

# Continuing CPS

COS 326

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# Last Time

Two key ideas:

- *continuation-passing style*
  - you can make functions tail-recursive by bottling up the stuff you might do after returning from a function call as a "continuation"
- sometimes you can't prove the theorem you want directly, because you need a powerful property in the middle of your proof.
- Solution: define and prove *a more general lemma*. eg:
  - instead of "for all x. property1(x, c)" for a constant c
  - prove "for all x. *for all y*. property2(x,y)"
    - where for all x. *property2(x,d) implies property1(x,c)*
    - so you can get the property1, which is what you wanted

## Challenge: CPS Convert the incr function

```
type tree = Leaf | Node of int * tree * tree ;;

let rec incr (t:tree) (i:int) : tree =
  match t with
  | Leaf -> Leaf
  | Node (j,left,right) ->
    Node (i+j, incr left i, incr right i)
;;
```

**Hint:** It is a little easier to put the continuations in the order in which they are called.

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```

} called *A-Normal Form*  
(intermediate computations given names; no function calls as args to other function calls)

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      let t2 = incr right i in  
      Node (i+j, t1, t2))
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      incr right i (fun t2 ->
        k (Node (i+j, t1, t2))))
```

# In general

```
let g input =  
  f3 (f2 (f1 input))
```

Direct Style

```
let g input =  
  let x1 = f1 input in  
  let x2 = f2 x1    in  
  f3 x2
```

A-normal Form

```
let g input k =  
  f1 input (fun x1 ->  
    f2 x1    (fun x2 ->  
      f3 x2 k))
```

CPS converted

# **GENERALIZING A THEOREM**

# sum vs sum\_cont

```
type cont = int -> int
```

```
let rec sum_cont (l:int list) (k:cont): int =  
  match l with  
  | [] -> k 0  
  | hd::tail -> sum_cont tail (fun s -> k (hd + s))
```

```
let rec sum (l:int list) : int =  
  match l with  
  | [] -> 0  
  | hd::tail -> hd + sum tail
```

**Theorem we tried to prove directly:**

for all l:int list,

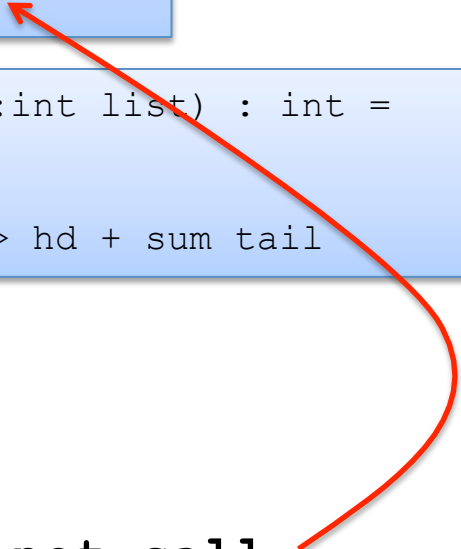
sum\_cont l (fun s -> s) == sum l

# sum vs sum\_cont

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```



**Theorem we tried to prove directly:**

for all `l:int list`,  
`sum_cont l (fun s -> s) == sum l`

**It didn't work because `sum_cont` does not call itself recursively using `(fun s -> s)`.**

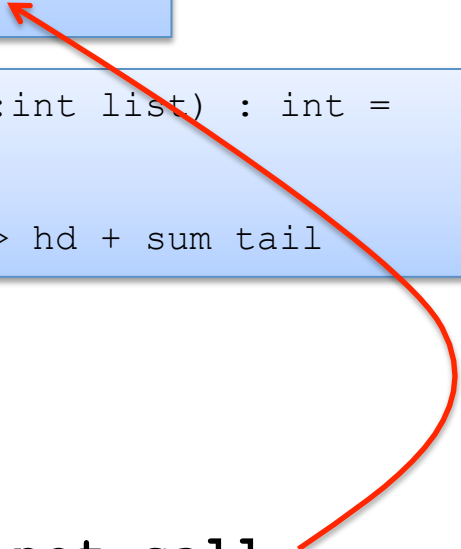
**To reason about recursive calls, we need to use the induction hypothesis, but we aren't allowed to here.**

# sum vs sum\_cont

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let rec sum (l:int list) : int =
  match l with
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```



**Theorem we tried to prove directly:**

for all `l:int list`,  
`sum_cont l (fun s -> s) == sum l`

It didn't work because `sum_cont` does not call itself recursively using `(fun s -> s)`.

To reason about recursive calls, we need to use the induction hypothesis, but we aren't allowed to here.

*Need to come up with IH that characterizes recursive calls*

# sum vs sum\_cont

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type cont = int -> int

let rec sum_cont (l:int list) (k:cont): int =
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```
let rec sum (l:int list) : int =
  match l with
  | [] -> 0
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```

**Theorem we tried to prove directly:**

for all l:int list,  
sum\_cont l (**fun s -> s**) == sum l

**New Theorem Attempt #1:**

for all l:int list,  
**for all k:int -> int,**  
sum\_cont l **k** == sum l



# sum vs sum\_cont

```
type cont = int -> int

let rec sum_cont (l:int list) (k:cont): int =
  match l with
  | [] -> k 0
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let rec sum (l:int list) : int =
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```

**Theorem we tried to prove directly:**

```
for all l:int list,
  sum_cont l (fun s -> s) == sum l
```

**New Theorem Attempt #1:**

```
for all l:int list,
for all k:int -> int,
  sum_cont l k == sum l
```

**But the theorem is false! :-)**  
**counter-example, choose:**  
**k = (fun x -> x + 1)**

# sum vs sum\_cont

```
type cont = int -> int

let rec sum_cont (l:int list) (k:cont): int =
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let rec sum (l:int list) : int =
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```

**Theorem we tried to prove directly:**

for all  $l:\text{int list}$ ,  
 $\text{sum\_cont } l \text{ (fun } s \text{ -> } s) == \text{sum } l$

**New Theorem Attempt #2:**

for all  $l:\text{int list}$ ,  
**for all  $k:\text{int} \rightarrow \text{int}$ ,**  
 $\text{sum\_cont } l \text{ } k == k \text{ (sum } l)$

Success!

# A Possible Strategy

Look at the recursive calls made within your function(s).

- If the arguments (other than the one you are doing induction on) are unchanged, you may have success with simple induction

```
let rec f (l:int list) (x: ...) (y:...) : int =  
  match l with  
  [] -> ...  
  | hd::tail -> ... (f tail x y)
```

- If they are different, you may have to search for a more general theorem that allows you to conclude something useful about those recursive calls.

```
let rec f (l:int list) (x: ...) (y:...) : int =  
  match l with  
  [] -> ...  
  | hd::tail -> ... (f tail (complex1) (complex2))
```

# Another Example

```
type exp =  
  Int of int  
| Add of exp * exp
```

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
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```

```
let rec eval1 (e:exp) : int =  
  match e with  
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## **Theorem:**

for all  $e:\text{exp}$ ,  
 $\text{eval1 } e == \text{eval2 } e \ 0$

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### **Theorem:**

for all  $e:\text{exp}$ ,  
 $\text{eval1 } e == \text{eval2 } e \ 0$

**What is going to go wrong if we try induction on the structure of  $e$  directly?**

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```

## Theorem:

```
for all e:exp,  
  eval1 e == eval2 e 0
```

In the case when  $e$  is  $\text{Add}(e_1, e_2)$ , we will need to reason that  **$\text{eval2 } e_1 (\text{eval2 } e_2 0) == ???$**  involving  **$\text{eval1}$**

But we won't be able to use IH. We'll have no facts to reason with. No way to make progress.

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## **Theorem:**

for all  $e:\text{exp}$ ,  
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## **Suggestions?**



# Another Example

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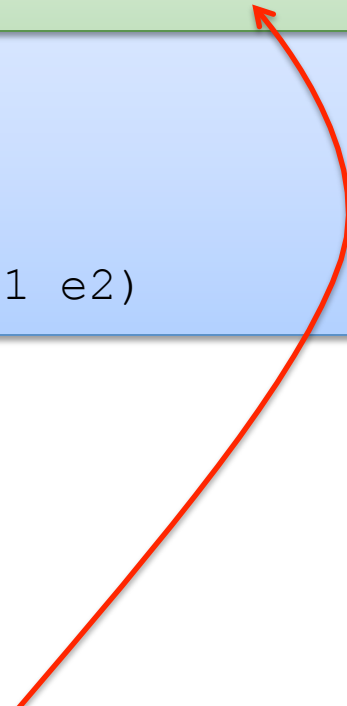
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## Suggestions?

We will need to reason about **eval2 e1 (...)**  
and to relate it to **eval1 e1** somehow.

What is the relationship?



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**Strategy: Introduce a new Lemma:**

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for all e:exp, for all n:int  
  (eval1 e) + n == eval2 e n
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**Proof: By induction on the structure of e.**

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case: e = int i

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for all e:exp, for all n:int  
 (eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

case: e = int i  
 eval1 (Int i) + n            (LHS)

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for all e:exp, for all n:int  
 (eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

```
case: e = int i  
  eval1 (Int i) + n      (LHS)  
== i + n                (by eval of eval1)  
== eval2 (Int i) n      (by reverse eval of eval2)
```

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**Strategy: Introduce a new Lemma:**

for all e:exp, for all n:int  
 (eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

case: e = Add(e1, e2)

# Another Example

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**Strategy: Introduce a new Lemma:**

for all  $e:\text{exp}$ , for all  $n:\text{int}$   
 $(\text{eval1 } e) + n == \text{eval2 } e \ n$

**Proof: By induction on the structure of  $e$ .**

case:  $e = \text{Add}(e1, e2)$

$\text{eval2 } (\text{Add}(e1, e2)) \ n$  (RHS)



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for all e:exp, for all n:int  
 (eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

case: e = Add(e1, e2)

```
  eval2 (Add(e1, e2)) n          (RHS)  
== eval2 e1 (eval2 e2 n)        (eval of eval2)
```

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for all e:exp, for all n:int  
(eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

case: e = Add(e1, e2)

eval2 (Add(e1, e2)) n	(RHS)
== eval2 e1 ( <b>eval2 e2 n</b> )	(eval of eval2)
== eval2 e1 ( <b>eval1 e2 + n</b> )	(by IH)

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**Strategy: Introduce a new Lemma:**

for all e:exp, for all n:int  
(eval1 e) + n == eval2 e n

**Proof: By induction on the structure of e.**

case: e = Add(e1, e2)

eval2 (Add(e1, e2)) n	(RHS)
== eval2 e1 (eval2 e2 n)	(eval of eval2)
== <b>eval2 e1 (eval1 e2 + n)</b>	(by IH)
== <b>eval1 e1 + (eval1 e2 + n)</b>	<b>(by IH)</b>
== (eval1 e1 + eval1 e2) + n	(associativity of +)
== eval1 (Add (e1, e2)) + n	(by eval in reverse)

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```

**Strategy: Introduce a new Lemma:**

for all  $e:\text{exp}$ , for all  $n:\text{int}$   
 $(\text{eval1 } e) + n == \text{eval2 } e \ n$

**Proof: By induction on the structure of  $e$ .**

```
case:  $e = \text{Add}(e1, e2)$   
   $\text{eval2 } (\text{Add}(e1, e2)) \ n$                 (RHS)  
==  $\text{eval2 } e1 \ (\text{eval2 } e2 \ n)$             (eval of eval2)  
==  $\text{eval2 } e1 \ (\text{eval1 } e2 + n)$             (by IH)  
==  $\text{eval1 } e1 + (\text{eval1 } e2 + n)$           (by IH)  
==  $(\text{eval1 } e1 + \text{eval1 } e2) + n$           (associativity of +)  
==  $\text{eval1 } (\text{Add } (e1, e2)) + n$           (by eval in reverse)
```

# Another Example

```
type exp =  
  Int of int  
| Add of exp * exp
```

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```

```
let rec eval1 (e:exp) : int =  
  match e with  
    Int i -> i  
  | Add (e1, e2) -> (eval1 e1) + (eval1 e2)
```

## Back to the Theorem:

for all  $e:\text{exp}$ ,  
 $\text{eval1 } e == \text{eval2 } e \ 0$

## Proof:

## Lemma:

for all  $e:\text{exp}$ , for all  $n:\text{int}$   
 $(\text{eval1 } e) + n == \text{eval2 } e \ n$

## Proof: Done!

# Another Example

```
type exp =  
  Int of int  
| Add of exp * exp
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```
let rec eval1 (e:exp) : int =  
  match e with  
    Int i -> i  
  | Add (e1, e2) -> (eval1 e1) + (eval1 e2)
```

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```

## Back to the Theorem:

for all  $e:\text{exp}$ ,  
 $\text{eval1 } e == \text{eval2 } e \ 0$

## Proof:

Pick any  $e$ .

$\text{eval2 } e \ 0$  (RHS)  
 $== \text{eval1 } e + 0$  (by Lemma, using 0 for  $n$ )  
 $== \text{eval1 } e$  (by math)

## Lemma:

for all  $e:\text{exp}$ , for all  $n:\text{int}$   
 $(\text{eval1 } e) + n == \text{eval2 } e \ n$

## Proof: Done!

## Quick Question

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```

Is eval2 tail recursive?

## Quick Question

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```

Is eval2 tail recursive?

No! Lot's of stuff happens after the first recursive call to eval2!



# Quick Question

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```



```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 ( _____ )
```

continuation of **eval2 e2 n**

# Quick Question

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```



```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (_____)
```



```
let rec eval2 (e:exp) (n:int) (k: int -> int) : int =  
  match e with  
    Int i -> k (i + n)  
  | Add (e1, e2) -> eval2 e2 n (fun m -> eval2 e1 m k)
```

# Quick Question

```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 (eval2 e2 n)
```



```
let rec eval2 (e:exp) (n:int) : int =  
  match e with  
    Int i -> i + n  
  | Add (e1, e2) -> eval2 e1 ( )
```



continuation of **eval2 e1**  
is whatever **eval2** does  
when it returns

```
let rec eval2 (e:exp) (n:int) (k: int -> int) : int =  
  match e with  
    Int i -> k (i + n)  
  | Add (e1, e2) -> eval2 e2 n (fun m -> eval2 e1 m k)
```

# Summary

Tail-recursive programs:

- do not do any computation after they make a recursive call
- conversion to CPS is one way to make any computation tail-recursive
  - bottle up the stuff you do after the call into a continuation

Proving programs correct can be arbitrarily hard:

- the difficult part comes in finding auxiliary lemmas to prove.
- these lemmas must be:
  - *strong enough* to imply the theorem you want
  - *weak enough* that they remain true and can be proven
  - insight is needed to find the right middle ground