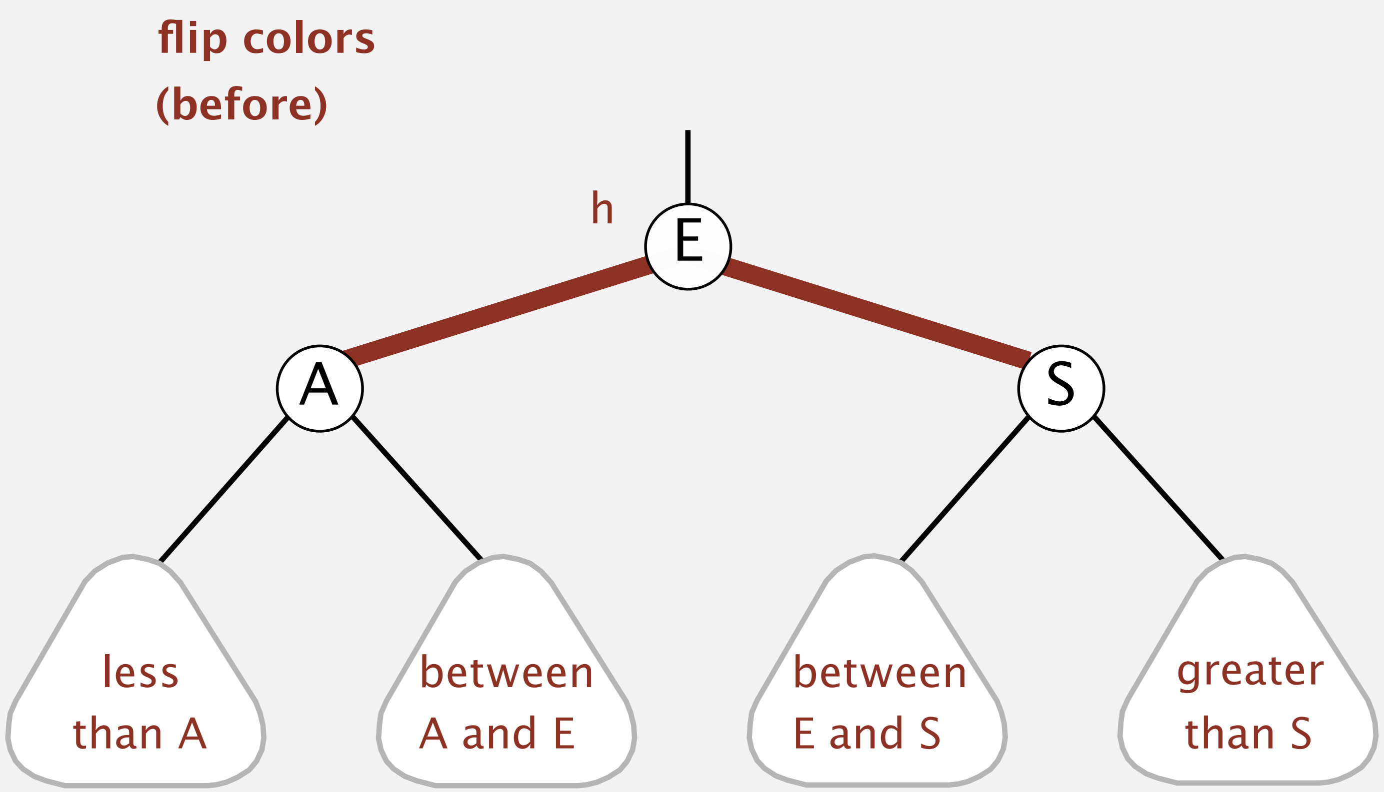



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

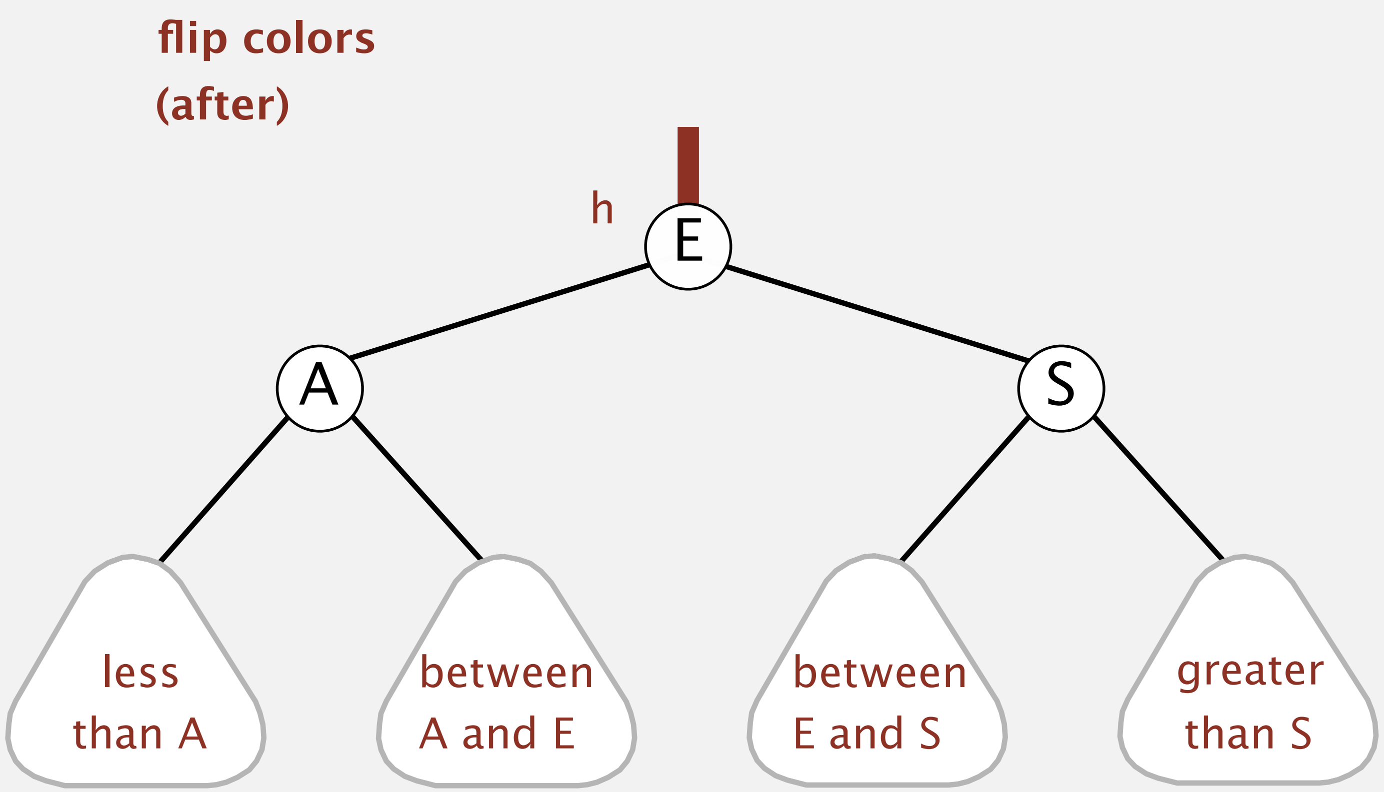



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

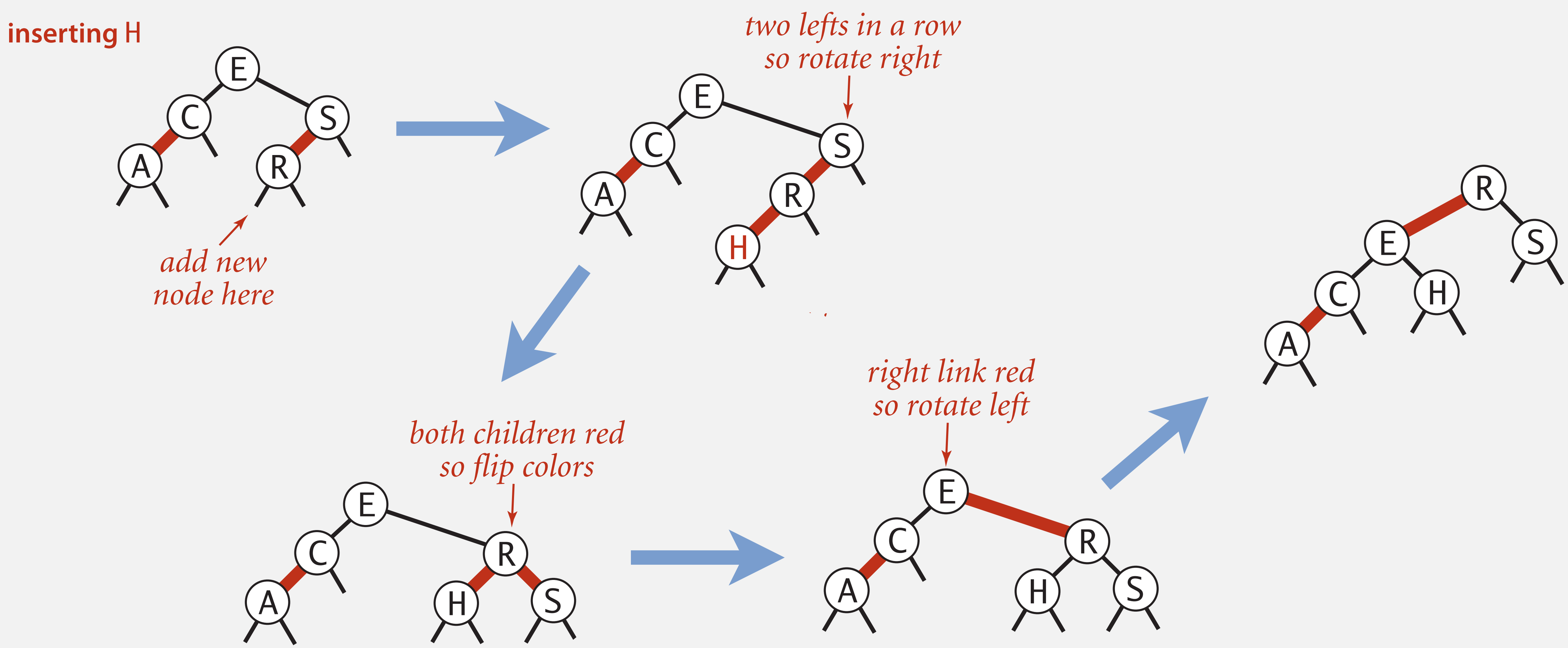



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Insertion into a LLRB tree

- Do standard BST insert. ⟵ to preserve symmetric order
- Color new link red. ⟵ to preserve perfect black balance
- Repeat up the tree until color invariants restored:
  - two left red links in a row? ⇒ rotate right
  - left and right links both red? ⇒ color flip
  - right link only red? ⇒ rotate left



Insertion into a LLRB tree

- Do standard BST insert.
- Color new link red.
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  - two left red links in a row? ⇒ rotate right
  - left and right links both red? ⇒ color flip
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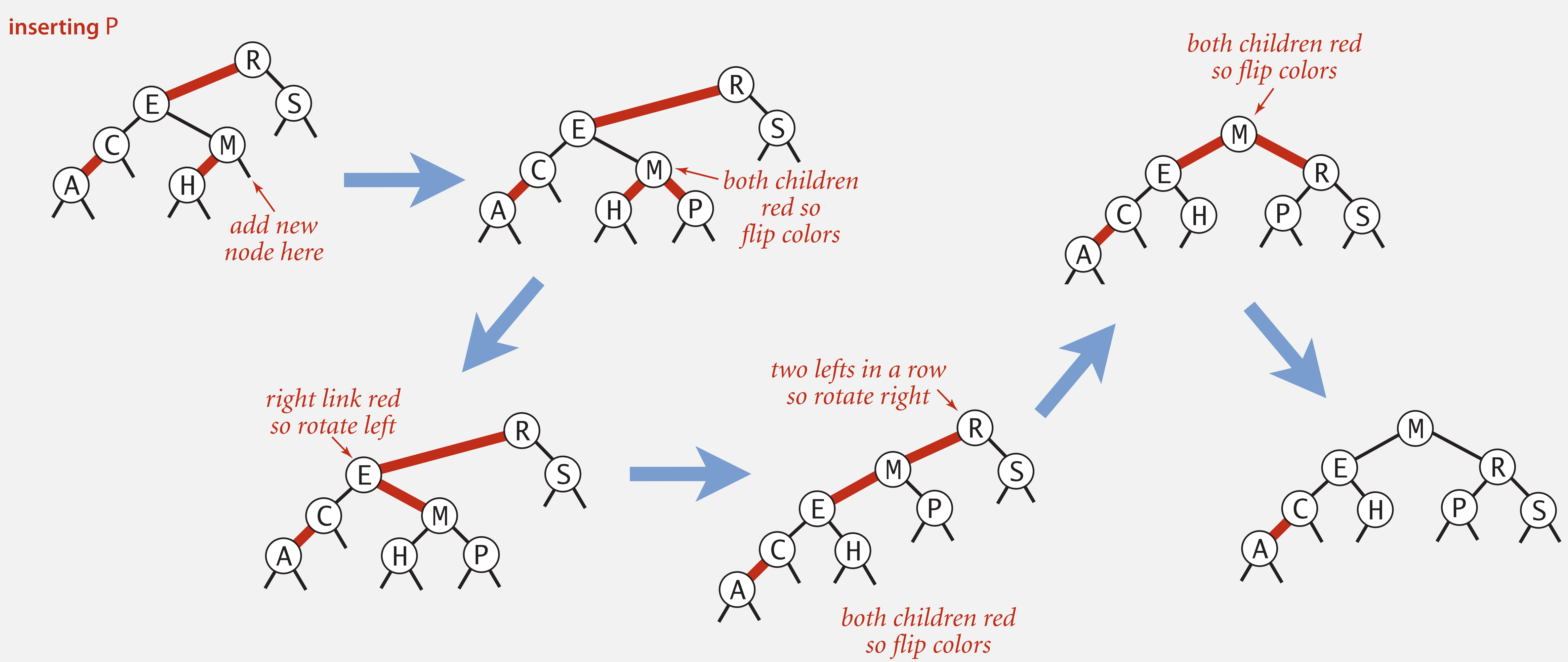
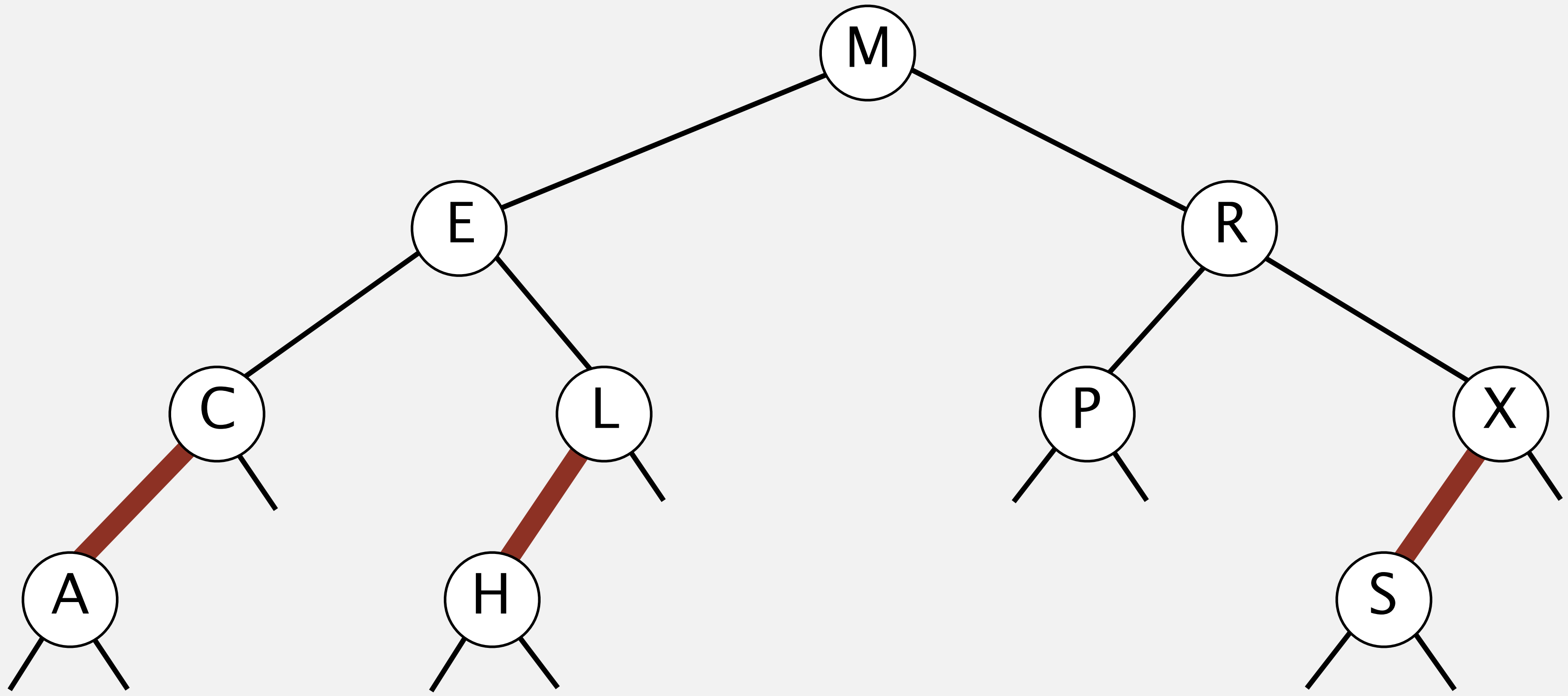


Red-black BST construction demo

insert S E A R C H X M P L



Insertion into a LLRB tree: Java implementation

- Do standard BST insert and color new link red.
- Repeat up the tree until color invariants restored:
  - right link only red? ⇒ rotate left
  - two left red links in a row? ⇒ rotate right
  - left and right links both red? ⇒ color flip

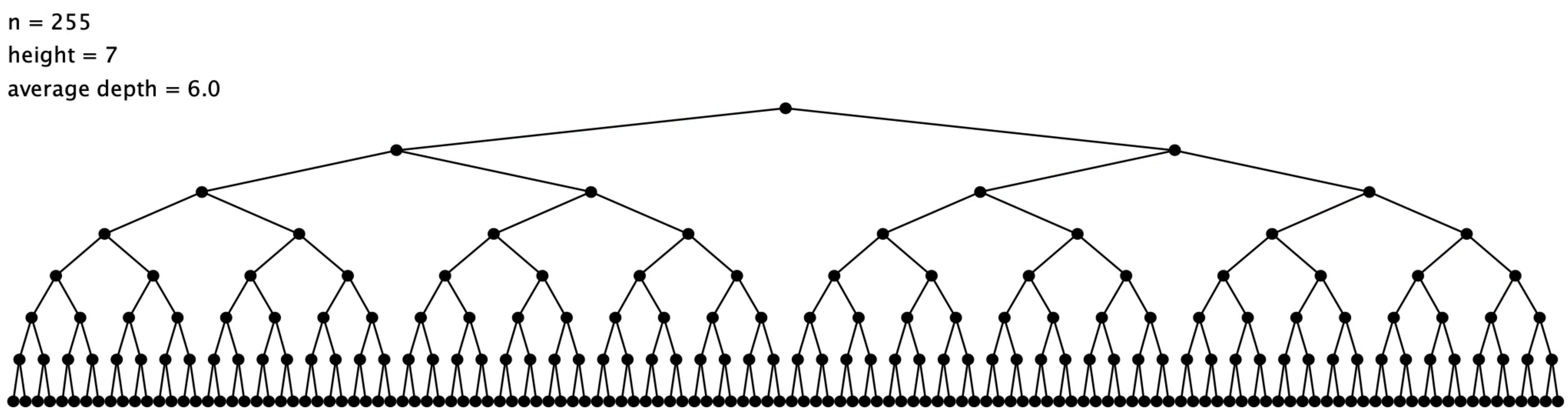
```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);

   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))      h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))      flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

restore color invariants

only a few extra lines of code provides near-perfect balance

Insertion into a LLRB tree: visualization



255 insertions in ascending order

Insertion into a LLRB tree: visualization

255 insertions in descending order

Insertion into a LLRB tree: visualization

255 insertions in random order

What is the maximum height of a LLRB tree with n keys?

A. $\sim \log_2 n$

B. $\sim 2 \log_3 n \approx 1.262 \log_2 n$

C. $\sim 2 \log_2 n$

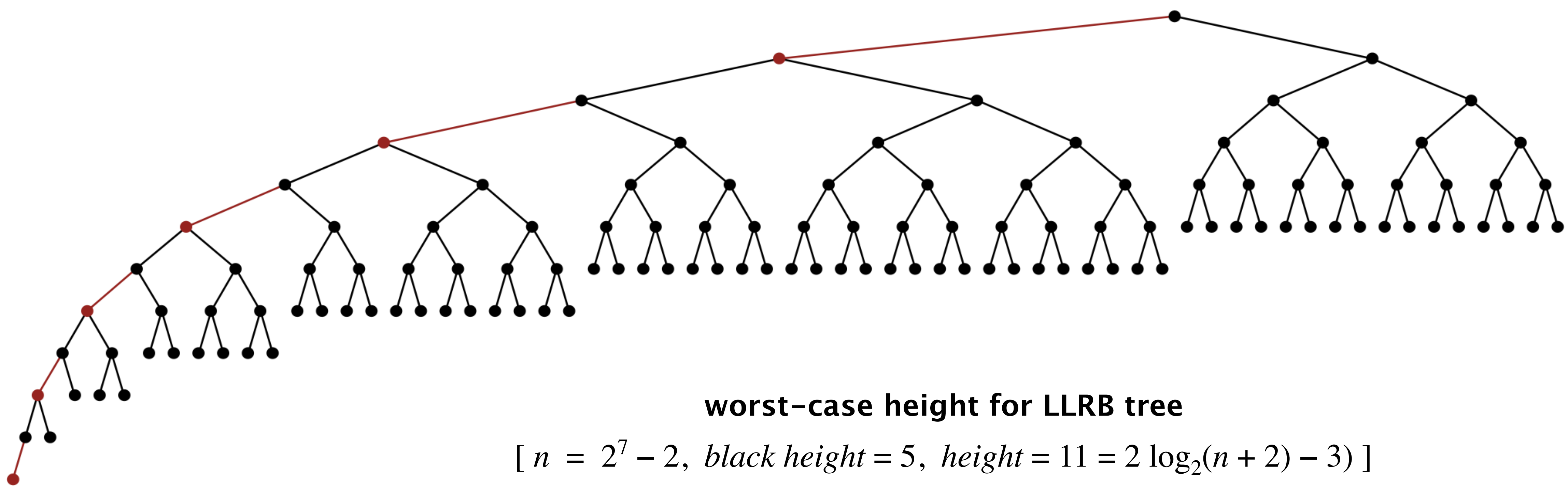
D. $\sim n$

Balance in LLRB trees

Proposition. Height of LLRB tree is $\leq 1 + 2 \log_2 n$.

Pf.

- Black height = height of corresponding 2–3 tree $\leq \log_2 n$.
- Never two red links in-a-row $\Rightarrow$ height $\leq$ 1 + 2 × black height. ▪



worst-case height for LLRB tree

[$n = 2^7 - 2$, *black height* $= 5$, *height* $= 11 = 2 \log_2(n + 2) - 3)$]

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

hidden constant c is small
(at most $2 \log_2 n$ compares)

Why named red-black BSTs?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

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ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

Balanced trees in the wild

Red–black BSTs are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: CFQ I/O scheduler, `linux/rbtree.h`.

Other balanced BSTs. AVL trees, splay trees, randomized BSTs, ….

B-trees (and cousins) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS, BTRFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

War story 1: red–black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.

- Red–black BST.
- Exceeding height limit of 80 triggered error-recovery process.

should allow for $\leq 2^{40}$ keys

Extended telephone service outage.

- Main cause = height bound exceeded!
- Telephone company sues database provider.
- Legal testimony:


[redacted]

> "*If implemented properly, the height of a red–black BST with n keys is at most* $2 \log_2 n$." — *expert witness*

War story 2: red-black BSTs

Celestine Omin @cyberomin

Follow

I was just asked to balance a Binary Search Tree by JFK's airport immigration. Welcome to America.

8:26 AM - 26 Feb 2017 from Manhattan, NY

8,025 Retweets **7,087** Likes

Celestine Omin @cyberomin · 26 Feb 2017

I was too tired to even think of a BST solution. I have e been travelling for 23hrs. But I was also asked about 10 CS questions.

8 164 244

Celestine Omin @cyberomin · 26 Feb 2017

sad thing is, if I didn't give the Wikipedia definition for these questions, it was considered a wrong answer.

19 324 703

Simon Sharwood @ssharwood · 26 Feb 2017

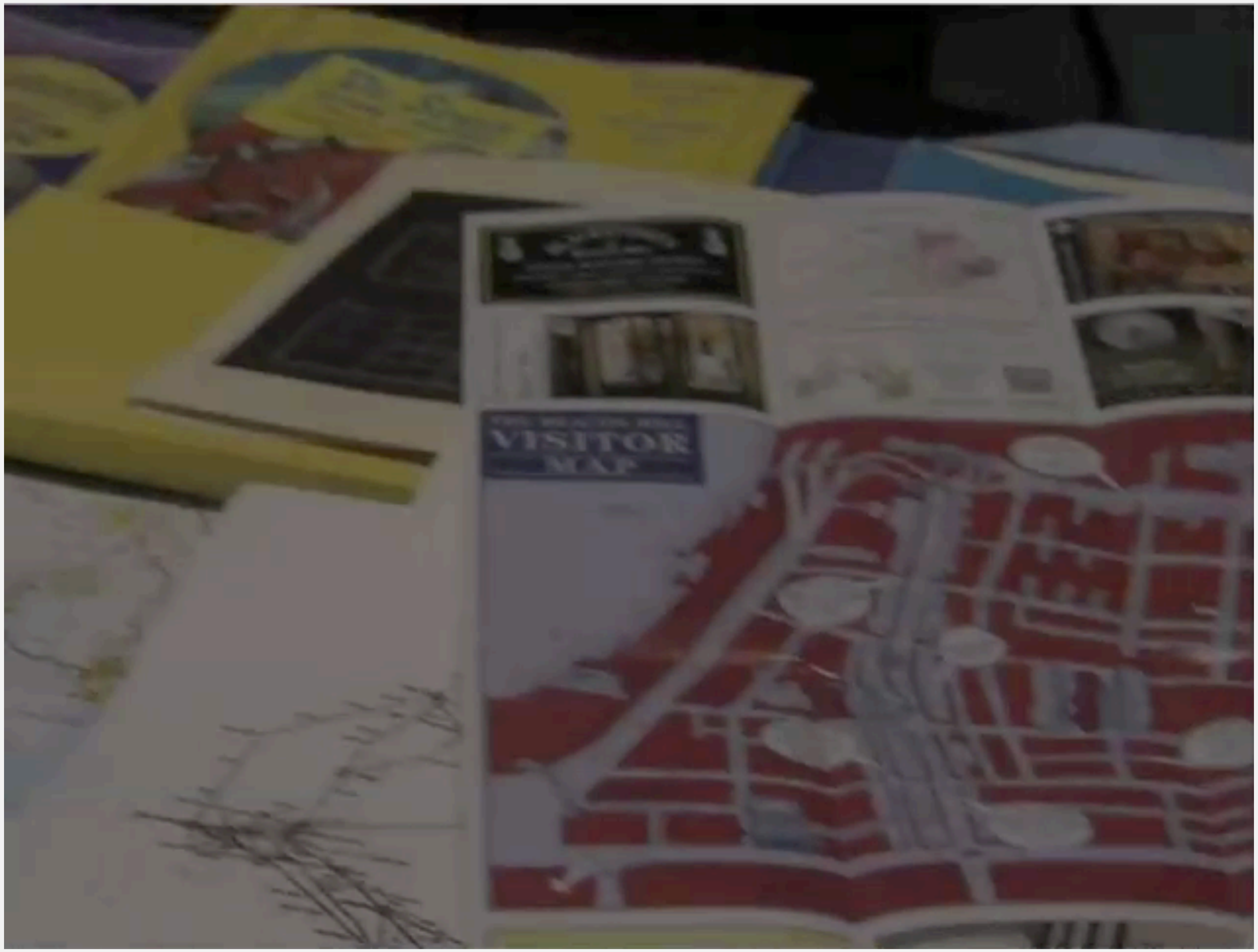
Replying to @cyberomin

seriously? am reporter for @theregister and would love to know more about your experience

2 22 171

https://twitter.com/cyberomin/status/835888786462625792

War story 3: red-black BSTs



Common sense. Sixth sense.
Together they're the
FBI's newest team.