https://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREES

- *BSTs*
- *ordered operations*
- *iteration*
- *deletion (see book or videos)*

https://algs4.cs.princeton.edu

3.2 Binary Search Trees

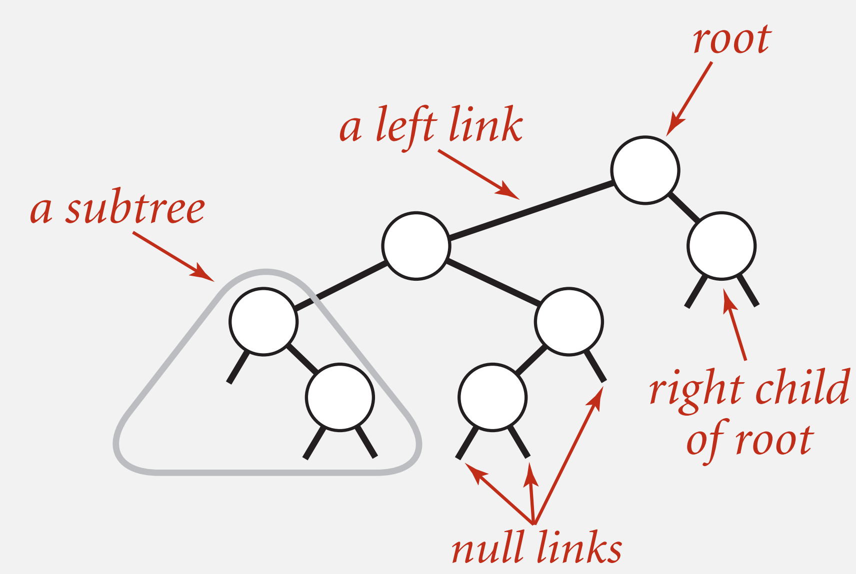
- *BSTs*
- *ordered operations*
- *iteration*
- *deletion (see book or videos)*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

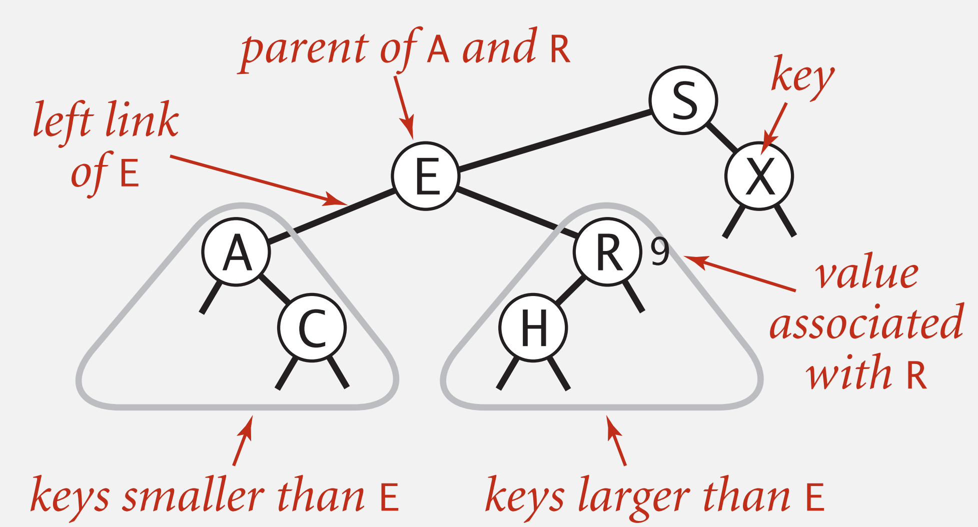
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



Symmetric order. Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search trees: quiz 1

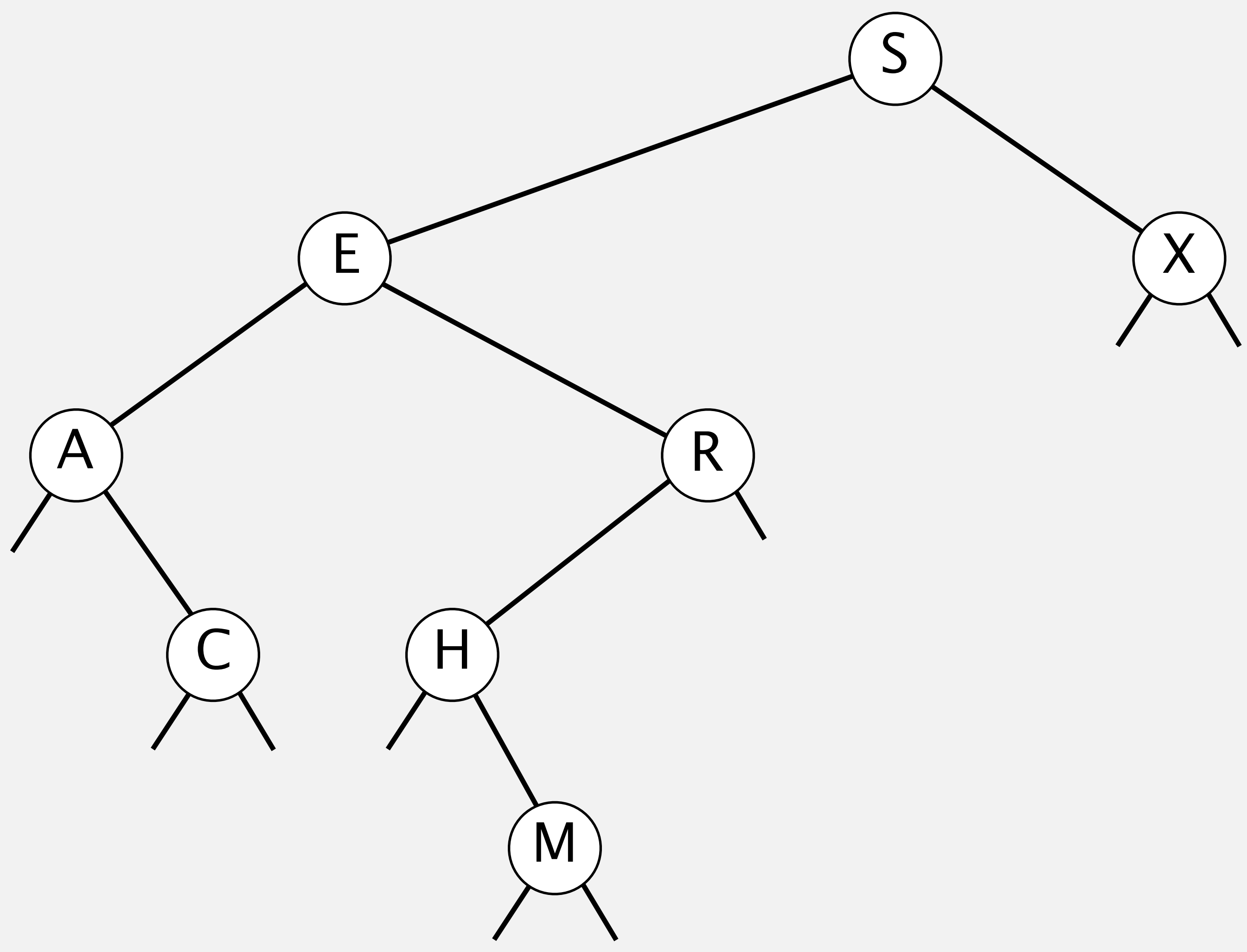
Which of the following properties hold?

- **A.** If a binary tree is heap ordered, then it is symmetrically ordered.
- **B.** If a binary tree is symmetrically ordered, then it is heap ordered.
- **C.** Both A and B.
- **D.** Neither A nor B.

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

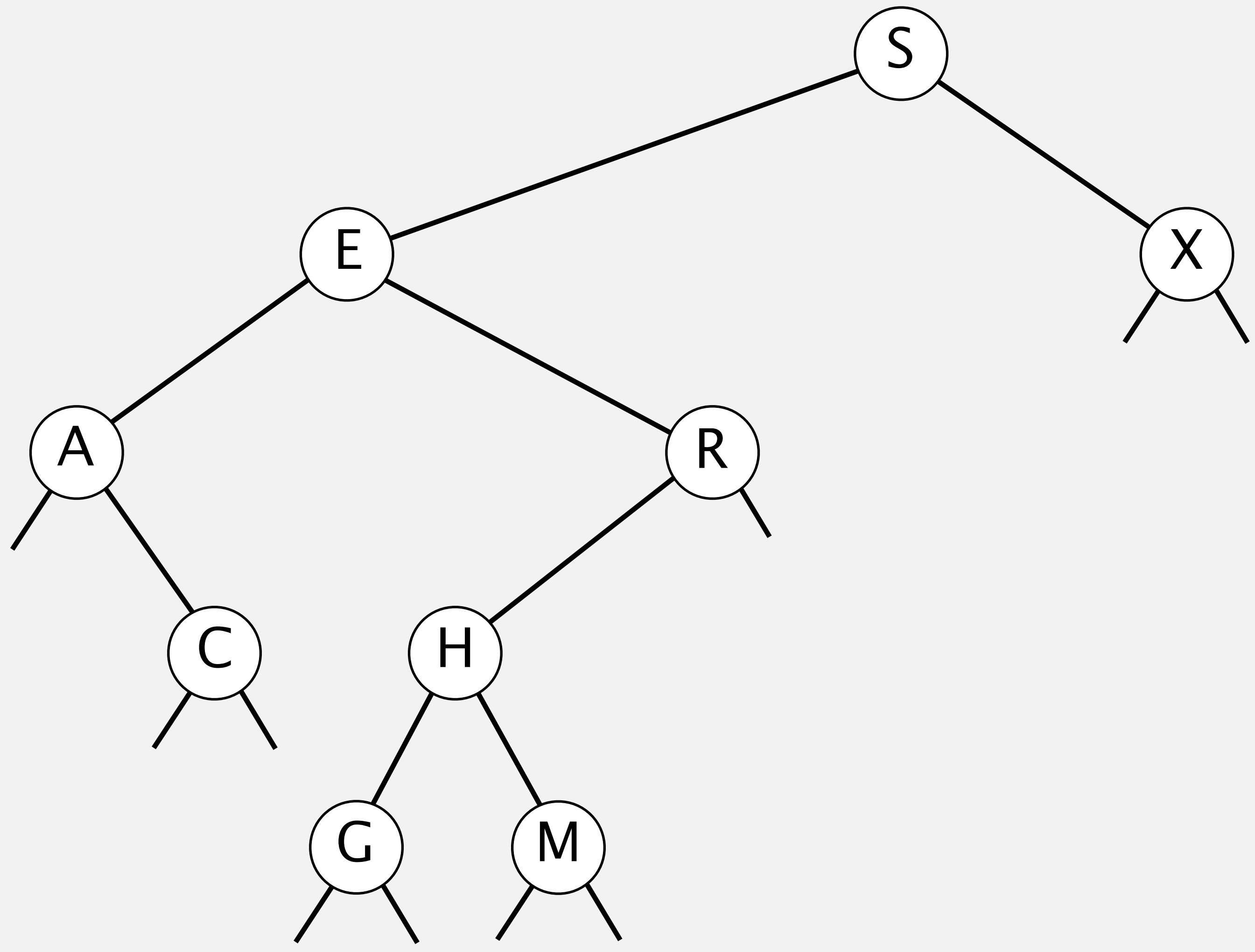
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if `null`, insert.

insert G



BST representation in Java

Java definition. A BST is a reference to a root `Node`.

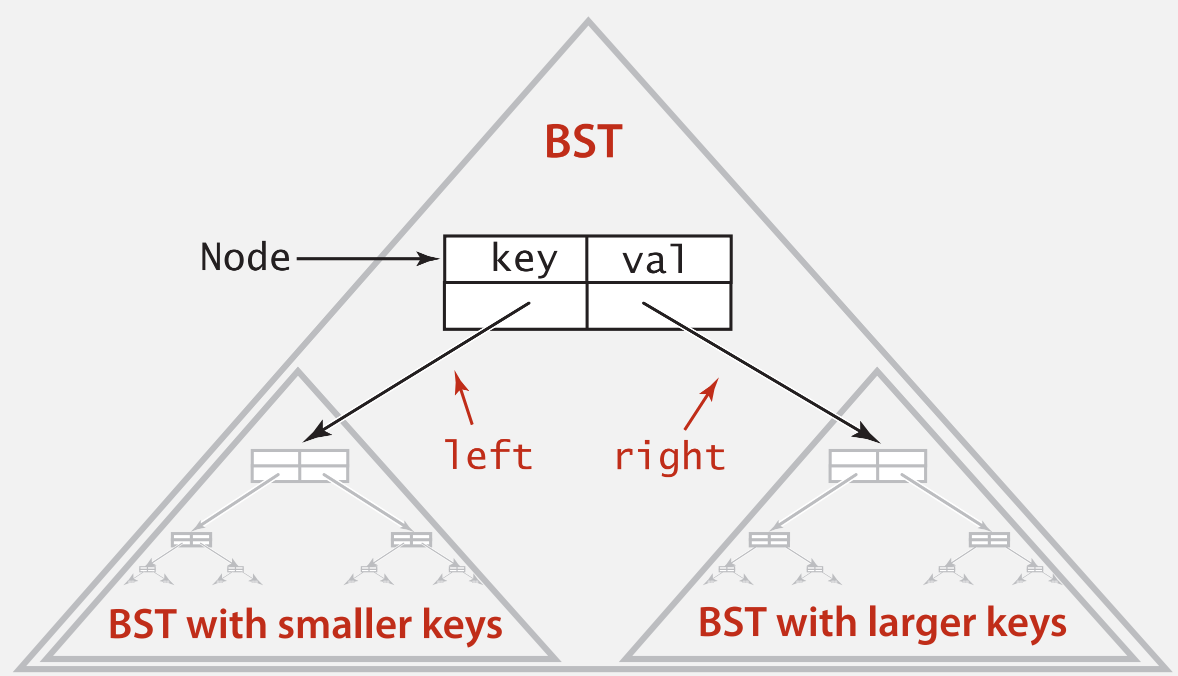
A `Node` is composed of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;

   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
   private Node root;

   private class Node
   {  /* see previous slide */  }

   public void put(Key key, Value val)
   {  /* see slide in this section */  }

   public Value get(Key key)
   {  /* see next slide */  }

   public Iterable<Key> keys()
   {  /* see slides in next section */  }

   public void delete(Key key)
   {  /* see textbook */  }

}
```

root of BST

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

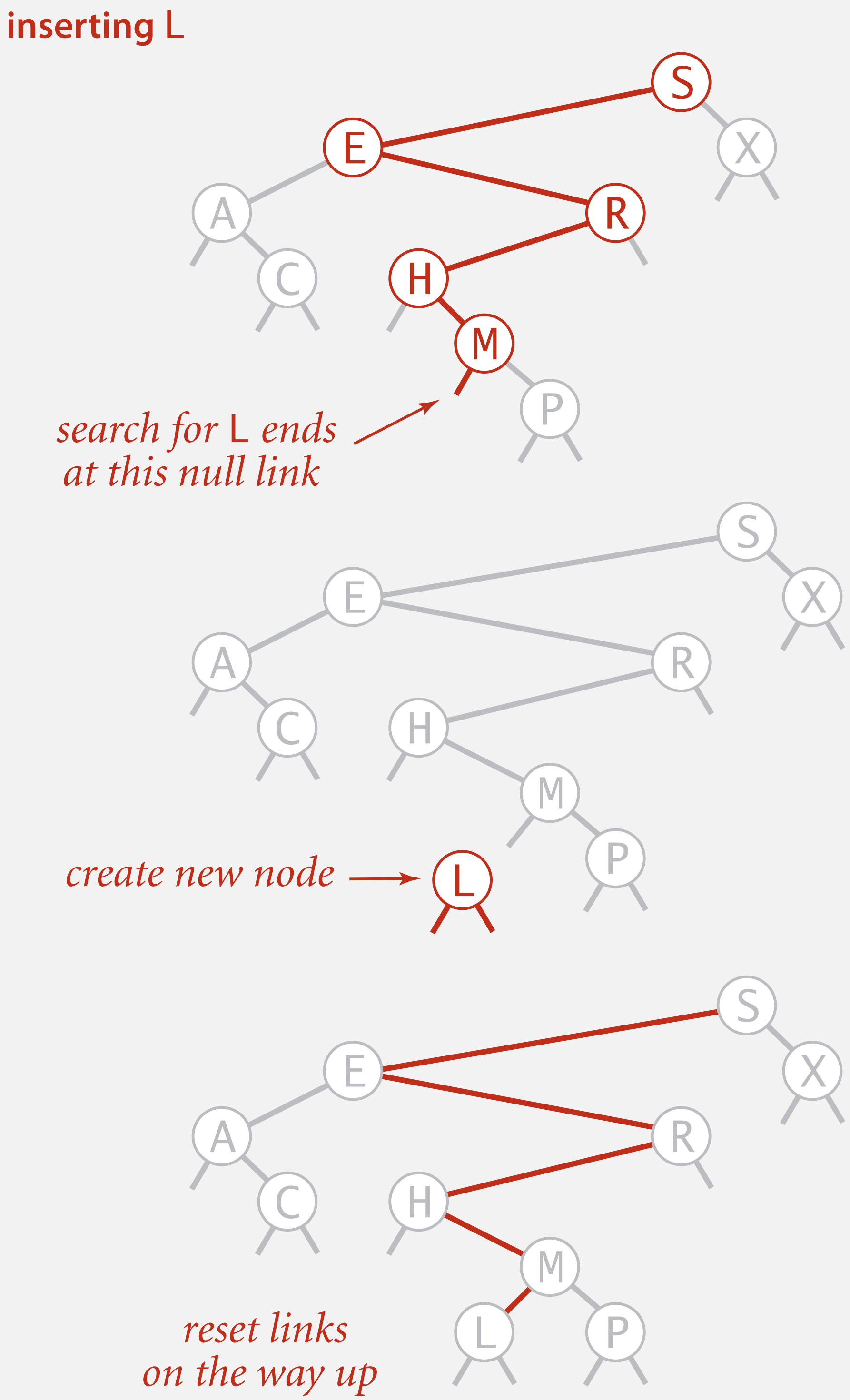
Cost. Number of compares = 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree $\Rightarrow$ reset value.
- Key not in tree $\Rightarrow$ add new node.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

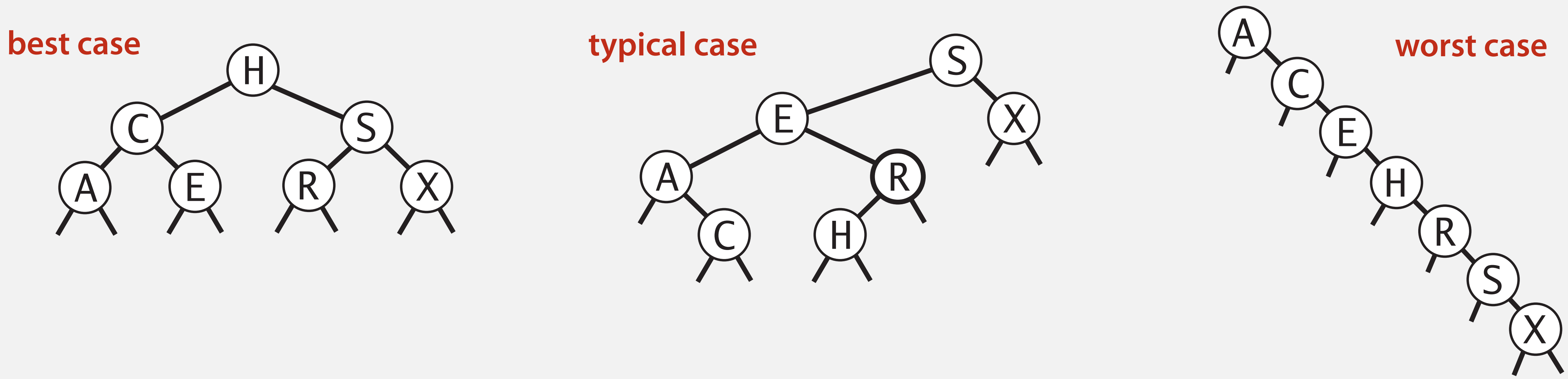
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val   = val;

   return x;
}
```

⚠ Warning: concise but tricky code; read carefully!

Cost. Number of compares = 1 + depth of node.

Tree shape

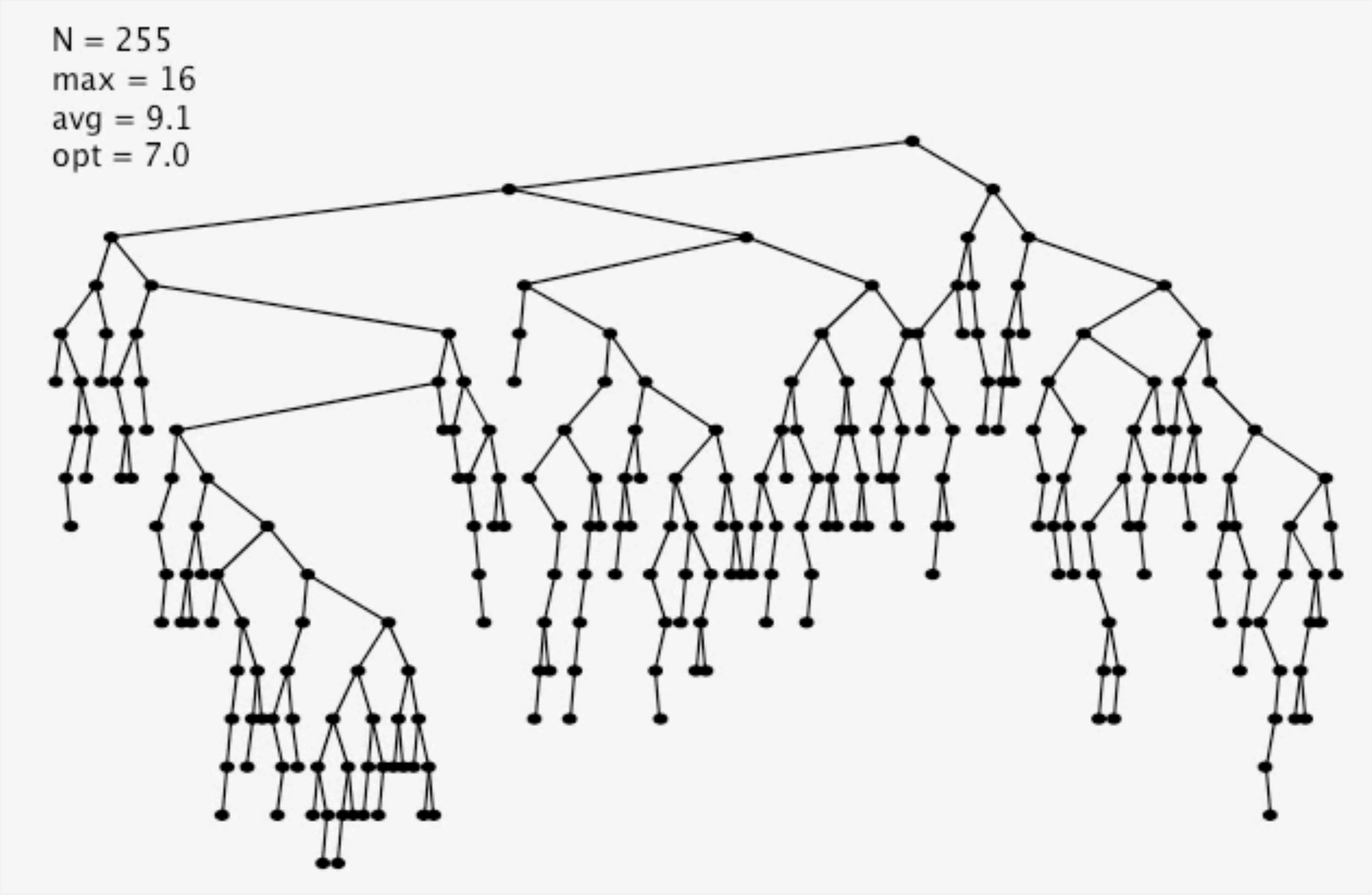
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert = 1 + depth of node.



Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.

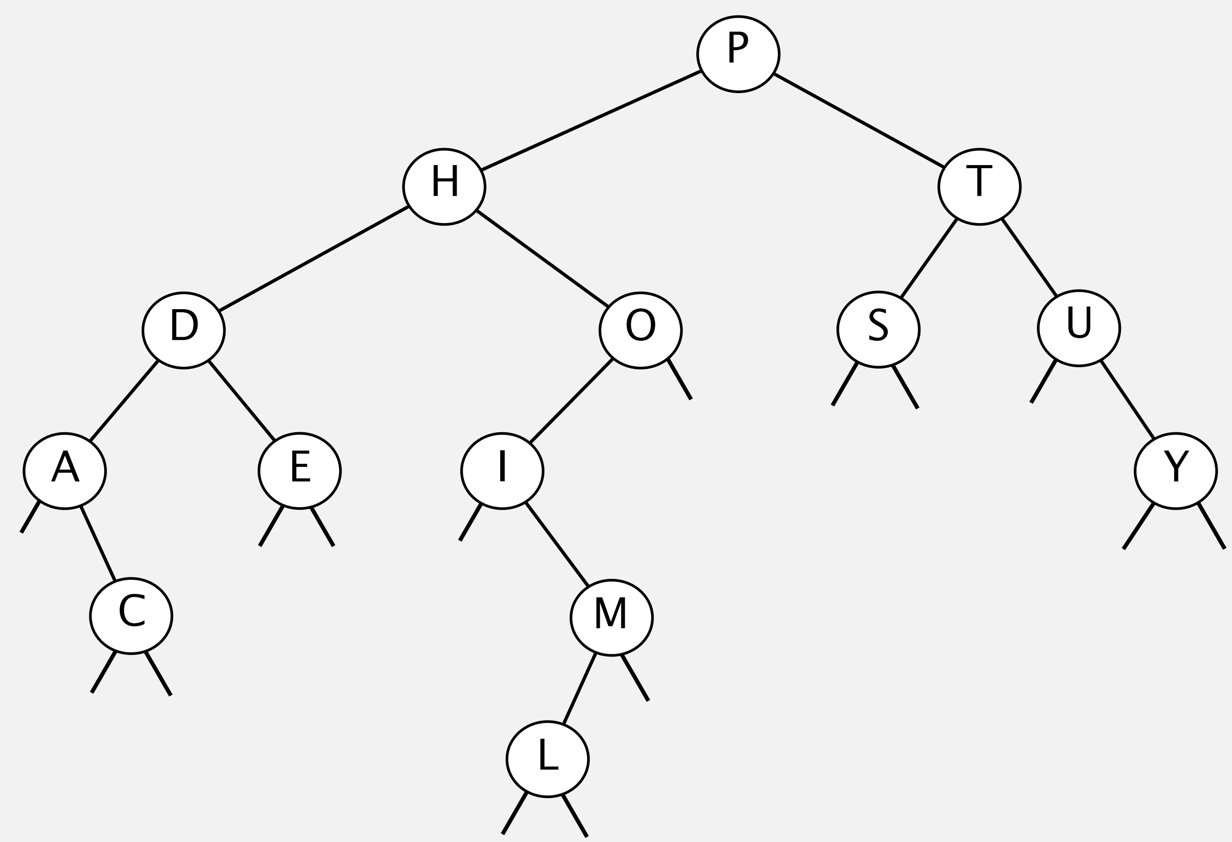


BSTs: mathematical analysis

Proposition. If n distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is $\sim 2 \ln n$.

Pf idea. 1–1 correspondence with quicksort partitioning.

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



Binary search trees: quiz 2

Suppose that you insert n keys in random order into a BST. What is the expected height of the resulting BST?

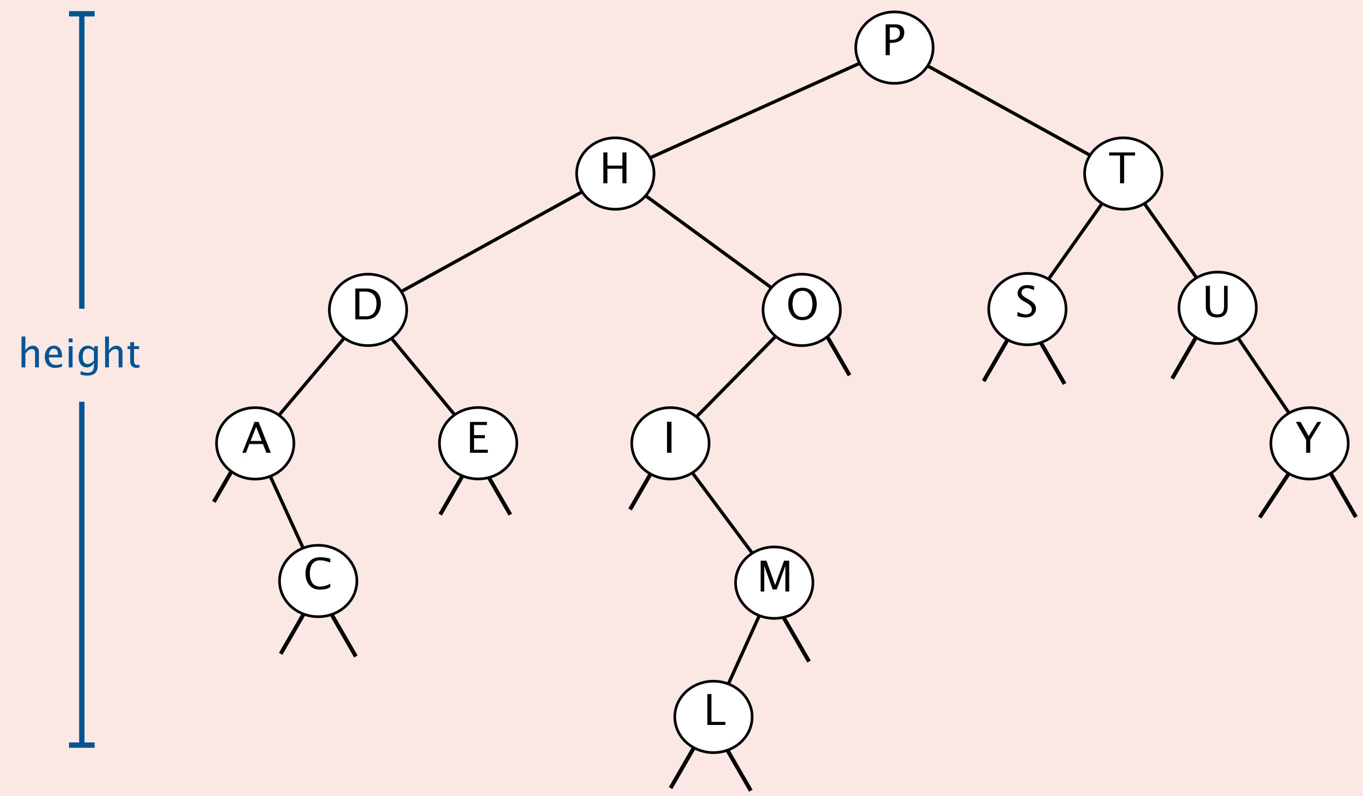
A. $\sim \lg n$

B. $\sim \ln n$

C. $\sim 2 \lg n$

D. $\sim 2 \ln n$

E. $\sim 4.31107 \ln n$



ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	n	n	n	n	`equals()`
binary search (ordered array)	$\log n$	n	$\log n$	n	`compareTo()`
BST	n	n	$\log n$	$\log n$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of $\log n$?

https://algs4.cs.princeton.edu

3.2 Binary Search Trees

- BSTs
- iteration
- ordered operations
- deletion

In which order does `traverse(root)` print the keys in the BST?

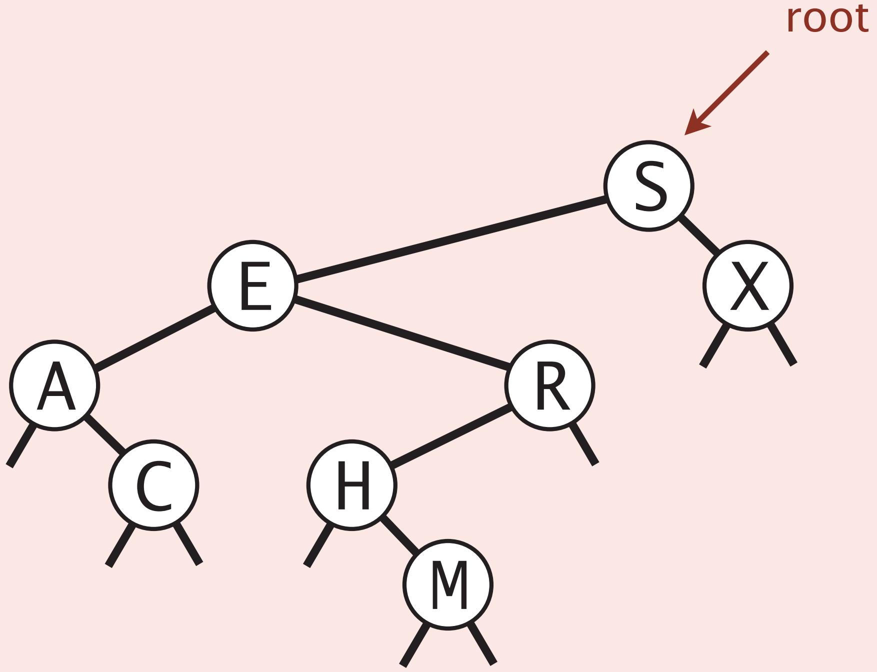
```
private void traverse(Node x)
{
   if (x == null) return;
   traverse(x.left);
   StdOut.println(x.key);
   traverse(x.right);
}
```

A. A C E H M R S X

B. S E A C R H M X

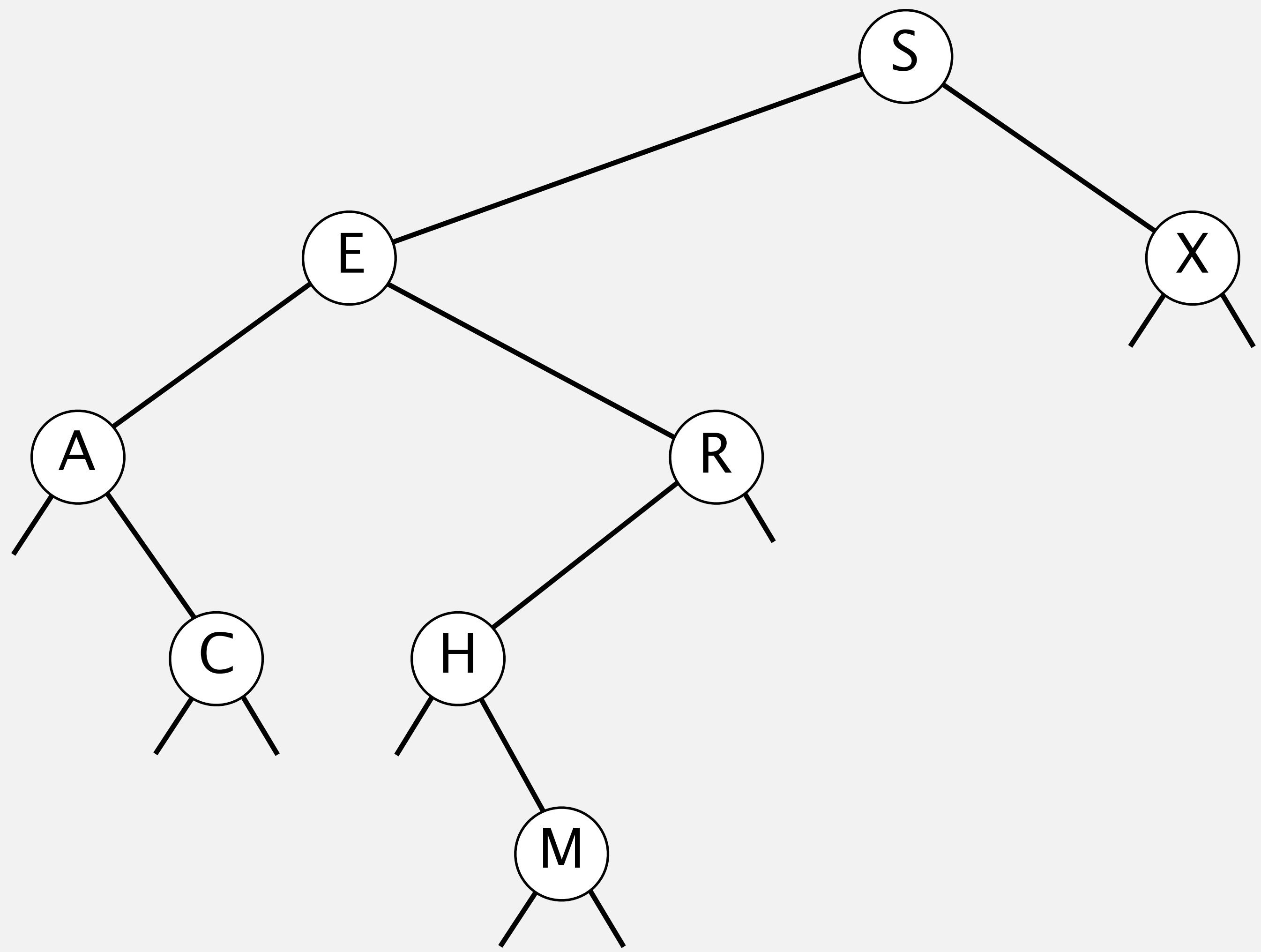
C. C A M H R E X S

D. S E X A R C H M



Inorder traversal

```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
      done X
   done S
```



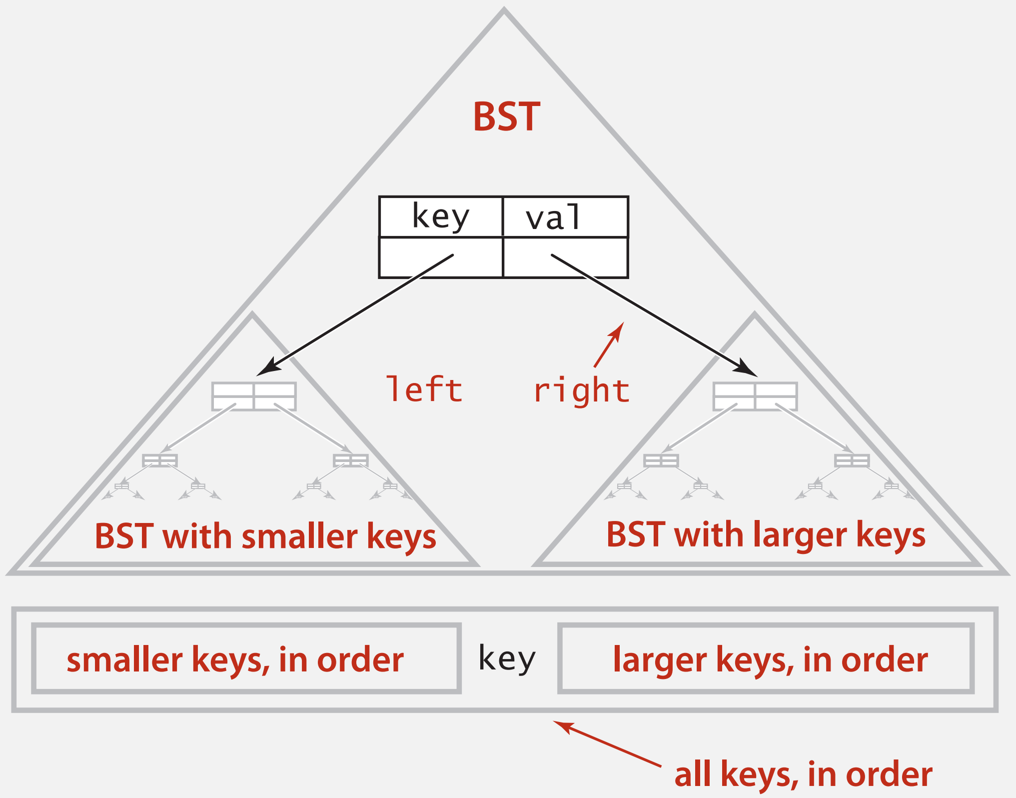
output: **A C E H M R S X**

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
   Queue<Key> q = new Queue<Key>();
   inorder(root, q);
   return q;
}
```

```
private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

Running time

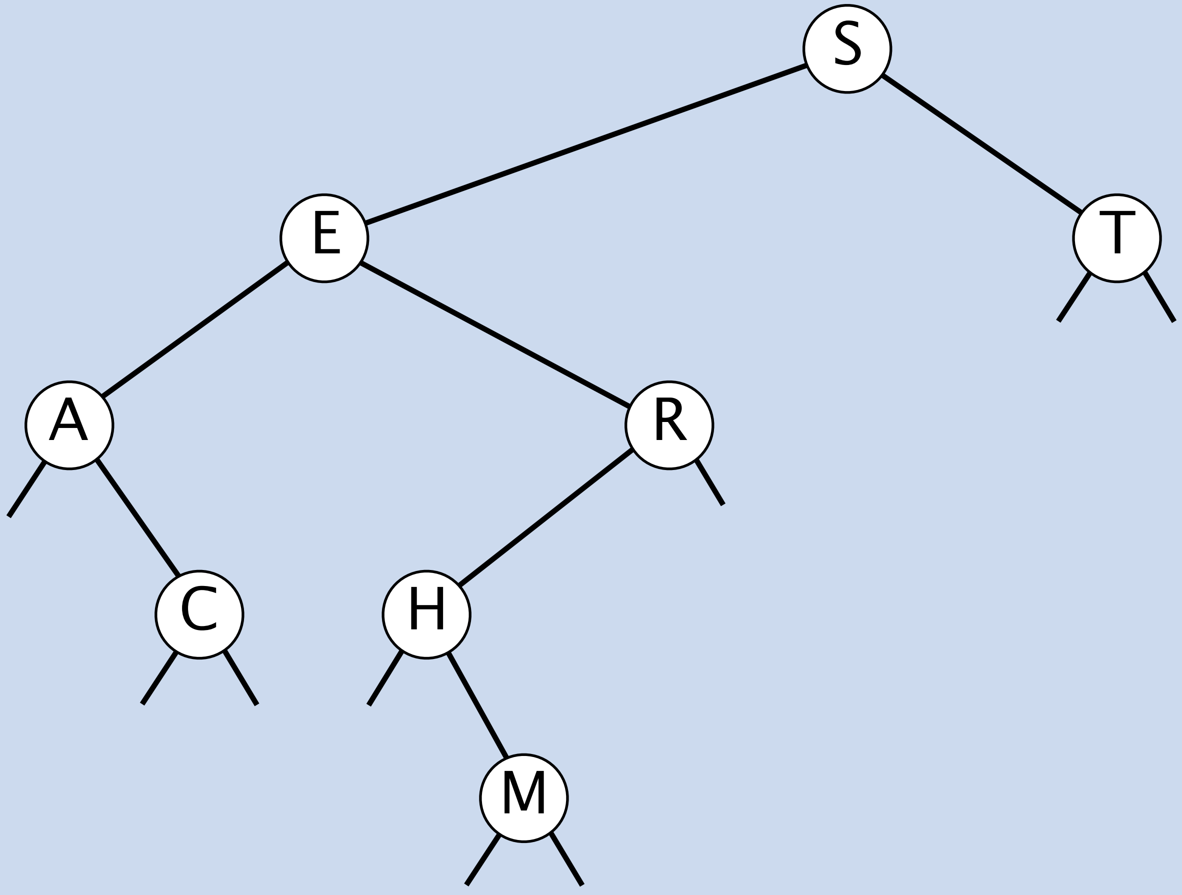
Property. Inorder traversal of a BST takes linear time.

Silicon Valley
(The Blood Boy)

Level-Order Traversal

Level-order traversal of a binary tree.

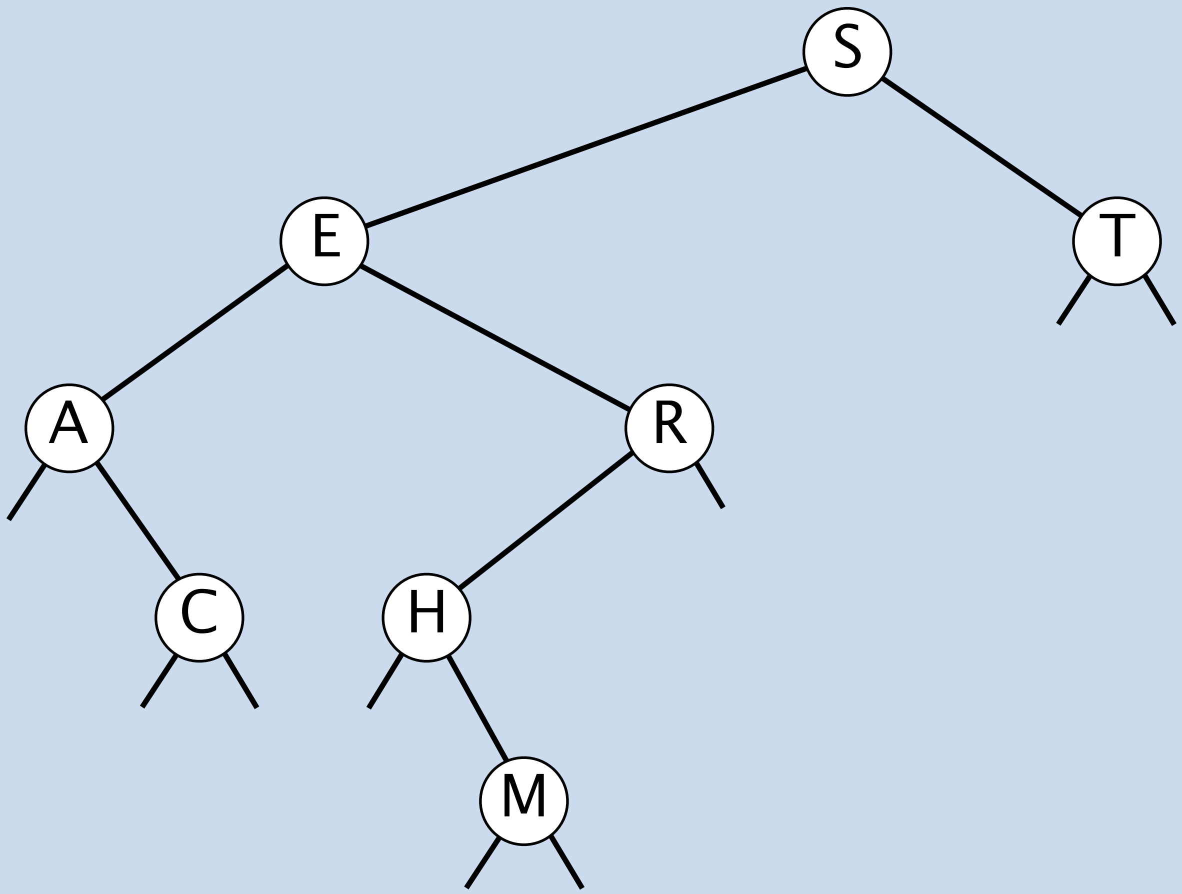
- Process root.
- Process children of root, from left to right.
- Process grandchildren of root, from left to right.
- ...



level-order traversal: S E T A R C H M

Level-Order Traversal

Q1. Given binary tree, how to compute level-order traversal?

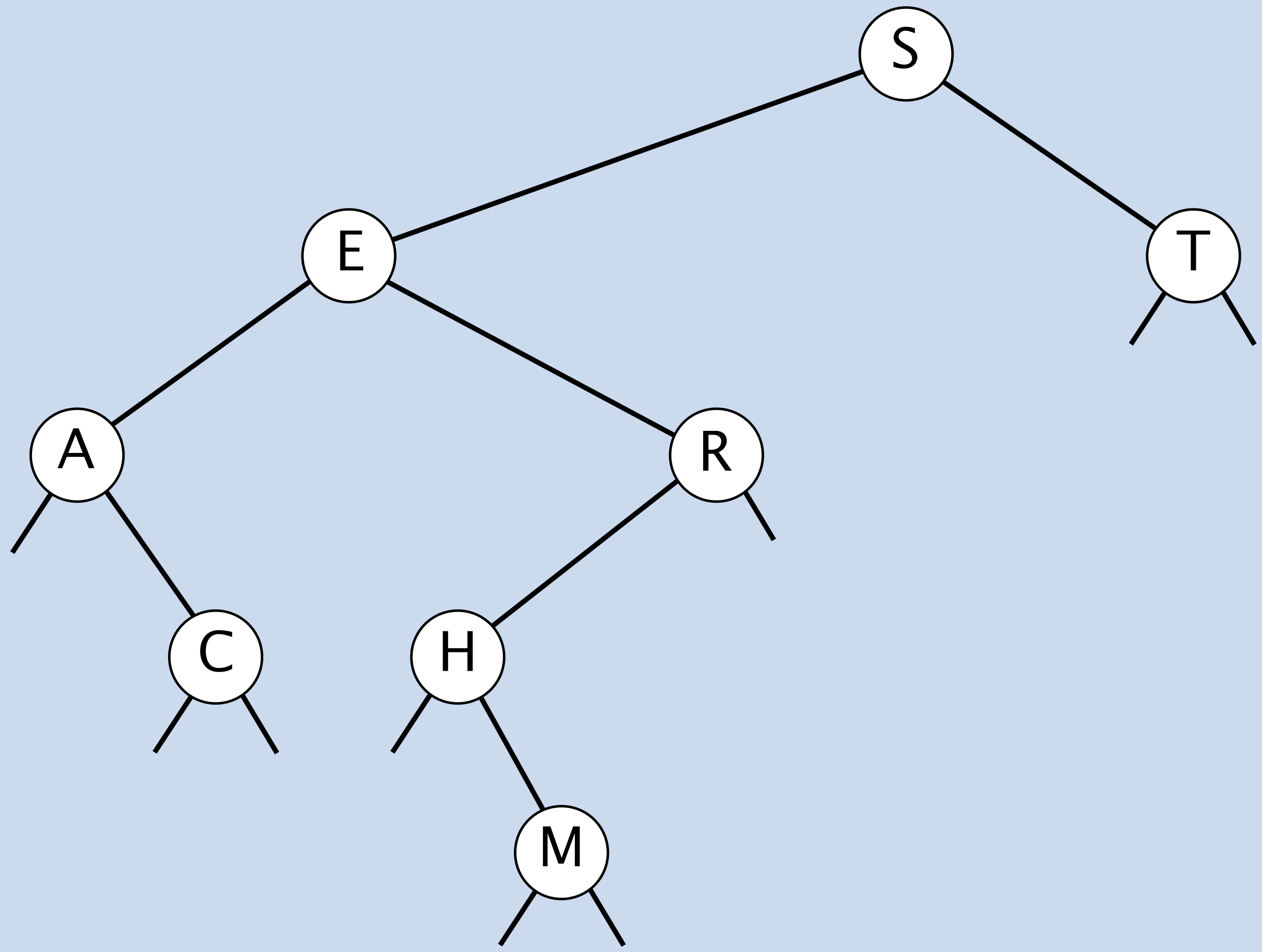


level-order traversal: S E T A R C H M

Level-Order Traversal

Q2. Given the level-order traversal of a BST, how to (uniquely) reconstruct?

Ex. ~~S~~ ~~E~~ ~~T~~ ~~A~~ ~~R~~ ~~C~~ ~~H~~ ~~M~~



https://algs4.cs.princeton.edu

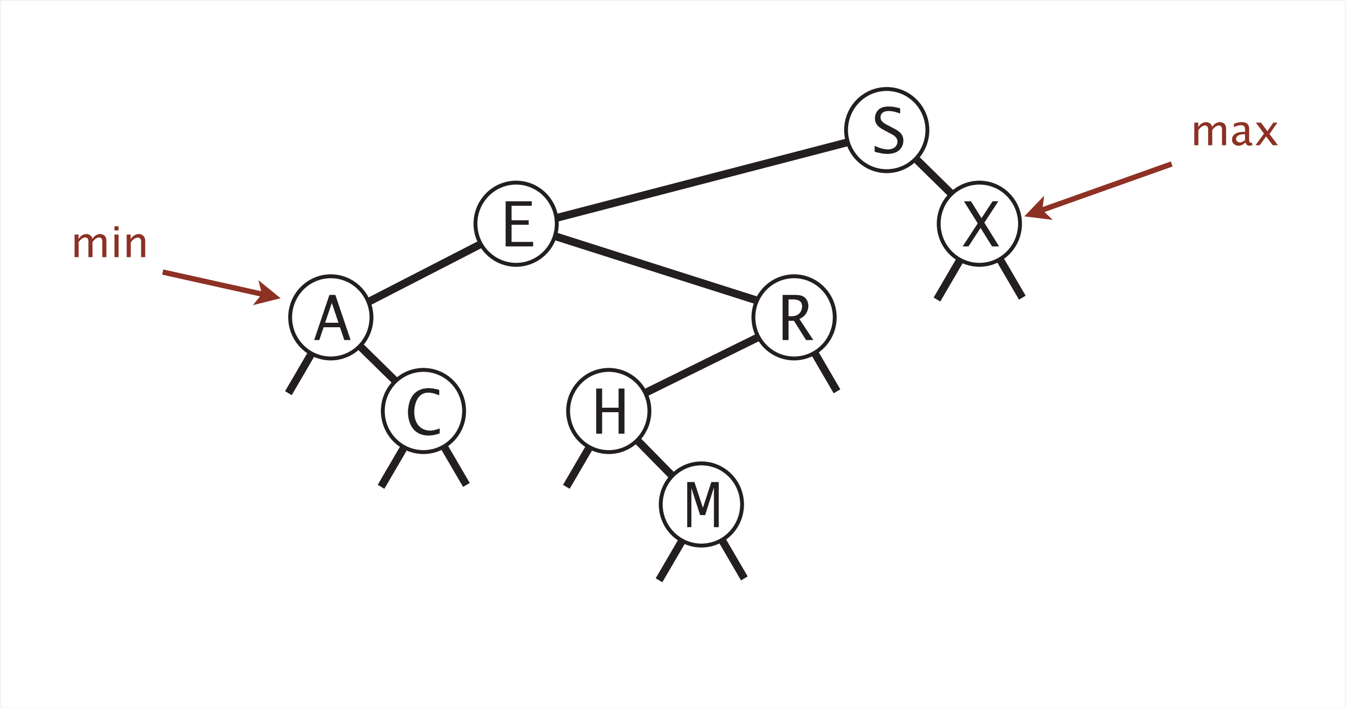
3.2 Binary Search Trees

- *BSTs*
- *iteration*
- ***ordered operations***
- *deletion*

Minimum and maximum

Minimum. Smallest key in BST.

Maximum. Largest key in BST.

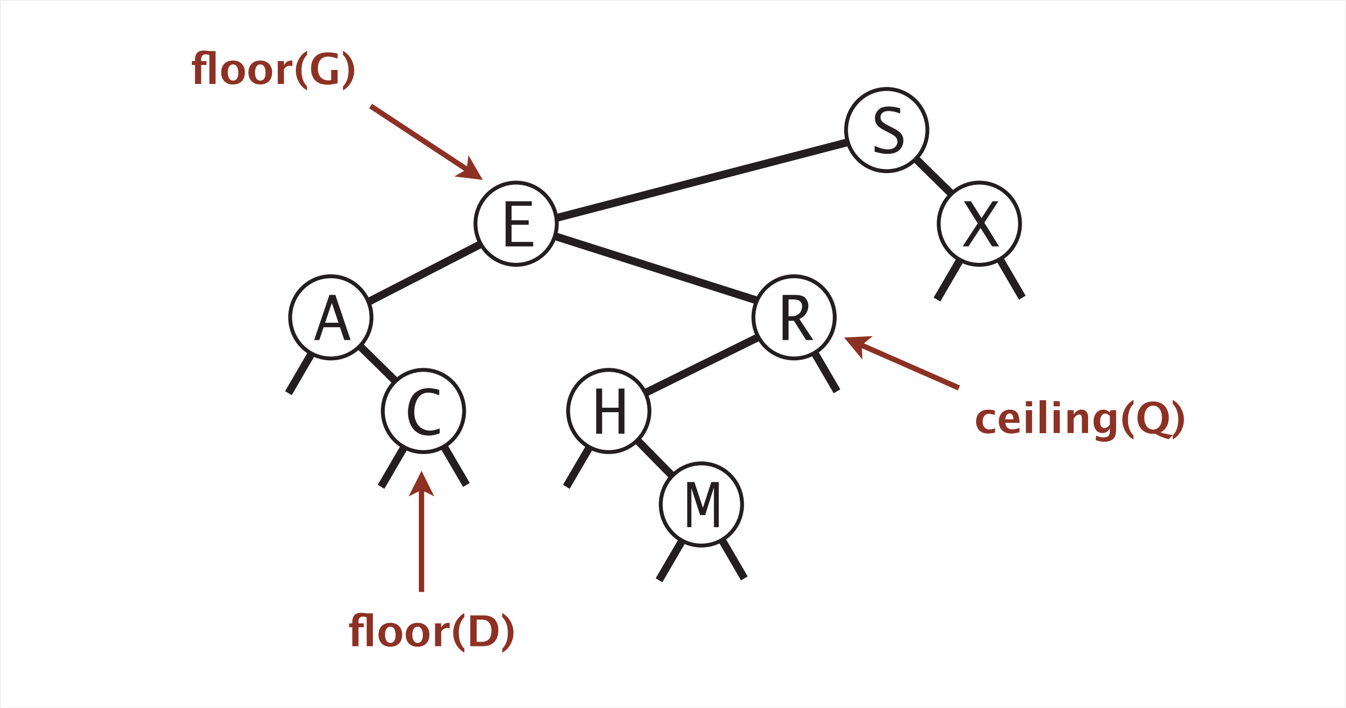


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key in BST $\leq$ query key.

Ceiling. Smallest key in BST $\geq$ query key.

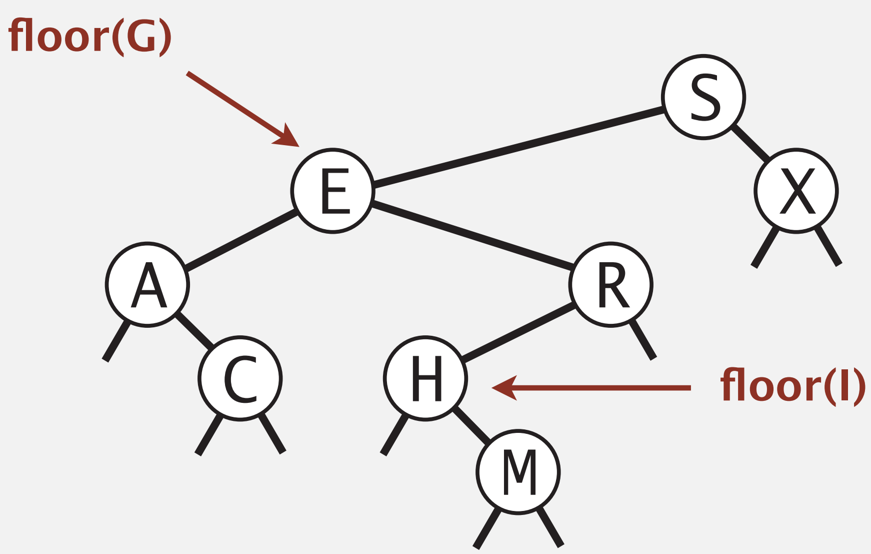


Computing the floor

Floor. Largest key in BST ≤ query key.

Key idea.

- To compute `floor(key)`, search for key.
- Both `floor(key)` and `ceiling(key)` must be on search path.
- Moreover, as you go down search path, any candidates get better and better.



Computing the floor: Java implementation

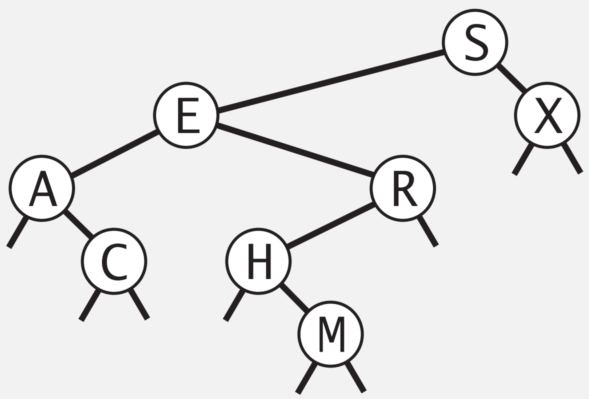
Invariant 1. The floor is either champ or in subtree rooted at x.

Invariant 2. Node x is in the right subtree of node containing champ.

champ must be floor

key in node is too large
(floor can't be in right subtree)

```
public Key floor(Key key)
{  return floor(root, key, null);  }

private Key floor(Node x, Key key, Key champ)
{
   if (x == null) return champ;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return floor(x.left, key, champ);
   else if (cmp  > 0) return floor(x.right, key, x.key);
   else if (cmp == 0) return x.key;
}
```



key in BST

key in node is a candidate for floor
(floor can't be in left subtree)

key in node is better candidate than champ

Rank and select

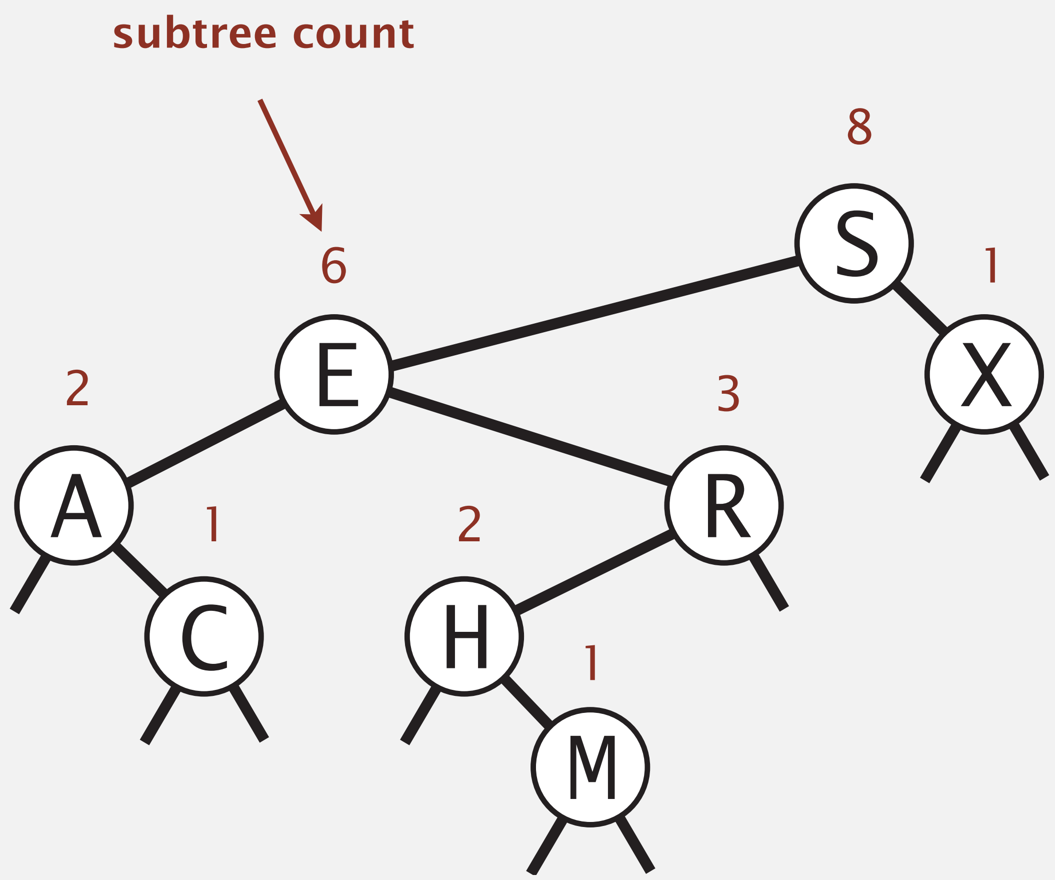
Rank. How many keys < *key* ?

Select. Key of rank *k*.

Q. How to implement `rank()` and `select()` efficiently for BSTs?

A. In each node, store the number of nodes in its subtree.

skipped in lecture
(see precept)



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

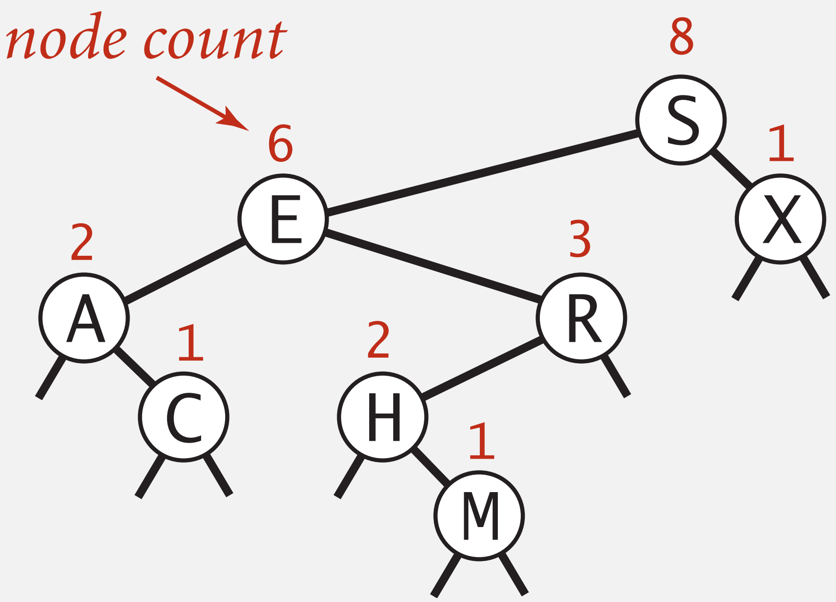
ok to call when x is null

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;

   x.count = 1 + size(x.left) + size(x.right);

   return x;
}
```

initialize subtree count to 1

Computing the rank

Rank. How many keys < *key* ?



Case 1. [*key* < key in node]

- Keys in left subtree? *count*
- Key in node? 0
- Keys in right subtree? 0

Case 2. [*key* > key in node]

- Keys in left subtree? *all*
- Key in node. 1
- Keys in right subtree? *count*

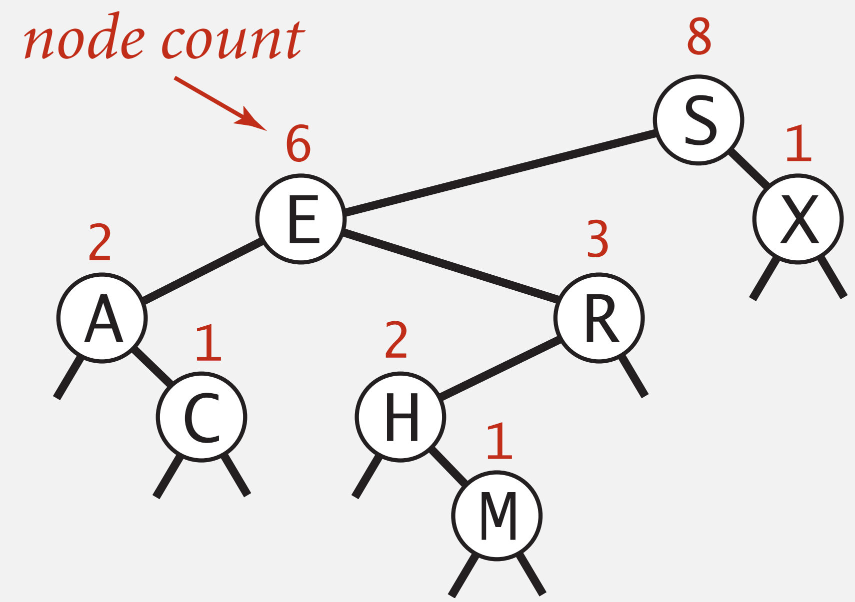
Case 3. [*key* = key in node]

- Keys in left subtree? *count*
- Key in node. 0
- Keys in right subtree? 0

Rank: Java implementation

Rank. How many keys < *key* ?

Easy recursive algorithm (3 cases!)




```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	n	$\log n$	h
insert	n	n	h
min / max	n	1	h
floor / ceiling	n	$\log n$	h
rank	n	$\log n$	h
select	n	1	h
ordered iteration	$n \log n$	n	n

h = height of BST

order of growth of running time of ordered symbol table operations

ST implementations: summary

implementation	guarantee		average case		ordered ops?	key interface
	search	insert	search hit	insert		
sequential search (unordered list)	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	$\log n$	n	✔	`compareTo()`
BST	n	n	$\log n$	$\log n$	✔	`compareTo()`
red–black BST	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

Next week. Guarantee logarithmic performance for all operations.