Did I get it right?

COS 326
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http://~cos326/notes/evaluation.php
http://~cos326/notes/reasoning.php
“Did I get it right?”
– Most fundamental question you can ask about a computer program

Techniques for answering:

**Grading**
- hand in program to TA
- check to see if you got an A
- (does not apply after school is out)

**Testing**
- create a set of sample inputs
- run the program on each input
- check the results
- how far does this get you?
  - has anyone ever tested a homework and not received an A?
  - why did that happen?

**Proving**
- consider all legal inputs
- show every input yields correct result
- how far does this get you?
  - has anyone ever proven a homework correct and not received an A?
  - why did that happen?
Program proving

The basic, overall *mechanics* of proving functional programs correct is not particularly hard.

- You are already doing it to some degree.
- The real goal of this lecture to help you further organize your thoughts and to give you a more systematic means of understanding your programs.
- Of course, it can certainly be hard to prove some specific program has some specific property -- just like it can be hard to write a program that solves some hard problem

We are going to focus on proving the correctness of *pure expressions*

- their meaning is determined exclusively by the value they return
- don’t print, don’t mutate global variables, don’t raise exceptions
- always terminate
- another word for “pure expression” is “valuable expression”
- but I want you to understand why the presence of possibly non-terminating programs complicates rigorous reasoning about program correctness
Two key concepts:

- A **valuable expression**
  - an expression that always terminates (without side effects) and produces a value

- A **total function** with type \( t_1 \rightarrow t_2 \)
  - a function that terminates on all arguments with type \( t_1 \), producing a value of type \( t_2 \)
  - the “opposite” of a total function is a **partial function**
    - terminates on some (possibly all) input values

Many reasoning rules depend on expressions being valuable and hence the functions that are applied being total.

*Unless told otherwise*, you can assume functions are total and expressions are valuable. (Such facts can typically be proven by induction.)
Example Theorems

We'll prove properties of OCaml expressions, starting with equivalence properties:

**Theorem:** easy 1 20 30 == 50

**Theorem:**
for all natural numbers \( n \),
\(\text{exp} \ n \equiv 2^n\)

**Theorem:**
for all lists \( \text{xs}, \text{ys} \),
\(\text{length} \ (\text{cat} \ \text{xs} \ \text{ys}) \equiv \text{length} \ \text{xs} + \text{length} \ \text{ys}\)
Things to Watch For

The types are going to guide us in our theorem proving, just like they guided us in our programming.
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- when *programming* with lists, *functions* (often) have 2 cases:
  
  - [ ]
  - hd :: tl

- when *proving* with lists, *proofs* (often) have 2 cases:
  
  - [ ]
  - hd :: tl
The types are going to guide us in our theorem proving, just like they guided us in our programming

— when *programming* with lists, *functions* (often) have 2 cases:
  • [ ]
  • hd :: tl
— when *proving* with lists, *proofs* (often) have 2 cases:
  • [ ]
  • hd :: tl
— when *programming* with natural numbers, *functions* have 2 cases:
  • 0
  • k + 1
— when *proving* with natural numbers, *proofs* have 2 cases:
  • 0
  • k + 1

This is not a fluke! Proof structure often related to program structure.
Things to Watch For

More structure:

– when *programming* with lists:
  
  • [ ] is often easy
  
  • hd :: tl often requires a *recursive function call* on tl
    – we *assume* our recursive function behaves correctly on tl

– when *proving* with lists:

  • [ ] is often easy
  
  • hd :: tl often requires appeal to an *induction hypothesis* for tl
    – we *assume* our property of interest holds for tl
Things to Watch For

• More structure:
  – when *programming* with lists:
    • [ ] is often easy
    • hd :: tl often requires a *recursive function call* on tl
      – we *assume* our recursive function behaves correctly on tl
  – when *proving* with lists:
    • [ ] is often easy
    • hd :: tl often requires appeal to an *induction hypothesis* for tl
      – we *assume* our property of interest holds for tl
  – when *programming* with natural numbers:
    • 0 is often easy
    • k + 1 often requires a *recursive call* on k
  – when *proving* with natural numbers:
    • 0 is often easy
    • k + 1 often requires appeal to an *induction hypothesis* for k
Idea 1: The fundamental definition of when programs are equal.

two expressions are equal if and only if:
• they both evaluate to the same value, or
• they both raise the same exception, or
• they both infinite loop

we will use what we learned about OCaml evaluation
Key Ideas

Idea 1: The fundamental definition of when programs are equal.

two expressions are equal if and only if:
• they both evaluate to the same value, or
• they both raise the same exception, or
• they both infinite loop

Idea 2: A fundamental proof principle.

if two expressions $e_1$ and $e_2$ are equal
and we have a third complicated expression $\text{FOO}(x)$
then $\text{FOO}(e_1)$ is equal to $\text{FOO}(e_2)$

super useful since we can do a small, local proof
and then use it in a big program: modularity!
The Workhorse: Substitution of Equals for Equals

if two expressions e1 and e2 are equal and we have a third complicated expression FOO (x) then FOO(e1) is equal to FOO (e2)

An example: I know 2+2 == 4.

I have a complicated expression: bar (foo ( ___ )) * 34

Then I also know that bar (foo (2+2)) * 34 == bar (foo (4)) * 34.

If expressions contain things like mutable references, this proof principle breaks down. That’s a big reason why I like functional programming and a big reason we are working primarily with pure expressions.
Important Properties of Expression Equality

Other important properties:

*(reflexivity)* every expression e is equal to itself: e == e

*(symmetry)* if e1 == e2 then e2 == e1

*(transitivity)* if e1 == e2 and e2 == e3 then e1 == e3

*(evaluation)* if e1 --> e2 then e1 == e2.

*(congruence, aka substitution of equals for equals)* if two expressions are equal, you can substitute one for the other inside any other expression:

- if e1 == e2 then e[e1/x] == e[e2/x]
EASY EXAMPLES
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

Given: \text{let easy } x \ y \ z = x \ast (y + z)
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

**Given:**

let easy x y z = x * (y + z)

**Theorem:** easy 1 20 30 == 50
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

Given: \( \text{let easy } x \ y \ z = x \times (y + z) \)

Theorem: \( \text{easy } 1 \ 20 \ 30 \equiv 50 \)

Proof:
\( \text{easy } 1 \ 20 \ 30 \) (left-hand side of equation)
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

Given: \( \text{let easy } x \ y \ z = x * (y + z) \)

Theorem: easy 1 20 30 == 50

Proof:
- easy 1 20 30 \hspace{100pt} \text{(left-hand side of equation)}
- == 1 * (20 + 30) \hspace{100pt} \text{(by evaluating easy 1 step)}
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

Given: \[ \text{let easy } x \ y \ z = x \ast (y + z) \]

Theorem: easy 1 20 30 == 50

Proof:
\[
\begin{align*}
\text{easy 1 20 30} & \quad \text{(left-hand side of equation)} \\
\text{== 1} \ast (20 + 30) & \quad \text{(by evaluating easy 1 step)} \\
\text{== 50} & \quad \text{(by math)} \\
\text{QED.}
\end{align*}
\]
Most of our proofs will use what we know about the substitution model of evaluation. Eg:

**Given:** let easy \( x \ y \ z \ = \ x \ * \ (y \ + \ z) \)

**Theorem:** easy \( 1 \ 20 \ 30 \ = \ = \ 50 \)

**Proof:**
- easy \( 1 \ 20 \ 30 \)
- \( = \ 1 \ * \ (20 \ + \ 30) \) (left-hand side of equation)
- \( = \ 50 \) (by evaluating easy 1 step)
- QED.

Facts go on the left

Justifications on the right

Notice the 2-column proof style
We can use *symbolic values* in our proofs too. Eg:

**Given:**  
let easy $x \ y \ z = x \ast (y + z)$

**Theorem:** for all integers $n$ and $m$, easy $1 \ n \ m \equiv n + m$

**Proof:**  
easy $1 \ n \ m$ (left-hand side of equation)
Easy Examples

We can use *symbolic values* in our proofs too. Eg:

**Given:** let easy x y z = x * (y + z)

**Theorem:** for all integers n and m, easy 1 n m == n + m

**Proof:**

easy 1 n m (left-hand side of equation)
== 1 * (n + m) (by evaluating easy)
We can use *symbolic values* in our proofs too. Eg:

**Given:** let easy \( x y z = x \times (y + z) \)

**Theorem:** for all integers \( n \) and \( m \), easy \( 1 n m \) == \( n + m \)

**Proof:**

\[
\begin{align*}
\text{easy } 1 n m & \quad \text{(left-hand side of equation)} \\
== 1 \times (n + m) & \quad \text{(by evaluating easy)} \\
== n + m & \quad \text{(by math)} \\
\end{align*}
\]

QED.
We can use *symbolic values* in our proofs too. Eg:

**Given:** \[\text{let easy } x \ y \ z = x \times (y + z)\]

**Theorem:** for all integers \(n, m, k\), easy \(k \ n \ m\) == easy \(k \ m \ n\)

**Proof:**

\[\text{easy } k \ n \ m\] \hspace{1cm} \text{(left-hand side of equation)}
We can use *symbolic values* in our proofs too. Eg:

**Given:**  
let easy x y z = x * (y + z)

**Theorem:**  
for all integers n, m, k, easy k n m == easy k m n

**Proof:**  
  easy k n m  
  == k * (n + m)  
  (left-hand side of equation)  
  (by evaluating easy)
We can use *symbolic values* in our proofs too. Eg:

**Given:** let easy \( x \ y \ z = x \times (y + z) \)

**Theorem:** for all integers \( n, m, k \), easy \( k \ n \ m = easy \ k \ m \ n \)

**Proof:**

\[
\begin{align*}
easy k n m & \quad \text{(left-hand side of equation)} \\
== k \times (n + m) & \quad \text{(by evaluating easy)} \\
== k \times (m + n) & \quad \text{(by math, subst of equals for equals)}
\end{align*}
\]

I'm not going to mention this from now on.
We can use *symbolic values* in our proofs too. Eg:

**Given:** let easy x y z = x * (y + z)

**Theorem:** for all integers n, m, k, easy k n m == easy k m n

**Proof:**

```plaintext
easy k n m            (left-hand side of equation)
== k * (n + m)        (by evaluating easy)
== k * (m + n)        (by math)
== easy k m n         (by evaluating easy)
QED.
```
Easy Examples

We can use *symbolic values* in our proofs too. Eg:

**Given:** let easy \( x \ y \ z = x \times (y + z) \)

**Theorem:** for all integers \( n, m, k \), easy \( k \ n \ m \) \( \equiv \) easy \( k \ m \ n \)

**Proof:**
- easy \( k \ n \ m \) \hspace{1cm} (left-hand side of equation)
- \( \equiv k \times (n + m) \) \hspace{1cm} (eval)
- \( \equiv k \times (m + n) \) \hspace{1cm} (by math)
- \( \equiv \) easy \( k \ m \ n \) \hspace{1cm} (eval)

QED.
An Aside: Symbolic Evaluation

One last thing: we sometimes find ourselves with a function, like `easy`, that has a symbolic argument like \( k+1 \) for some \( k \) and we would like to evaluate it in our proof. eg:

\[
\text{easy } x \ y \ (k+1) \\
\equiv x \times (y + (k+1)) \quad \text{(by evaluation of easy .... I hope)}
\]

However, that is not how OCaml evaluation works. OCaml evaluates it’s arguments to a `value` first, and then calls the function.

Don’t worry: if you know that the expression `will` evaluate to a value (and will not infinite loop or raise an exception) then you can substitute the symbolic expression for the parameter of the function

*To be rigorous, you should prove it will evaluate to a value, not just guess ... typically we will take this for granted ...*
An Aside: Symbolic Evaluation

An interesting example:

```javascript
let const x = 7
```

const ( exp ) == 7  (By evaluation of const?)

does this work for any expression?
An Aside: Symbolic Evaluation

An interesting example:

```javascript
let const x = 7
```

const ( n / 0 ) == 7  (By careless, wrong! evaluation of const)
An interesting example:

```javascript
let const x = 7
```

\[
\text{const } (\ n / 0\ ) == 7 \quad \text{(By careless, wrong! evaluation of const)}
\]

- n / 0 raises an exception
- so const (n / 0) raises an exception
- but 7 is just 7 and doesn’t raise an exception
- an expression that raises an exception is not equal to one that returns a value!
An Aside: Symbolic Evaluation

An interesting example:

```javascript
let const x = 7
```

```javascript
const ( n / 0 ) == 7  (By careless, wrong! evaluation of const)
```

what to remember:

f (e) == body_of_f_with_e_substituted_for_f_parameter

whenever e evaluates to a value (not an exception or infinite loop)
Summary so far: Proof by simple calculation

- Some proofs are very easy and can be done by:
  - eval definitions (ie: using forwards evaluation)
  - using lemmas or facts we already know (eg: math)
  - folding definitions back up (ie: using reverse evaluation)

- Eg:

<table>
<thead>
<tr>
<th>Definition:</th>
<th>Theorem: easy a b c == easy a c b</th>
</tr>
</thead>
<tbody>
<tr>
<td>let easy x y z = x * (y + z)</td>
<td>Proof:</td>
</tr>
<tr>
<td>given this</td>
<td>easy a b c</td>
</tr>
<tr>
<td>we do this proof</td>
<td>== a * (b + c) (by def of easy)</td>
</tr>
<tr>
<td></td>
<td>== a * (c + b) (by math)</td>
</tr>
<tr>
<td></td>
<td>== easy a c b (by def of easy)</td>
</tr>
</tbody>
</table>
INDUCTIVE PROOFS
Theorem: For all natural numbers n, 
exp(n) == 2^n.

let rec exp n = 
match n with 
| 0 -> 1 
| n -> 2 * exp (n-1)
Theorem: For all natural numbers $n$,
\[ \exp(n) = 2^n. \]

Recall: Every natural number $n$ is either 0 or it is $k+1$ (where $k$ is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

```occam
let rec exp n =
  match n with
  | 0 -> 1
  | n -> 2 * exp (n-1)
```
Theorem: For all natural numbers n,
\( \text{exp}(n) = 2^n \).

Recall: Every natural number n is either 0 or it is \( k+1 \) (where k is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: \( n = 0 \):

\[ \text{exp} 0 \]
Theorem: For all natural numbers n,
exp(n) == 2^n.

Recall: Every natural number n is
either 0 or it is k+1 (where k is also a natural number).
Hence, we follow the structure of the data and do
our proof in two cases.

Proof:

Case: \( n = 0 \):
exp 0
== match 0 with 0 -> 1 | n -> 2 * exp (n-1) (by eval exp)
Theorem: For all natural numbers n, 
\( \text{exp}(n) = 2^n. \)

Recall: Every natural number n is either 0 or it is \( k+1 \) (where \( k \) is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: \( n = 0 \):

\[
\begin{align*}
\text{exp} 0 &= \text{match } 0 \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \ast \text{exp}(n-1) \\
&= 1 \\
&= 2^0
\end{align*}
\]
(by eval exp)  
(by evaluating match)  
(by math)
Theorem: For all natural numbers n,
\[ \text{exp}(n) = 2^n. \]

Recall: Every natural number n is either 0 or it is k+1 (where k is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: n == k+1:
\[ \text{exp}(k+1) \]
Theorem: For all natural numbers n,
exp(n) == 2^n.

Recall: Every natural number n is either 0 or it is k+1 (where k is also a natural number).
Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: n == k+1:
exp (k+1)
== match (k+1) with 0 -> 1 | n -> 2 * exp (n -1) (by eval exp)
Theorem: For all natural numbers n,
exp(n) == 2^n.

Recall: Every natural number n is either 0 or it is k+1 (where k is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: n == k+1:
exp (k+1)
== match (k+1) with 0 -> 1 | n -> 2 * exp (n -1) (by eval exp)
== 2 * exp (k+1 - 1) (by evaluating case)
Theorem: For all natural numbers n,

\[ \exp(n) = 2^n. \]

Recall: Every natural number n is either 0 or it is \( k+1 \) (where \( k \) is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: \( n = k+1 \):

\[
\begin{align*}
\exp(k+1) \\
= & \text{match } (k+1) \text{ with } 0 \rightarrow 1 | n \rightarrow 2 \times \exp(n-1) \quad \text{(by eval } \exp) \\
= & 2 \times \exp(k+1 - 1) \quad \text{(by evaluating case)} \\
= & ??
\end{align*}
\]

let rec exp n =
  match n with
  | 0 -> 1
  | n -> 2 * exp (n-1)
Theorem: For all natural numbers \( n \),
\[ \text{exp}(n) = 2^n. \]

Recall: Every natural number \( n \) is either 0 or it is \( k+1 \) (where \( k \) is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: \( n = k+1 \):

\[
\begin{align*}
\text{exp}(k+1) &= \text{match } (k+1) \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \times \text{exp}(n-1) \quad \text{(by eval exp)} \\
&= 2 \times \text{exp}(k+1-1) \quad \text{(by evaluating case)} \\
&= 2 \times (\text{match } (k+1-1) \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \times \text{exp}(n-1)) \quad \text{(by eval exp)}
\end{align*}
\]
A problem

Theorem: For all natural numbers $n$, 
$\exp(n) = 2^n$.

Recall: Every natural number $n$ is either 0 or it is $k+1$ (where $k$ is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: $n = k+1$:

$\exp (k+1)$

$= \text{match } (k+1) \text{ with } 0 \rightarrow 1 | n \rightarrow 2 * \exp (n-1)$ (by eval $\exp$)

$= 2 * \exp (k+1 - 1)$ (by evaluating case)

$= 2 * (\text{match } (k+1-1) \text{ with } 0 \rightarrow 1 | n \rightarrow 2 * \exp (n-1))$ (by eval $\exp$)

$= 2 * (2 * \exp ((k+1) - 1 - 1))$ (by evaluating case)
Theorem: For all natural numbers n,
exp(n) == 2^n.

Recall: Every natural number n is either 0 or it is k+1 (where k is also a natural number).
Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: n == k+1:

exp (k+1)
== match (k+1) with 0 -> 1 | n -> 2 * exp (n -1) \hfill (by eval exp)
== 2 * exp (k+1 - 1) \hfill (by evaluating case)
== 2 * (match (k+1-1) with 0 -> 1 | n -> 2 * exp (n -1)) \hfill (by eval exp)
== 2 * (2 * exp ((k+1) - 1 - 1)) \hfill (by evaluating case)
== ... we aren’t making progress ... just unrolling the loop forever ...
Induction

When proving theorems about recursive functions, we usually need to use *induction*.

– In inductive proofs, in a case for object $X$, we assume that the theorem holds *for all objects smaller than $X*
  
  • this assumption is called the *inductive hypothesis* (IH for short)

– Eg: When proving a theorem about natural numbers by induction, and considering the case for natural number $k+1$, we get to assume our theorem is true for natural number $k$ (because $k$ is smaller than $k+1$)

– Eg: When proving a theorem about lists by induction, and considering the case for a list $x::xs$, we get to assume our theorem is true for the list $xs$ (which is a shorter list than $x::xs$)
Theorem: For all natural numbers n,

\[ \text{exp}(n) = 2^n. \]

Recall: Every natural number n is either 0 or it is k+1 (where k is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: n == k+1:

\[ \text{exp}(k+1) \]
\[ = \text{match } (k+1) \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \times \text{exp}(n-1) \]  \hspace{5cm} \text{(by eval exp)}
\[ = 2 \times \text{exp}(k+1-1) \]  \hspace{5cm} \text{(by evaluating case)}
Theorem: For all natural numbers $n$,

$$\exp(n) = 2^n.$$ 

Recall: Every natural number $n$ is either 0 or it is $k+1$ (where $k$ is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: $n = k+1$:

$$\exp(k+1)$$

$$= \text{match } (k+1) \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \ast \exp(n-1)$$

(by eval $\exp$)

$$= 2 \ast \exp(k+1 - 1)$$

(by evaluating case)

$$= 2 \ast \exp(k)$$

(by math)
Theorem: For all natural numbers $n$, 
$\exp(n) = 2^n$.

Recall: Every natural number $n$ is either $0$ or it is $k+1$ (where $k$ is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: $n = k+1$:

\[
\begin{align*}
\exp(k+1) & = \text{match } (k+1) \text{ with } 0 \rightarrow 1 \mid n \rightarrow 2 \times \exp(n-1) \\
& = 2 \times \exp(k+1 - 1) \\
& = 2 \times \exp(k) \\
& = 2 \times 2^k
\end{align*}
\]

(by eval $\exp$)

(by evaluating case)

(by math)

(by IH!)

let rec $\exp n =$ match $n$ with
| 0 -> 1 \\
| $n$ -> $2 \times \exp(n-1)$
Theorem: For all natural numbers \( n \),
\[
\exp(n) = 2^n.
\]

Recall: Every natural number \( n \) is either \( 0 \) or it is \( k+2 \) (where \( k \) is also a natural number). Hence, we follow the structure of the data and do our proof in two cases.

Proof:

Case: \( n = k+1 \):
\[
\begin{align*}
\exp(k+1) &= \text{match (k+1) with 0 -> 1 | n -> 2 * exp (n-1)} \\
&= 2 * \exp(k+1 - 1) \\
&= 2 * \exp(k) \\
&= 2 * 2^k \\
&= 2^{(k+1)}
\end{align*}
\] (by \text{IH!})

QED!
Another example

**Theorem:** For all natural numbers $n$,
\[
\text{even}(2*n) == \text{true}.
\]

**Recall:** Every natural number $n$ is either 0 or $k+1$, where $k$ is also a natural number.

Case: $n == 0$:

...  

Case: $n == k+1$:

...  

let rec even n =
match n with
| 0 -> true
| 1 -> false
| n -> even (n-2)
Another example

**Theorem:** For all natural numbers \( n \),
\[ \text{even}(2*n) == \text{true}. \]

**Recall:** Every natural number \( n \) is either 0 or \( k + 1 \), where \( k \) is also a natural number.

**Case:** \( n == 0 \):
\[ \text{even} (2*0) \]

```ocaml
let rec even n =
  match n with
  | 0 -> true
  | 1 -> false
  | n -> even(n-2)
```

==
Theorem: For all natural numbers n, even(2*n) == true.

Recall: Every natural number n is either 0 or k+1, where k is also a natural number.

Case: n == 0:
  even (2*0)
== even (0)
==

let rec even n = match n with
  | 0 -> true
  | 1 -> false
  | n -> even (n-2) (by math)
Theorem: For all natural numbers n,
\[ \text{even}(2\times n) = \text{true}. \]

Recall: Every natural number \( n \) is either 0 or \( k+1 \), where \( k \) is also a natural number.

Case: \( n = 0 \):
\[
\text{even} (2\times 0) \\
= \text{even} (0) \\
= \text{match} 0 \text{ of } (0 \rightarrow \text{true} | 1 \rightarrow \text{false} | n \rightarrow \text{even} (n-2)) \\
= \text{true}
\]

(by math) \quad (by eval even) \quad (by evaluation)
Another example

**Theorem:** For all natural numbers \( n \),
\[ \text{even}(2*n) == \text{true}. \]

**Recall:** Every natural number \( n \) is either \( 0 \) or \( k+1 \), where \( k \) is also a natural number.

**Case:** \( n == k+1: \)
\[ \text{even } (2*(k+1)) \]

```ocaml
let rec even n =
  match n with
  | 0 -> true
  | 1 -> false
  | n -> even (n-2)
```
Another example

Theorem: For all natural numbers $n$, 
even(2*n) == true.

Recall: Every natural number $n$ is either 0 or $k+1$, where $k$ is also a natural number.

Case: $n == k+1$:
  even (2*(k+1))
== even (2*k+2)  
== (by math)
Another example

Theorem: For all natural numbers n, even(2*n) == true.

Recall: Every natural number n is either 0 or k+1, where k is also a natural number.

Case: n == k+1:
   even (2*(k+1))
== even (2*k+2) (by math)
== match 2*k+2 with (0 -> true | 1 -> false | n -> even (n-2)) (by eval even)
== even ((2*k+2)-2) (by evaluation)
== even (2*k) (by math)
Another example

Theorem: For all natural numbers n, even(2*n) == true.

Recall: Every natural number n is either 0 or k+1, where k is also a natural number.

Case: n == k+1:
    even (2*(k+1)) == even (2*k+2) == match 2*k+2 with (0 -> true | 1 -> false | n -> even (n-2)) == even ((2*k+2)-2) == even (2*k) == true
QED.
Template for Inductive Proofs on Natural Numbers

**Theorem:** For all natural numbers \( n \), property of \( n \).

**Proof:** By induction on natural numbers \( n \).

- **Case:** \( n = 0 \):
  
  ...  

- **Case:** \( n = k+1 \):
  
  ...  

Cases must cover all natural numbers.

Proof methodology. Write this down.

Justifications to use:
- simple math
- eval, reverse eval, "by def"
- IH
Theorem: For all natural numbers n, property of n.

Proof: By induction on natural numbers n.

Case: \( n = 0 \):
...

Case: \( n = k+1 \):
...

Note there are other ways to cover all natural numbers:
• eg: case for 0, case for 1, case for \( k+2 \)
PROOFS ABOUT LIST-PROCESSORS
A Couple of Useful Functions

let rec length xs =
    match xs with
    | [] -> 0
    | x::xs -> 1 + length xs

let rec cat xs1 xs2 =
    match xs1 with
    | [] -> xs2
    | hd::tl -> hd :: cat tl xs2
**Proofs About Lists**

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length(cat } xs \ text{ ys)} = \text{length } xs + \text{length } ys$$

**Proof strategy:**

- **Proof by induction on the list $xs$**
  - recall, a list may be of these two things:
    - $[]$ (the empty list)
    - $\text{hd} :: \text{tl}$ (a non-empty list, where $\text{tl}$ is shorter)
  - a proof must cover both cases: $[]$ and $\text{hd} :: \text{tl}$
  - in the second case, you will often use the *inductive hypothesis* on the smaller list $\text{tl}$
  - otherwise as before:
    - use folding/eval of OCaml definitions
    - use your knowledge of OCaml evaluation
    - use lemmas/properties you know of basic operations like :: and +
Proofs About Lists

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length}(\text{cat} \, xs \, ys) = \text{length} \, xs + \text{length} \, ys$$

**Proof strategy:**

- **Proof by induction on the list $xs$? why not on the list $ys$?**
  - answering that question, may be the hardest part of the proof!
  - it tells you how to split up your cases
  - sometimes you just need to do some trial and error

```plaintext
let rec length xs =
  match xs with
  | [] -> 0
  | x::xs -> 1 + length xs

let rec cat xs1 xs2 =
  match xs1 with
  | [] -> xs2
  | x::xsl -> hd::tl -> hd :: cat tl xs2
```

---

**a clue:** pattern matching on first argument.

In the theorem: $\text{cat} \, xs \, ys$

Hence induction on $xs$. Case split the same way as the program.
Theorem: For all lists $xs$ and $ys$,
\[
\text{length(cat } xs \text{ ys)} = \text{length } xs + \text{length } ys
\]

Proof: By induction on $xs$.

\[
\text{case } xs = [ ]:
\]

\[
\begin{align*}
\text{let rec } & \text{ length } xs = \\
& \text{ match } xs \text{ with } \\
& \mid [] \rightarrow 0 \\
& \mid x::xs \rightarrow 1 + \text{length } xs \\
\end{align*}
\]

\[
\begin{align*}
\text{let rec } & \text{ cat } xs1 \text{ xs2 } = \\
& \text{ match } xs1 \text{ with } \\
& \mid [] \rightarrow xs2 \\
& \mid \text{hd}::\text{tl} \rightarrow \text{hd} :: \text{cat } \text{tl } xs2 \\
\end{align*}
\]
Proofs About Lists

Theorem: For all lists xs and ys,

\[ \text{length(cat xs ys)} = \text{length xs} + \text{length ys} \]

Proof: By induction on xs.

\[
\text{case } \text{xs} = [ ]: \\
\text{length (cat [ ] ys)} \quad \text{(LHS of theorem)}
\]

\[
\text{let rec length xs =} \\
\text{match xs with} \\
| [] -> 0 \\
| x::xs -> 1 + length xs
\]

\[
\text{let rec cat xs1 xs2 =} \\
\text{match xs1 with} \\
| [] -> xs2 \\
| hd::tl -> hd :: cat tl xs2
\]
Theorem: For all lists $xs$ and $ys$,
$$\text{length}(\text{cat } xs \; ys) = \text{length } xs + \text{length } ys$$

Proof: By induction on $xs$.

\begin{align*}
\text{case } xs = [ ]: \\
& \text{length (cat } [ ] \; ys) \quad (\text{LHS of theorem}) \\
& = \text{length } ys \quad (\text{evaluate cat})
\end{align*}

let rec length $xs$ = 
match $xs$ with 
| [] -> 0 
| $x$::$xs$ -> 1 + length $xs$

let rec cat $xs1$ $xs2$ = 
match $xs1$ with 
| [] -> $xs2$ 
| hd::tl -> hd :: cat tl $xs2$
Theorem: For all lists $xs$ and $ys$,
\[
\text{length}(\text{cat} \; xs \; ys) = \text{length} \; xs + \text{length} \; ys
\]

Proof: By induction on $xs$.

case $xs = []$:
\[
\begin{align*}
\text{length} \; (\text{cat} \; [] \; ys) &= \text{length} \; ys \quad \text{(evaluate cat)} \\
&= 0 \quad \text{(arithmetic)}
\end{align*}
\]

let rec \[\]
length \; xs = 
match xs with 
| [] -> 0 
| x::xs -> 1 + length xs
[/let]

let rec \[\]
cat \; xs1 \; xs2 = 
match xs1 with 
| [] -> xs2 
| hd::tl -> hd :: cat \; tl \; xs2
[/let]
Proofs About Lists

Theorem: For all lists xs and ys,
\[ \text{length}(\text{cat } xs \ ys) = \text{length } xs + \text{length } ys \]

Proof: By induction on xs.

case \( xs = [] \):
  \[ \text{length } (\text{cat } [] \ ys) = \text{length } ys \]
  \[ = 0 + (\text{length } ys) \]
  \[ = (\text{length } []) + (\text{length } ys) \]

case done!

let rec length xs =
  match xs with
  | [] -> 0
  | x::xs -> 1 + length xs

let rec cat xs1 xs2 =
  match xs1 with
  | [] -> xs2
  | hd::tl -> hd :: cat tl xs2
Proofs About Lists

Theorem: For all lists xs and ys,

\[ \text{length}(\text{cat} \; \text{xs} \; \text{ys}) = \text{length} \; \text{xs} + \text{length} \; \text{ys} \]

Proof: By induction on xs.

\[
\text{case} \; \text{xs} = \text{hd}::\text{tl}
\]

let rec length xs =
  \text{match} \; \text{xs} \; \text{with}\n  | [] -> 0
  | x::xs -> 1 + length xs

let rec \text{cat} \; \text{xs1} \; \text{xs2} =
  \text{match} \; \text{xs1} \; \text{with}\n  | [] -> \text{xs2}
  | hd::tl -> hd :: \text{cat} \; \text{tl} \; \text{xs2}
**Proofs About Lists**

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length} (\text{cat} \; xs \; ys) = \text{length} \; xs + \text{length} \; ys$$

**Proof:** By induction on $xs$.

Case $xs = \text{hd}::\text{tl}$

IH: $\text{length} (\text{cat} \; \text{tl} \; ys) = \text{length} \; \text{tl} + \text{length} \; ys$

```plaintext
let rec length xs = 
  match xs with 
  | [] -> 0 
  | x::xs -> 1 + length xs

let rec cat xs1 xs2 = 
  match xs1 with 
  | [] -> xs2 
  | hd::tl -> hd :: cat tl xs2
```
Proofs About Lists

Theorem: For all lists xs and ys,

\[ \text{length}(\text{cat } xs \ ys) = \text{length } xs + \text{length } ys \]

Proof: By induction on xs.

case xs = hd::tl

IH: length (cat tl ys) = length tl + length ys

\[ \text{length (cat (hd::tl) ys)} \] (LHS of theorem)

==

let rec length xs =
    match xs with
    | [] -> 0
    | x::xs -> 1 + length xs

let rec cat xs1 xs2 =
    match xs1 with
    | [] -> xs2
    | hd::tl -> hd :: cat tl xs2
Proofs About Lists

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length}(\text{cat } x_0 y) = \text{length } x_0 + \text{length } y$$

**Proof:** By induction on $xs$.

\[
\begin{align*}
\text{case } xs &= \text{hd}::\text{tl} \\
\text{IH} &\quad \text{length } (\text{cat} \ text{tl} y_0) = \text{length } \text{tl} + \text{length } y_0
\end{align*}
\]

\[
\begin{align*}
\text{length } (\text{cat } (\text{hd}::\text{tl}) y_0) &= \text{(LHS of theorem)} \\
\text{== length } (\text{hd} :: (\text{cat } \text{tl} y_0)) &= \text{(evaluate cat, take 2}^{\text{nd}} \text{ branch)}
\end{align*}
\]

\[
\begin{align*}
\text{let rec length } x_0 &= \\
\text{match } x_0 \text { with} \\
\text{| } [] \rightarrow 0 \\
\text{| } x::x_0 \rightarrow 1 + \text{length } x_0
\end{align*}
\]

\[
\begin{align*}
\text{let rec cat } x_0 x_1 x_2 &= \\
\text{match } x_0 \text { with} \\
\text{| } [] \rightarrow x_2 \\
\text{| } \text{hd}::\text{tl} \rightarrow \text{hd} :: \text{cat } x_1 \text{tl } x_2
\end{align*}
\]
Proofs About Lists

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length}(\text{cat } xs \text{ ys}) = \text{length } xs + \text{length } ys$$

**Proof:** By induction on $xs$.

 case $xs = \text{hd}::\text{tl}$

 IH: $\text{length } (\text{cat } \text{tl } ys) = \text{length } \text{tl } + \text{length } ys$

$$\text{length } (\text{cat } (\text{hd}::\text{tl}) \text{ ys}) \quad \text{(LHS of theorem)}$$

$$== \text{length } (\text{hd} :: (\text{cat } \text{tl } ys)) \quad \text{(evaluate cat, take 2\textsuperscript{nd} branch)}$$

$$== 1 + \text{length } (\text{cat } \text{tl } ys) \quad \text{(evaluate length, take 2\textsuperscript{nd} branch)}$$

$$==$$

let rec length xs =
    match xs with
    | [] -> 0
    | x::xs -> 1 + length xs

let rec cat xs1 xs2 =
    match xs1 with
    | [] -> xs2
    | hd::tl -> hd :: cat tl xs2
Proofs About Lists

**Theorem:** For all lists \( xs \) and \( ys \),

\[
\text{length(cat } xs \text{ } ys) = \text{length } xs + \text{length } ys
\]

**Proof:** By induction on \( xs \).

```plaintext
case xs = hd::tl
  IH: length (cat tl ys) = length tl + length ys

  length (cat (hd::tl) ys)   (LHS of theorem)
  == length (hd :: (cat tl ys))  (evaluate cat, take 2\text{nd} branch)
  == 1 + length (cat tl ys)  (evaluate length, take 2\text{nd} branch)
  == 1 + (length tl + length ys)  (by IH)
  ==
```

```plaintext
let rec length xs =
  match xs with
  | [] -> 0
  | x::xs -> 1 + length xs
let rec cat xs1 xs2 =
  match xs1 with
  | [] -> xs2
  | hd::tl -> hd :: cat tl xs2
```
Proofs About Lists

**Theorem:** For all lists $xs$ and $ys$,

$$\text{length}(\text{cat}\ xs\ ys) = \text{length}\ xs + \text{length}\ ys$$

**Proof:** By induction on $xs$.

**case** $xs = \text{hd}::\text{tl}$

IH: $\text{length}\ (\text{cat}\ \text{tl}\ ys) = \text{length}\ \text{tl} + \text{length}\ ys$

$$\text{length}\ (\text{cat}\ (\text{hd}::\text{tl})\ ys)$$

(LHS of theorem)

$=\text{length}\ (\text{hd} :: (\text{cat}\ \text{tl}\ ys))$  
(evaluate cat, take 2\textsuperscript{nd} branch)

$=1 + \text{length}\ (\text{cat}\ \text{tl}\ ys)$  
(evaluate length, take 2\textsuperscript{nd} branch)

$=1 + (\text{length}\ \text{tl} + \text{length}\ ys)$  
(by IH)

$=\text{length}\ (\text{hd}::\text{tl}) + \text{length}\ ys$  
(reparenthesizing and evaling length in reverse
we have RHS with hd::tl for $xs$)

**case done!**

let rec length $xs$ =
match $xs$ with
| [] -> 0
| x::xs -> 1 + length xs

let rec cat $xs1$ $xs2$ =
match $xs1$ with
| [] -> $xs2$
| hd::tl -> hd :: cat tl $xs2$
Be careful with the Induction Hypothesis!

**Theorem:** For all lists \( xs \) and \( ys \),

\[
\text{length}(\text{cat } xs \ ys) = \text{length } xs + \text{length } ys
\]

**Proof:** By induction on \( xs \).

- case \( xs = \text{hd} :: \text{tl} \)
  - IH: \( \text{length } (\text{cat } \text{tl } ys) = \text{length } \text{tl} + \text{length } ys \)
  
  \[
  \text{length } (\text{cat } (\text{hd} :: \text{tl}) \ ys)
  \]
  
  \[
  = \text{length } (\text{hd} :: (\text{cat } \text{tl } ys))
  \]
  
  \[
  = 1 + \text{length } (\text{cat } \text{tl } ys)
  \]
  
  \[
  = 1 + (\text{length } \text{tl} + \text{length } ys)
  \]
  
  (by IH)
  
  \[
  = \text{length } (\text{hd} :: \text{tl}) + \text{length } ys
  \]
  
  (reparenthesizing and evaling length in reverse)

- case done!

Induction hypothesis is a function of one variable (in this case, \( xs \))

The use of the IH must be at a smaller value (in this case, “\( \text{tl} \)” is smaller than “\( xs \)”)

In your proofs, it should be really obvious
- which variable the IH is supposed to be a function of
- that your induction is on that variable
- that you’re applying the IH at smaller values

If you’re not sure it’s obvious, just say explicitly in your proof: which variable it is, and why you claim you’re applying it at smaller values.
Theorem: For all lists \( xs \) and \( ys \),

\[
\text{length(cat } xs \text{ } ys) = \text{length } xs + \text{length } ys
\]

Proof: By induction on \( xs \).

Induction hypothesis is a function of one variable (in this case, \( xs \)).

In more complicated proofs, the induction hypothesis is a function of one structure where the ordering of elements in the structure is well-founded (there are no infinite descending chains). Eg, we could do induction on pairs of naturals \((x, y)\) where pairs are ordered lexicographically. ie:

\[
(x1, y1) > (x2, y2)
\]

iff \( x1 > x2 \) or \((x1 = x2 \text{ and } y1 > y2)\)
Theorem: For all lists \(xs\),

\[
\text{add\_all}\ (\text{add\_all}\ \text{xs}\ a)\ b\ =\ =\ \text{add\_all}\ \text{xs}\ (a+b)
\]
Another List example

**Theorem:** For all lists $xs$,

$$
\text{add\_all}\ (\text{add\_all}\ xs\ a)\ b = a + b
$$

**Proof:** By induction on $xs$.

```ocaml
let rec add_all xs c =
  match xs with
  | [] -> []
  | hd::tl -> (hd+c)::add_all tl c
```
Another List example

**Theorem:** For all lists $xs$, 
\[
add\_all\ (add\_all\ xs\ a)\ b\ ==\ add\_all\ xs\ (a+b)
\]

**Proof:** By induction on $xs$.

case $xs = [\ ]$:

\[
add\_all\ (add\_all\ [\ ]\ a)\ b
\]

\[
==
\]

let rec add\_all\ xs\ c =
  match\ xs\ with
  | [ ]\ -> [ ]
  | hd::tl\ ->\ (hd+c)::add\_all\ tl\ c
Another List example

**Theorem:** For all lists xs,
\[
\text{add\_all (add\_all xs a) b == add\_all xs (a+b)}
\]

**Proof:** By induction on xs.

```
case xs = []:
    add_all (add_all [] a) b  (LHS of theorem)
== add_all [] b            (by evaluation of add\_all)
==
```

```
let rec add_all xs c =
    match xs with
    | []      -> []
    | hd::tl  -> (hd+c)::add_all tl c
```
Another List example

Theorem: For all lists $xs$, 
\[ \text{add\_all} \ (\text{add\_all} \ xs \ a) \ b \ =\ = \ \text{add\_all} \ xs \ (a+b) \]

Proof: By induction on $xs$.

case $xs = [ ]$:

\[
\begin{align*}
\text{add\_all} \ (\text{add\_all} \ [ ] \ a) \ b & \quad \text{(LHS of theorem)} \\
= \text{add\_all} \ [ ] \ b & \quad \text{(by evaluation of add\_all)} \\
= [ ] & \quad \text{(by evaluation of add\_all)} \\
\end{align*}
\]
**Theorem:** For all lists \( xs \),

\[
\text{add\_all} (\text{add\_all} \ xs \ a) \ b =\equiv \text{add\_all} \ xs \ (a+b)
\]

**Proof:** By induction on \( xs \).

\[
\text{case } xs = [ ]:\n\]

\[
\text{add\_all} (\text{add\_all} [ ] a) \ b =\equiv \text{add\_all} [ ] b =\equiv [ ] =\equiv \text{add\_all} [ ] (a + b)
\]

(by evaluation of \text{add\_all})

(by evaluation of \text{add\_all})

(by evaluation of \text{add\_all})

---

```
let rec add_all xs c =
    match xs with
    | [ ] -> [ ]
    | hd::tl -> (hd+c)::add_all tl c
```
Another List example

**Theorem:** For all lists \( xs \),

\[
\text{add\_all} \ (\text{add\_all} \ xs \ a) \ b \ \text{==} \ \text{add\_all} \ xs \ (a+b)
\]

**Proof:** By induction on \( xs \).

\[
\text{case } xs = \text{hd} :: \text{tl}:
\]

\[
\text{add\_all} \ (\text{add\_all} \ (\text{hd} :: \text{tl}) \ a) \ b \ \text{==} \ \text{(LHS of theorem)}
\]

\[
\text{let rec add\_all} \ xs \ c = \text{match} \ xs \ \text{with}
| [ ] \rightarrow [ ]
| \text{hd}::\text{tl} \rightarrow (\text{hd}+c)::\text{add\_all} \ \text{tl} \ c
\]
Another List example

Theorem: For all lists xs,

\[ \text{add\_all}\ (\text{add\_all}\ \text{xs}\ a)\ b \equiv \text{add\_all}\ \text{xs}\ (a+b) \]

Proof: By induction on xs.

case \(\text{xs} = \text{hd} :: \text{tl}:\)

\[
\text{add\_all}\ (\text{add\_all}\ (\text{hd} :: \text{tl})\ a)\ b \\
\equiv \text{add\_all}\ ((\text{hd}+a) :: \text{add\_all}\ \text{tl}\ a)\ b \\
\equiv
\]

let rec add_all xs c =
match xs with
| [ ] -> [ ]
| hd::tl -> (hd+c)::add_all tl c
Another List example

Theorem: For all lists \(xs\),
\[
\text{add\_all} \ (\text{add\_all} \ xs \ a) \ b \ =\ = \ \text{add\_all} \ xs \ (a+b)
\]

Proof: By induction on \(xs\).

case \(xs = \text{hd} :: \text{tl}\):

\[
\begin{align*}
\text{add\_all} \ (\text{add\_all} \ (\text{hd} :: \text{tl}) \ a) \ b \ &= \ \text{add\_all} \ ((\text{hd}+a) :: \text{add\_all} \ \text{tl} \ a) \ b \quad \text{(LHS of theorem)}\\
&= \ (\text{hd}+a+b) :: (\text{add\_all} \ (\text{add\_all} \ \text{tl} \ a) \ b) \quad \text{(by eval inner add\_all)}\\
&= \quad \text{(by eval outer add\_all)}
\end{align*}
\]

let rec add_all xs c =
match xs with
| [] -> []
| hd::tl -> (hd+c)::add_all tl c
Theorem: For all lists \(xs\),

\[
\text{add\_all}\ (\text{add\_all}\ xs\ a)\ b\ ==\ \text{add\_all}\ xs\ (a+b)
\]

Proof: By induction on \(xs\).

case \(xs = \text{hd} :: \text{tl}\):

\[
\begin{align*}
\text{add\_all}\ (\text{add\_all}\ (\text{hd} :: \text{tl})\ a)\ b & \quad \text{(LHS of theorem)} \\
==\ \text{add\_all}\ ((\text{hd}+a) :: \text{add\_all}\ \text{tl}\ a)\ b & \quad \text{(by eval inner add\_all)} \\
==\ (\text{hd}+a+b) :: (\text{add\_all}\ (\text{add\_all}\ \text{tl}\ a)\ b) & \quad \text{(by eval outer add\_all)} \\
==\ (\text{hd}+a+b) :: \text{add\_all}\ \text{tl}\ (a+b) & \quad \text{(by IH)}
\end{align*}
\]

let rec add_all xs c =
match xs with
  | [ ] -> [ ]
  | hd::tl -> (hd+c)::add_all tl c
Theorem: For all lists xs,

\[ \text{add\_all} \left( \text{add\_all} \; \text{xs} \; a \right) \; b \; == \; \text{add\_all} \; \text{xs} \; (a+b) \]

Proof: By induction on xs.

case \( \text{xs} = \text{hd} :: \text{tl} \):

\[
\begin{align*}
\text{add\_all} \left( \text{add\_all} \; (\text{hd} :: \text{tl}) \; a \right) \; b & \quad \text{(LHS of theorem)} \\
== \text{add\_all} \; ((\text{hd}+a) :: \text{add\_all} \; \text{tl} \; a) \; b & \quad \text{(by eval inner add\_all)} \\
== (\text{hd}+a+b) :: \text{add\_all} \; (\text{add\_all} \; \text{tl} \; a) \; b & \quad \text{(by eval outer add\_all)} \\
== (\text{hd}+a+b) :: \text{add\_all} \; \text{tl} \; (a+b) & \quad \text{(by IH)} \\
== (\text{hd}+(a+b)) :: \text{add\_all} \; \text{tl} \; (a+b) & \quad \text{(associativity of + )}
\end{align*}
\]

let rec add_all xs c =
  match xs with
  | [] -> []
  | hd::tl -> (hd+c)::add_all tl c
Another List example

Theorem: For all lists \( xs \),
\[
\text{add\_all (add\_all \, xs \, a) \, b} \equiv \text{add\_all \, xs \, (a+b)}
\]

Proof: By induction on \( xs \).

case \( xs = hd :: tl \):

\[
\begin{align*}
\text{add\_all (add\_all (hd :: tl) a) \, b} & \quad \text{(LHS of theorem)} \\
\equiv & \text{add\_all ((hd+a) :: add\_all \, tl \, a) \, b} \quad \text{(by eval inner add\_all)} \\
\equiv & (hd+a+b) :: (add\_all (add\_all \, tl \, a) \, b) \quad \text{(by eval outer add\_all)} \\
\equiv & (hd+a+b) :: add\_all \, tl \, (a+b) \quad \text{(by IH)} \\
\equiv & (hd+(a+b)) :: add\_all \, tl \, (a+b) \quad \text{(associativity of + )} \\
\equiv & \text{add\_all (hd::tl) \, (a+b)} \quad \text{(by (reverse) eval of add\_all)}
\end{align*}
\]

let rec add_all xs c =
match xs with
| [ ] -> [ ]
| hd::tl -> (hd+c)::add_all tl c
Template for Inductive Proofs on Lists

**Theorem:** For all lists $xs$, property of $xs$.

**Proof:** By induction on lists $xs$.

- Case: $xs \equiv [ ]$
  
  \[ ... \]

- Case: $xs \equiv \text{hd} :: \text{tl}$
  
  \[ ... \]

Note there are other ways to cover all lists:

- eg: case for $[]$, case for $x1::[]$, case for $x1::x2::\text{tl}$
Template for Inductive Proofs on *any datatype*

type ty = A of ... | B of ... | C of ... | D ;;

**Theorem:** For all ty \( x \), property of \( x \).

**Proof:** By induction on the constructors of ty.

Case: \( x == A(...) \):
    ...
Case: \( x == B(...) \):
    ...
Case: \( x == C(...) \):
    ...
Case: \( x == D \):
    ...

Cases must cover all the constructors of the datatype
SUMMARY
Proofs about programs are structured similarly to the programs:
- types tell you the kinds of values your proofs/programs operate over
- types suggest how to break down proofs/programs in to cases
- when programs that use recursion on smaller values, their proofs appeal to the inductive hypothesis on smaller values

Key proof ideas:
- two expressions that evaluate to the same value are equal
- substitute equals for equals
- use calculation (evaluation) to reason about simple equalities
- use well-established axioms about primitives (+, -, %, etc)
- use proof by induction to prove correctness of recursive functions