1.5 Union-Find

- union-find data type
- quick-find
- quick-union
- improvements

Applications

see precept

https://algs4.cs.princeton.edu
Steps to developing a usable algorithm to solve a computational problem.

1. Model the problem
2. Design an algorithm
3. Is the algorithm efficient?
   - Yes: Solve the problem
   - No: Understand why not
     - Try again

Subtext of today’s lecture (and this course)
1.5 Union–Find

- union–find data type
- quick-find
- quick-union
- improvements
- applications
Union–find data type

Disjoint sets. A collection of sets containing $n$ elements; each element in exactly one set.

Find. Return a “canonical” element in the set containing $p$.

Union. Merge the set containing $p$ with the set containing $q$.

\[
\text{find}(1) = \text{find}(4) = \text{find}(5) = 4
\]

\[
\{ 0 \} \{ 1, 4, 5 \} \{ 2, 3, 6, 7 \}
\]

8 elements, 3 disjoint sets

union(2, 5)

\[
\{ 0 \} \{ 1, 2, 3, 4, 5, 6, 7 \}
\]

2 disjoint sets

Simplifying assumption. The $n$ elements are named 0, 1, ..., $n - 1$. 
Union–find data type (API)

Goal. Design an efficient union–find data type.
- Number of elements $n$ can be huge.
- Number of operations $m$ can be huge.
- Union and find operations can be intermixed.

```java
public class UF {

    UF(int n)  // initialize union–find data structure with n singleton sets (0 to n – 1)
    void union(int p, int q)  // merge sets containing elements p and q
    int find(int p)  // canonical element in set containing p (0 to n – 1)

```
An application: dynamic connectivity

Given \( n \) vertices, support two operations:

- Add edge: directly connect two vertices with an edge.
- Connection query: is there a path connecting two vertices?

\[
\begin{align*}
\text{add edge} & \quad 4\rightarrow 3 \\
\text{add edge} & \quad 3\rightarrow 8 \\
\text{add edge} & \quad 6\rightarrow 5 \\
\text{add edge} & \quad 9\rightarrow 4 \\
\text{add edge} & \quad 2\rightarrow 1 \\
\text{are 8 and 9 connected?} & \quad \checkmark \\
\text{are 5 and 7 connected?} & \quad \times \\
\text{add edge} & \quad 5\rightarrow 0 \\
\text{add edge} & \quad 7\rightarrow 2 \\
\text{add edge} & \quad 6\rightarrow 1 \\
\text{add edge} & \quad 1\rightarrow 0 \\
\text{are 5 and 7 connected?} & \quad \checkmark
\end{align*}
\]


A larger connectivity example

Q. Is there a path connecting vertices $v$ and $w$?

finding a path is a slightly harder problem
(stay tuned for graph algorithms in Chapter 4)

A. Yes.
Modeling the dynamic-connectivity problem

Q. How to model the dynamic-connectivity problem using union–find?
A. Maintain disjoint sets that correspond to connected components.

Connected component. Maximal set of vertices that are mutually connected.

{ 0 } { 1, 4, 5 } { 2, 3, 6, 7 }
3 disjoint sets
Q. How to model the dynamic-connectivity problem using union–find?

A. Maintain disjoint sets that correspond to connected components.
   - Add edge between vertices $v$ and $w$.
   - Are vertices $v$ and $w$ connected?

add edge 2–5

$\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
\end{array}
\end{array}$

3 connected components

are vertices 5 and 6 connected?

$\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
\end{array}
\end{array}$

2 connected components

union(2, 5)

$\begin{array}{c}
\begin{array}{c}
\{ 0 \} \ { 1, \ 4, \ 5 \} \ { 2, \ 3, \ 6, \ 7 \} \\
\end{array}
\end{array}$

3 disjoint sets

$\begin{array}{c}
\begin{array}{c}
\{ 0 \} \ { 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7 \} \\
\end{array}
\end{array}$

2 disjoint sets
1.5 UNION–FIND

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Quick-find  [eager approach]

Data structure.
- Integer array \( id[] \) of length \( n \).
- Interpretation: \( id[p] \) is canonical element in the set containing \( p \).

```
    0 1 2 3 4 5 6 7 8 9
id[]   0 1 1 8 8 0 0 1 8 8
```

\[ id[i] = 0 \]
\{ 0, 5, 6 \}

\[ id[i] = 1 \]
\{ 1, 2, 7 \}

\[ id[i] = 8 \]
\{ 3, 4, 8, 9 \}

3 disjoint sets

Q. How to implement \( \text{find}(p) \)?
A. Easy, just return \( id[p] \).
Quick-find  [eager approach]

Data structure.

- Integer array \( \text{id}[] \) of length \( n \).
- Interpretation: \( \text{id}[p] \) is canonical element in the set containing \( p \).

\[
\text{union}(6, 1)
\]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>id[]</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

![Diagram](image)

**Problem:** many values can change

Q. How to implement \( \text{union}(p, q) \)?

A. Change all entries whose identifier equals \( \text{id}[p] \) to \( \text{id}[q] \) (or vice versa).
Quick-find: Java implementation

```java
public class QuickFindUF {
    private int[] id;

    public QuickFindUF(int n) {
        id = new int[n];
        for (int i = 0; i < n; i++)
            id[i] = i;
    }

    public int find(int p) {
        return id[p];
    }

    public void union(int p, int q) {
        int pid = id[p];
        int qid = id[q];
        for (int i = 0; i < id.length; i++)
            if (id[i] == pid) id[i] = qid;
    }
}
```

set id of each element to itself
(n array accesses)

return the id of p
(1 array access)

change all entries with id[p] to id[q]
(n + 2 to 2n + 2 array accesses)

https://algs4.cs.princeton.edu/15uf/QuickFindUF.java.html
Quick-find is too slow

Cost model. Number of array accesses (for read or write).

<table>
<thead>
<tr>
<th>algorithm</th>
<th>initialize</th>
<th>union</th>
<th>find</th>
</tr>
</thead>
<tbody>
<tr>
<td>quick-find</td>
<td>$n$</td>
<td>$n$</td>
<td>1</td>
</tr>
</tbody>
</table>

number of array accesses (ignoring leading constant)

Union is too expensive. Processing a sequence of $n$ union operations on $n$ elements takes more than $n^2$ array accesses.
1.5 Union–Find

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Quick-union  [lazy approach]

Data structure.
- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.

Q. How to implement `find(p)` operation?
A. Return `root` of tree containing `p`. 
Quick-union quiz

Data structure.

- Integer array `parent[]` of length \( n \), where `parent[i]` is parent of \( i \) in tree.
- Interpretation: elements in one tree correspond to one set.

How to implement `union(3, 5)`?


Quick-union [lazy approach]

Data structure.
- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.

Q. How to implement `union(p, q)`?
A. Set parent of `p`'s root to `q`'s root.
Quick-union  [lazy approach]

**Data structure.**
- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.

```
    union(3, 5)
    0 1 2 3 4 5 6 7 8 9
    0 1 9 4 9 6 6 7 8 6
```

**Q.** How to implement `union(p, q)`?
**A.** Set parent of `p`’s root to `q`’s root.
Quick-union demo
Quick-union: Java implementation

```java
public class QuickUnionUF {
    private int[] parent;

    public QuickUnionUF(int n) {
        parent = new int[n];
        for (int i = 0; i < n; i++)
            parent[i] = i;
    }

    public int find(int p) {
        while (p != parent[p])
            p = parent[p];
        return p;
    }

    public void union(int p, int q) {
        int r1 = find(p);
        int r2 = find(q);
        parent[r1] = r2;
    }
}
```

- Set parent of each element to itself \((n\) array accesses)
- Chase parent pointers until reach root \((\text{depth of } p \text{ array accesses})\)
- Change root of \(p\) to point to root of \(q\) \((\text{depth of } p \text{ and } q \text{ array accesses})\)

https://algs4.cs.princeton.edu/15uf/QuickUnionUF.java.html
Quick-union is also too slow

**Cost model.** Number of array accesses (for read or write).

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<td>$n$</td>
<td>$n$</td>
<td>$1$</td>
</tr>
<tr>
<td>quick-union</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

number of array accesses (ignoring leading constant)

**Quick-find defect.**
- Union too expensive (more than $n$ array accesses).
- Trees are flat, but too expensive to keep them flat.

**Quick-union defect.**
- Trees can get tall.
- Find too expensive (could be more than $n$ array accesses).
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https://algs4.cs.princeton.edu
Weighted quick-union quiz

When merging two trees, which strategy is most effective?

A. Link the root of the smaller tree to the root of the larger tree.
B. Link the root of the larger tree to the root of the smaller tree.
C. Link the root of the shorter tree to the root of the taller tree.
D. Link the root of the taller tree to the root of the shorter tree.

![Diagram of trees]

- Shorter and larger tree (height = 2, size = 14)
- Taller and smaller tree (height = 5, size = 9)
**Improvement 1: weighting**

**Weighted quick-union.**

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of elements).
- Always link root of smaller tree to root of larger tree.
Suppose that the parent[] array during weighted quick-union is:

```
parent[] = [0, 0, 0, 0, 0, 0, 7, 8, 8, 8]
```

Which parent[] entry changes during `union(2, 6)`?

A. parent[0]  
B. parent[2]  
C. parent[6]  
D. parent[8]
Weighted quick-union demo

```
parent[]
0  1  2  3  4  5  6  7  8  9
```
Quick-union vs. weighted quick-union: larger example

Quick-union and weighted quick-union (100 sites, 88 union() operations)

average distance to root: 5.11

average distance to root: 1.52
Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array size[i] to count number of elements in the tree rooted at i, initially 1.

- Find: identical to quick-union.
- Union: link root of smaller tree to root of larger tree; update size[].

```java
public void union(int p, int q)
{
    int r1 = find(p);
    int r2 = find(q);
    if (r1 == r2) return;

    if (size[r1] >= size[r2])
    {
        int temp = r1; r1 = r2; r2 = temp;
    }

    parent[r1] = r2;
    size[r2] += size[r1];
}
```

https://algs4.cs.princeton.edu/15uf/WeightedQuickUnionUF.java.html
Weighted quick-union analysis

Running time.
- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given two roots.

Proposition. Depth of any node $x$ is at most $\lg n$.

in computer science, $\lg$ means base-2 logarithm
Running time.

- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given two roots.

**Proposition.** Depth of any node $x$ is at most $\lg n$.

**Pf.** What causes the depth of element $x$ to increase?

- Increases by 1 when root of tree $T_1$ containing $x$ is linked to root of tree $T_2$.
- The size of the tree containing $x$ at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing $x$ can double at most $\lg n$ times. Why?
Weighted quick-union analysis

Running time.
- Find: takes time proportional to depth of $p$.
- Union: takes constant time, given two roots.

Proposition. Depth of any node $x$ is at most $\lg n$.

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<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>weighted quick-union</td>
<td>$n$</td>
<td>$\log n$</td>
<td>$\log n$</td>
</tr>
</tbody>
</table>

number of array accesses (ignoring leading constant)

log mean logarithm, for some constant base
Summary

Key point. Weighted quick-union makes it possible to solve problems that could not otherwise be addressed.

<table>
<thead>
<tr>
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<th>worst-case time</th>
</tr>
</thead>
<tbody>
<tr>
<td>quick-find</td>
<td>( mn )</td>
</tr>
<tr>
<td>quick-union</td>
<td>( mn )</td>
</tr>
<tr>
<td>weighted quick-union</td>
<td>( m \log n )</td>
</tr>
<tr>
<td>QU + path compression</td>
<td>( m \log n )</td>
</tr>
<tr>
<td>weighted QU + path compression</td>
<td>( m \alpha(n) )</td>
</tr>
</tbody>
</table>

order of growth for \( m \geq n \) union–find operations on a set of \( n \) elements

Ex. [\( 10^9 \) unions and finds with \( 10^9 \) elements]
- Weighted quick-union reduces run time from 30 years to 6 seconds.
- Supercomputer won’t help much; good algorithm enables solution.