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# 5.1 String Sorts

- *strings in Java*
- *key-indexed counting*
- *LSD radix sort*
- *MSD radix sort*
- *3-way radix quicksort*
- *suffix arrays*


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# 5.1 String Sorts

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# String processing

String. Sequence of characters.

Important fundamental abstraction.

- Programming systems (e.g., Java programs).
- Communication systems (e.g., email).
- Information processing.
- Genomic sequences.
- ...

*" The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. "* — *M. V. Olson*

# The char data type

C char data type. Typically an 8-bit integer.

- Supports 7-bit ASCII.
- Can represent at most 256 characters.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | NUL | SOH | STX | ETX | EOT | ENQ | ACK | BEL | BS | HT | LF | VT | FF | CR | SO | SI |
| 1 | DLE | DC1 | DC2 | DC3 | DC4 | NAK | SYN | ETB | CAN | EM | SUB | ESC | FS | GS | RS | US |
| 2 | SP | ! | " | # | $ | % | & | ' | ( | ) | * | + | , | - | . | / |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | : | ; | < | = | > | ? |
| 4 | @ | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 5 | P | Q | R | S | T | U | V | W | X | Y | Z | [ | \ | ] | ^ | _ |
| 6 | ` | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o |
| 7 | p | q | r | s | t | u | v | w | x | y | z | { | \| | } | ~ | DEL |

all 128 ASCII characters

| A | á | ∂ | 💩 |
|---|---|---|---|
| U+0041 | U+00E1 | U+2202 | U+1F4A9 |

some Unicode characters

Java char data type. A 16-bit unsigned integer.

- Supports 16-bit Unicode 1.0.
- Supports 21-bit Unicode 8.0 (awkwardly). ← 120,737 characters and emoji

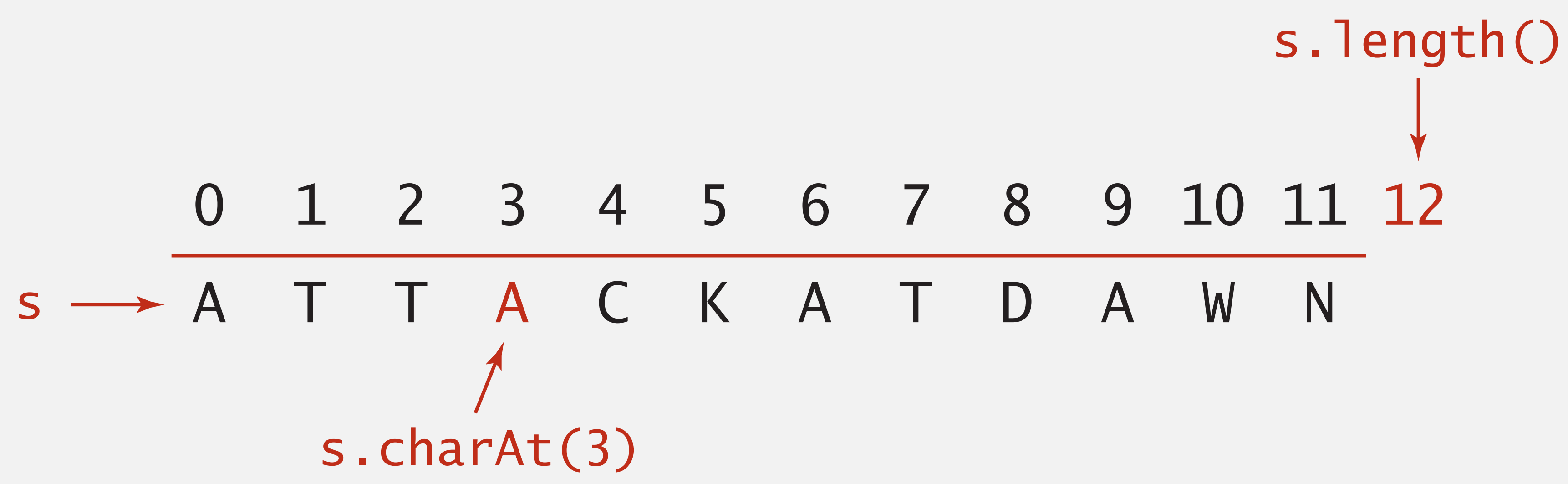
# The String data type

String data type in Java. Immutable sequence of characters.

Length. Number of characters.

Indexing. Get the $i^{\text{th}}$ character.

Concatenation. Concatenate one string to the end of another.



# The String data type: representation

Representation (Java 7 and 8). Immutable `char[]` array + cache of hash.

| operation | Java | running time |
|---|---|---|
| **length** | `s.length()` | $1$ |
| **indexing** | `s.charAt(i)` | $1$ |
| **concatenation** | `s + t` | $m + n$ |
| ⋮ | ⋮ | ⋮ |

# String performance trap

Q. How to build a long string, one character at a time?

```
public static String reverse(String s)
{
   String reverse = "";
   for (int i = s.length() - 1; i >= 0; i--)
      rev += s.charAt(i);
   return reverse;
}
```

quadratic time
$(1 + 2 + 3 + \ldots + n)$

A. Use `StringBuilder` data type (mutable `char[]` resizing array).

```
public static String reverse(String s)
{
   StringBuilder reverse = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      reverse.append(s.charAt(i));
   return reverse.toString();
}
```

linear time

# THE STRING DATA TYPE: IMMUTABILITY

Q. Why are Java strings immutable?

# Alphabets

Digital key. Sequence of digits over fixed alphabet.

Radix. Number of digits $R$ in alphabet.

| name | R() | lgR() | characters |
|---|---|---|---|
| BINARY | 2 | 1 | 01 |
| OCTAL | 8 | 3 | 01234567 |
| DECIMAL | 10 | 4 | 0123456789 |
| HEXADECIMAL | 16 | 4 | 0123456789ABCDEF |
| DNA | 4 | 2 | ACTG |
| LOWERCASE | 26 | 5 | abcdefghijklmnopqrstuvwxyz |
| UPPERCASE | 26 | 5 | ABCDEFGHIJKLMNOPQRSTUVWXYZ |
| PROTEIN | 20 | 5 | ACDEFGHIKLMNPQRSTVWY |
| BASE64 | 64 | 6 | ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef ghijklmnopqrstuvwxyz0123456789+/ |
| ASCII | 128 | 7 | *ASCII characters* |
| EXTENDED_ASCII | 256 | 8 | *extended ASCII characters* |
| UNICODE16 | 65536 | 16 | *Unicode characters* |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# 5.1 String Sorts

- strings in Java
- **key-indexed counting**
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

# Program optimization

" *The first rule of program optimization: don't do it.*
*The second rule of program optimization (for experts only!): don't do it yet.* " – *Michael A. Jackson*

# Review: summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| **insertion sort** | ½ $n^2$ | ¼ $n^2$ | 1 | ✔ | `compareTo()` |
| **mergesort** | $n \lg n$ | $n \lg n$ | $n$ | ✔ | `compareTo()` |
| **quicksort** | $1.39\, n \lg n$ * | $1.39\, n \lg n$ | $c \lg n$ * | | `compareTo()` |
| **heapsort** | $2\, n \lg n$ | $2\, n \lg n$ | 1 | | `compareTo()` |

* probabilistic

`compareTo()` not constant time for strings

Lower bound. $\sim n \lg n$ compares required by any compare-based algorithm.

Q. Can we sort strings faster (despite lower bound)?

A. Yes, by exploiting access to individual characters.

use characters to make R-way decisions (instead of binary decisions)

# Key-indexed counting: assumptions about keys

Assumption. Keys are integers between $0$ and $R - 1$.

Implication. Can use key as an array index.

Applications.

- Sort string by first letter.
- Sort class roster by section.
- Sort phone numbers by area code.
- Subroutine in a string sorting algorithm.

Remark. Keys may have associated data $\Rightarrow$ can't simply count keys of each value.

| input *name* | *section* | sorted result *(by section)* | |
|---|---|---|---|
| Anderson | 2 | Harris | 1 |
| Brown | 3 | Martin | 1 |
| Davis | 3 | Moore | 1 |
| Garcia | 4 | Anderson | 2 |
| Harris | 1 | Martinez | 2 |
| Jackson | 3 | Miller | 2 |
| Johnson | 4 | Robinson | 2 |
| Jones | 3 | White | 2 |
| Martin | 1 | Brown | 3 |
| Martinez | 2 | Davis | 3 |
| Miller | 2 | Jackson | 3 |
| Moore | 1 | Jones | 3 |
| Robinson | 2 | Taylor | 3 |
| Smith | 4 | Williams | 3 |
| Taylor | 3 | Garcia | 4 |
| Thomas | 4 | Johnson | 4 |
| Thompson | 4 | Smith | 4 |
| White | 2 | Thomas | 4 |
| Williams | 3 | Thompson | 4 |
| Wilson | 4 | Wilson | 4 |

↑ *keys are small integers*

# Key-indexed counting demo

Goal. Sort an array a[] of $n$ integers between $0$ and $R - 1$.

R = 6

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int n = a.length;
int[] count = new int[R+1];

for (int i = 0; i < n; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < n; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < n; i++)
   a[i] = aux[i];
```

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

use a for 0
b for 1
c for 2
d for 3
e for 4
f for 5

# Key-indexed counting demo

Goal. Sort an array `a[]` of $n$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

count frequencies →

```
int n = a.length;
int[] count = new int[R+1];

for (int i = 0; i < n; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < n; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < n; i++)
   a[i] = aux[i];
```

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

offset by 1
[stay tuned]

| r | count[r] |
|---|---|
| a | 0 |
| b | 2 |
| c | 3 |
| d | 1 |
| e | 2 |
| f | 1 |
| - | 3 |

# Key-indexed counting demo

Goal. Sort an array a[] of $n$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

compute cumulates →

```
int n = a.length;
int[] count = new int[R+1];

for (int i = 0; i < n; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < n; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < n; i++)
   a[i] = aux[i];
```

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

| r | count[r] |
|---|---|
| a | 0 |
| b | 2 |
| c | 5 |
| d | 6 |
| e | 8 |
| f | 9 |
| - | 12 |

6 keys < d, 8 keys < e
so d's go in a[6] and a[7]

# Key-indexed counting demo

Goal. Sort an array `a[]` of $n$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int n = a.length;
int[] count = new int[R+1];

for (int i = 0; i < n; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < n; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < n; i++)
   a[i] = aux[i];
```

move items → `aux[count[a[i]]++] = a[i];`

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

| r | count[r] |
|---|---|
| a | 2 |
| b | 5 |
| c | 6 |
| d | 8 |
| e | 9 |
| f | 12 |
| - | 12 |

| i | aux[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

# Key-indexed counting demo

Goal. Sort an array a[] of $n$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int n = a.length;
int[] count = new int[R+1];

for (int i = 0; i < n; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < n; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < n; i++)
   a[i] = aux[i];
```

copy back

| i | a[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

| r | count[r] |
|---|---|
| a | 2 |
| b | 5 |
| c | 6 |
| d | 8 |
| e | 9 |
| f | 12 |
| - | 12 |

| i | aux[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

# Radix sorting: quiz 1

**Which of the following are properties of key-indexed counting?**

A. Running time proportional to $n + R$.

B. Extra space proportional to $n + R$.

C. Stable.

D. All of the above.


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- *suffix arrays*

# Least-significant-digit-first (LSD) radix sort

- Consider characters from right to left.
- Stably sort using $d^{\text{th}}$ character as the key (using key-indexed counting).

| | | sort key (d = 2) | sort key (d = 1) | sort key (d = 0) |
|---|---|---|---|---|
| 0 | d a b | d a b | d a b | a c e |
| 1 | a d d | c a b | c a b | a d d |
| 2 | c a b | e b b | f a d | b a d |
| 3 | f a d | a d d | b a d | b e d |
| 4 | f e e | f a d | d a d | b e e |
| 5 | b a d | b a d | e b b | c a b |
| 6 | d a d | d a d | a c e | d a b |
| 7 | b e e | f e d | a d d | d a d |
| 8 | f e d | b e d | f e d | e b b |
| 9 | b e d | f e e | b e d | f a d |
| 10 | e b b | b e e | f e e | f e d |
| 11 | a c e | a c e | b e e | f e e |

sort is stable
(arrows do not cross)

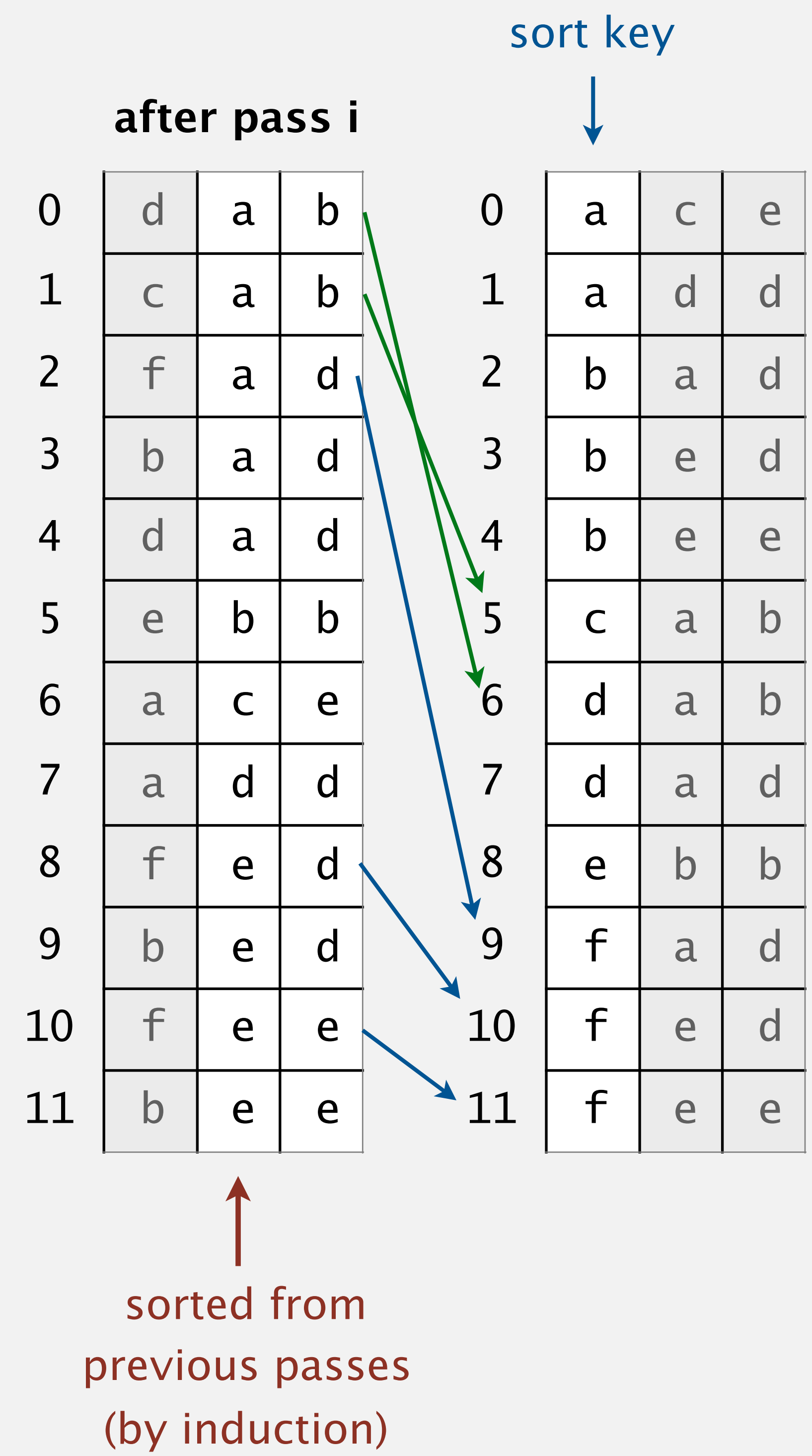
# LSD string sort: correctness proof

**Proposition.** LSD sorts fixed-length strings in ascending order.

**Pf.** [ by induction on $i$ ]

After pass $i$, strings are sorted by last $i$ characters.

- If two strings differ on sort key, key-indexed counting puts them in proper relative order.
- If two strings agree on sort key, stability of key-indexed counting keeps them in proper relative order.




**Proposition.** LSD sort is stable.

**Pf.** Key-indexed counting is stable.

# LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256;
      int n = a.length;
      String[] aux = new String[n];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < n; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < n; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < n; i++)
            a[i] = aux[i];

      }
   }
}
```

- fixed-length W strings (annotates `public static void sort(String[] a, int W)`)
- radix R (annotates `int R = 256;`)
- do key-indexed counting for each digit from right to left (annotates `for (int d = W-1; d >= 0; d--)`)
- key-indexed counting (annotates the inner block)

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| **insertion sort** | $\frac{1}{2}\, n^2$ | $\frac{1}{4}\, n^2$ | $1$ | ✔ | `compareTo()` |
| **mergesort** | $n \lg n$ | $n \lg n$ | $n$ | ✔ | `compareTo()` |
| **quicksort** | $1.39\, n \lg n$ * | $1.39\, n \lg n$ | $c \lg n$ | | `compareTo()` |
| **heapsort** | $2\, n \lg n$ | $2\, n \lg n$ | $1$ | | `compareTo()` |
| **LSD sort** † | $2\, W (n + R)$ | $2\, W (n + R)$ | $n + R$ | ✔ | `charAt()` |

* probabilistic
† fixed-length W keys

1 call to `compareTo()` can involve as many as $W$ calls to `charAt()`

Q. What if strings are not all of same length $W$?

# Radix sorting: quiz 2

**Which sorting method to use to sort 1 million 32-bit integers?**

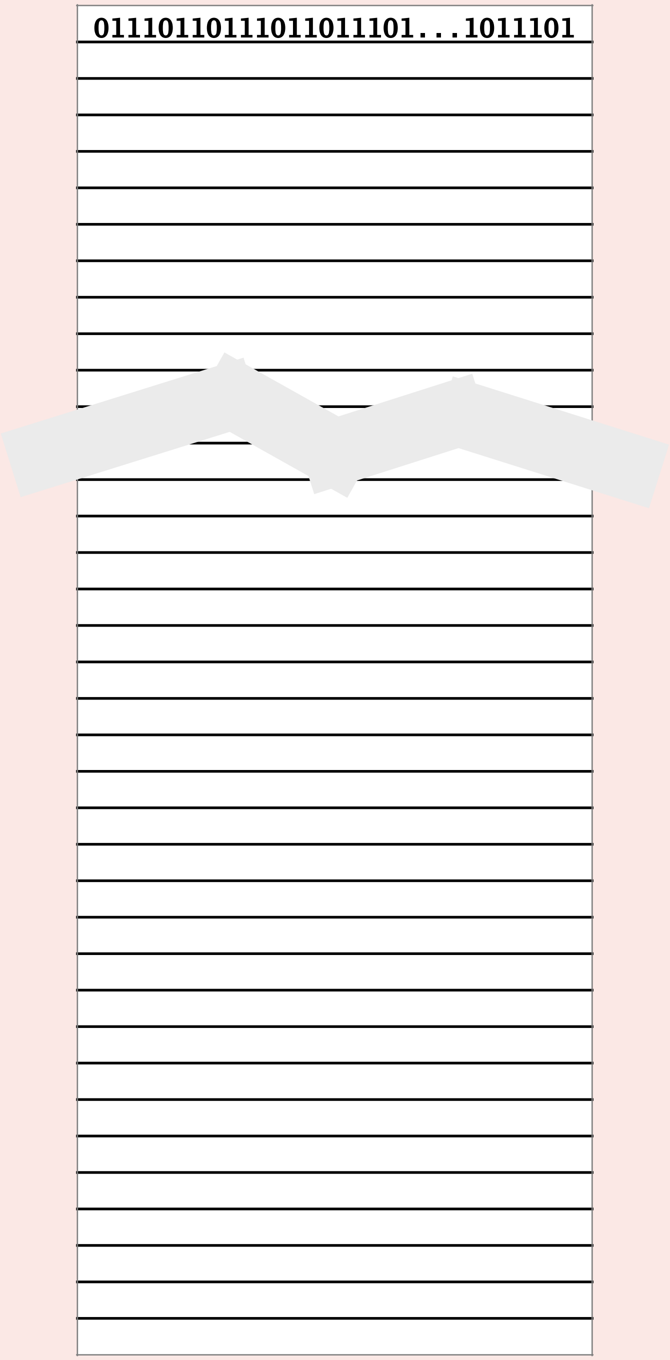
**A.** Insertion sort.

**B.** Mergesort.

**C.** Quicksort.

**D.** Heapsort.

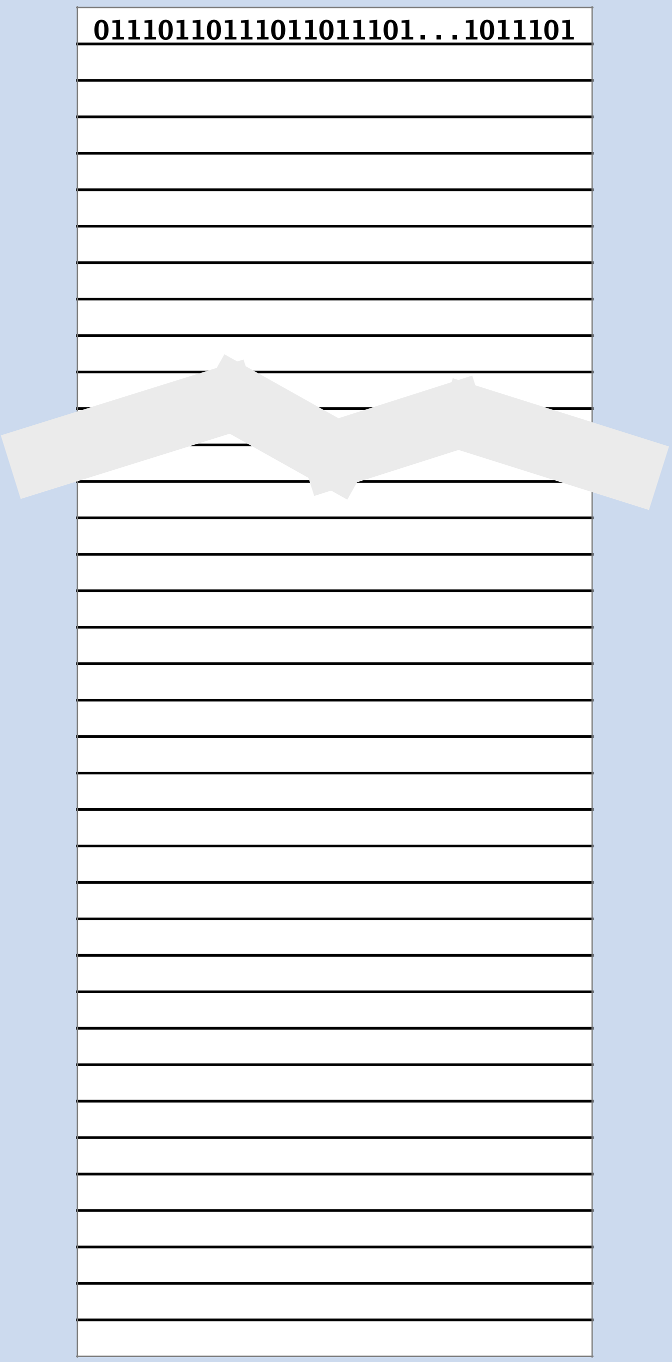
**E.** LSD radix sort.



# Sort Array of 128-Bit Numbers

Problem. Sort huge array of random 128-bit numbers.

Ex. Supercomputer sort, internet router.




Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD radix sort.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# 5.1 STRING SORTS

- *strings in Java*
- *key-indexed counting*
- *LSD radix sort*
- ***MSD radix sort***
- *3-way radix quicksort*
- *suffix arrays*

# Reverse LSD

- Consider characters from left to right.
- Stably sort using $d^{\text{th}}$ character as the key (using key-indexed counting).

| | | | |
|---|---|---|---|
| 0 | d | a | b |
| 1 | a | d | d |
| 2 | c | a | b |
| 3 | f | a | d |
| 4 | f | e | e |
| 5 | b | a | d |
| 6 | d | a | d |
| 7 | b | e | e |
| 8 | f | e | d |
| 9 | b | e | d |
| 10 | e | b | b |
| 11 | a | c | e |

sort key (d = 0)

| | | | |
|---|---|---|---|
| 0 | a | d | d |
| 1 | a | c | e |
| 2 | b | a | d |
| 3 | b | e | e |
| 4 | b | e | d |
| 5 | c | a | b |
| 6 | d | a | b |
| 7 | d | a | d |
| 8 | e | b | b |
| 9 | f | a | d |
| 10 | f | e | e |
| 11 | f | e | d |

sort key (d = 1)

| | | | |
|---|---|---|---|
| 0 | b | a | d |
| 1 | c | a | b |
| 2 | d | a | b |
| 3 | d | a | d |
| 4 | f | a | d |
| 5 | e | b | b |
| 6 | a | c | e |
| 7 | a | d | d |
| 8 | b | e | e |
| 9 | b | e | d |
| 10 | f | e | e |
| 11 | f | e | d |

sort key (d = 2)

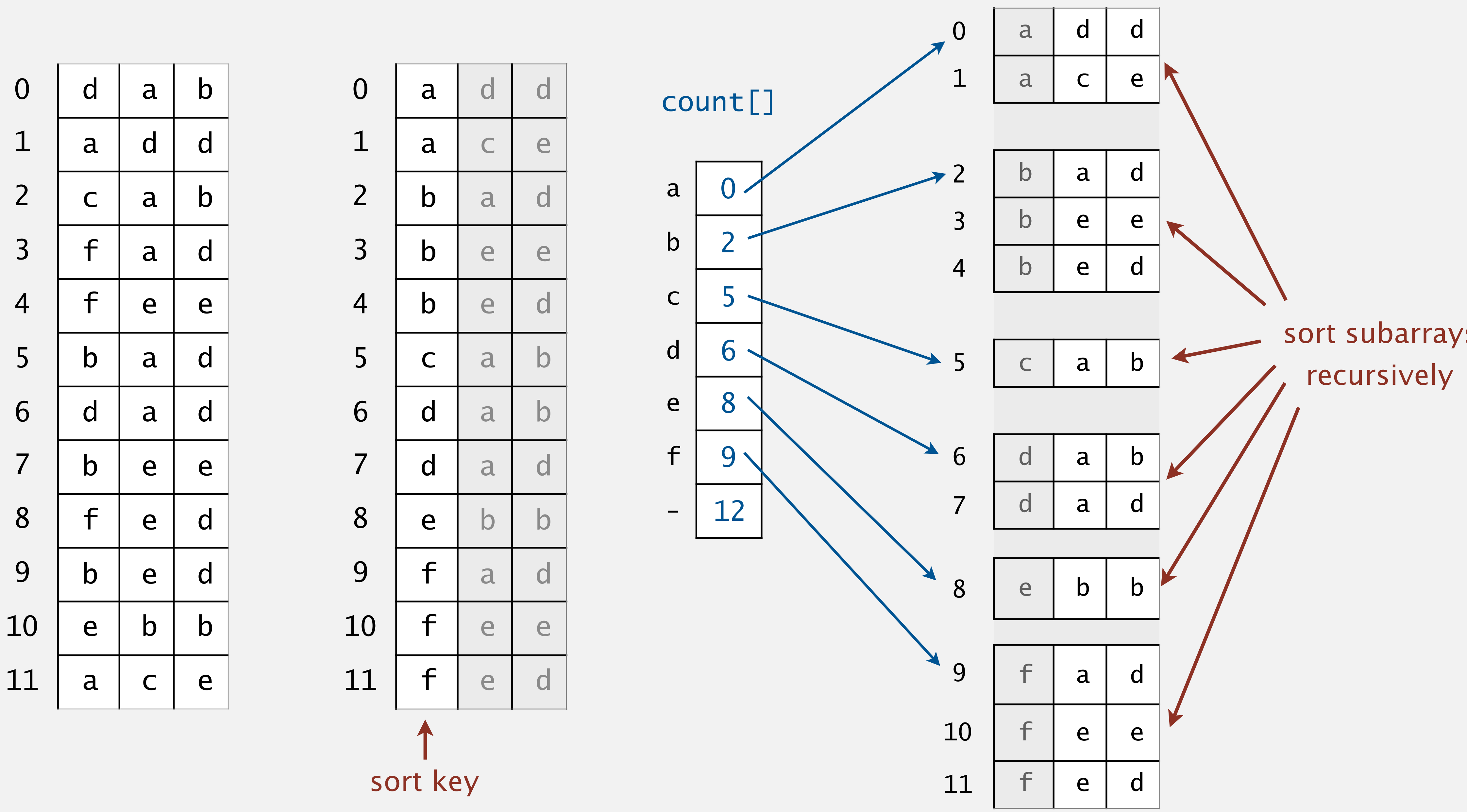
| | | | |
|---|---|---|---|
| 0 | c | a | b |
| 1 | d | a | b |
| 2 | e | b | b |
| 3 | b | a | d |
| 4 | d | a | d |
| 5 | f | a | d |
| 6 | a | d | d |
| 7 | b | e | d |
| 8 | f | e | d |
| 9 | a | c | e |
| 10 | b | e | e |
| 11 | f | e | e |

**not sorted!**

# Most-significant-digit-first string sort

## MSD string (radix) sort.

- Partition array into $R$ pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



# Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | s | e | a | -1 | | | | | | |
| 1 | s | e | a | s | h | e | l | l | s | -1 |
| 2 | s | e | l | l | s | -1 | | | | |
| 3 | s | h | e | -1 | | | | | | |
| 4 | s | h | e | -1 | | | | | | |
| 5 | s | h | e | l | l | s | -1 | | | |
| 6 | s | h | o | r | e | -1 | | | | |
| 7 | s | u | r | e | l | y | -1 | | | |

she before shells

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char '\0' at end ⇒ no extra work needed.

# MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length - 1, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
    int[] count = new int[R+2];
    for (int i = lo; i <= hi; i++)
       count[charAt(a[i], d) + 2]++;
    for (int r = 0; r < R+1; r++)
       count[r+1] += count[r];
    for (int i = lo; i <= hi; i++)
       aux[count[charAt(a[i], d) + 1]++] = a[i];
    for (int i = lo; i <= hi; i++)
       a[i] = aux[i - lo];

    for (int r = 0; r < R; r++)
       sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);

}
```

recycles aux[] array but not count[] array

key-indexed counting

sort R subarrays recursively

# MSD string sort: performance

## Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear in input size!

compareTo() based sorts can also be sublinear!

| Random (sublinear) | Non-random with duplicates (nearly linear) | Worst case (linear) |
|---|---|---|
| 1EIO402 | are | 1DNB377 |
| 1HYL490 | by | 1DNB377 |
| 1ROZ572 | sea | 1DNB377 |
| 2HXE734 | seashells | 1DNB377 |
| 2IYE230 | seashells | 1DNB377 |
| 2XOR846 | sells | 1DNB377 |
| 3CDB573 | sells | 1DNB377 |
| 3CVP720 | she | 1DNB377 |
| 3IGJ319 | she | 1DNB377 |
| 3KNA382 | shells | 1DNB377 |
| 3TAV879 | shore | 1DNB377 |
| 4CQP781 | surely | 1DNB377 |
| 4QGI284 | the | 1DNB377 |
| 4YHV229 | the | 1DNB377 |

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| **insertion sort** | $\frac{1}{2} n^2$ | $\frac{1}{4} n^2$ | $1$ | ✔ | `compareTo()` |
| **mergesort** | $n \lg n$ | $n \lg n$ | $n$ | ✔ | `compareTo()` |
| **quicksort** | $1.39\, n \lg n$ * | $1.39\, n \lg n$ | $c \lg n$ * | | `compareTo()` |
| **heapsort** | $2\, n \lg n$ | $2\, n \lg n$ | $1$ | | `compareTo()` |
| **LSD sort** † | $2\, W (n + R)$ | $2\, W (n + R)$ | $n + R$ | ✔ | `charAt()` |
| **MSD sort** ‡ | $2\, W (n + R)$ | $n \log_R n$ | $n + D R$ | ✔ | `charAt()` |

D = function-call stack depth
(length of longest prefix match)

* probabilistic
† fixed-length W keys
‡ average-length W keys

# Engineering a radix sort (American flag sort)

Optimization 0. Cutoff to insertion sort.

- MSD is much too slow for small subarrays.
- Essential for performance.

Optimization 1. Replace recursion with explicit stack.

- Push subarrays to be sorted onto stack.
- One `count[]` array now suffices.

Optimization 2. Do $R$-way partitioning in place.

- Eliminates `aux[]` array.
- Sacrifices stability.

**American national flag problem**

**Dutch national flag problem**

*Engineering Radix Sort*

Peter M. McIlroy and Keith Bostic
University of California at Berkeley;
and M. Douglas McIlroy
AT&T Bell Laboratories

ABSTRACT: Radix sorting methods have excellent asymptotic performance on string data, for which comparison is not a unit-time operation. Attractive for use in large byte-addressable memories, these methods have nevertheless long been eclipsed by more easily programmed algorithms. Three ways to sort strings by bytes left to right—a stable list sort, a stable two-array sort, and an in-place "American flag" sort—are illustrated with practical C programs. For heavy-duty sorting, all three perform comparably, usually running at least twice as fast as a good quicksort. We recommend American flag sort for general use.

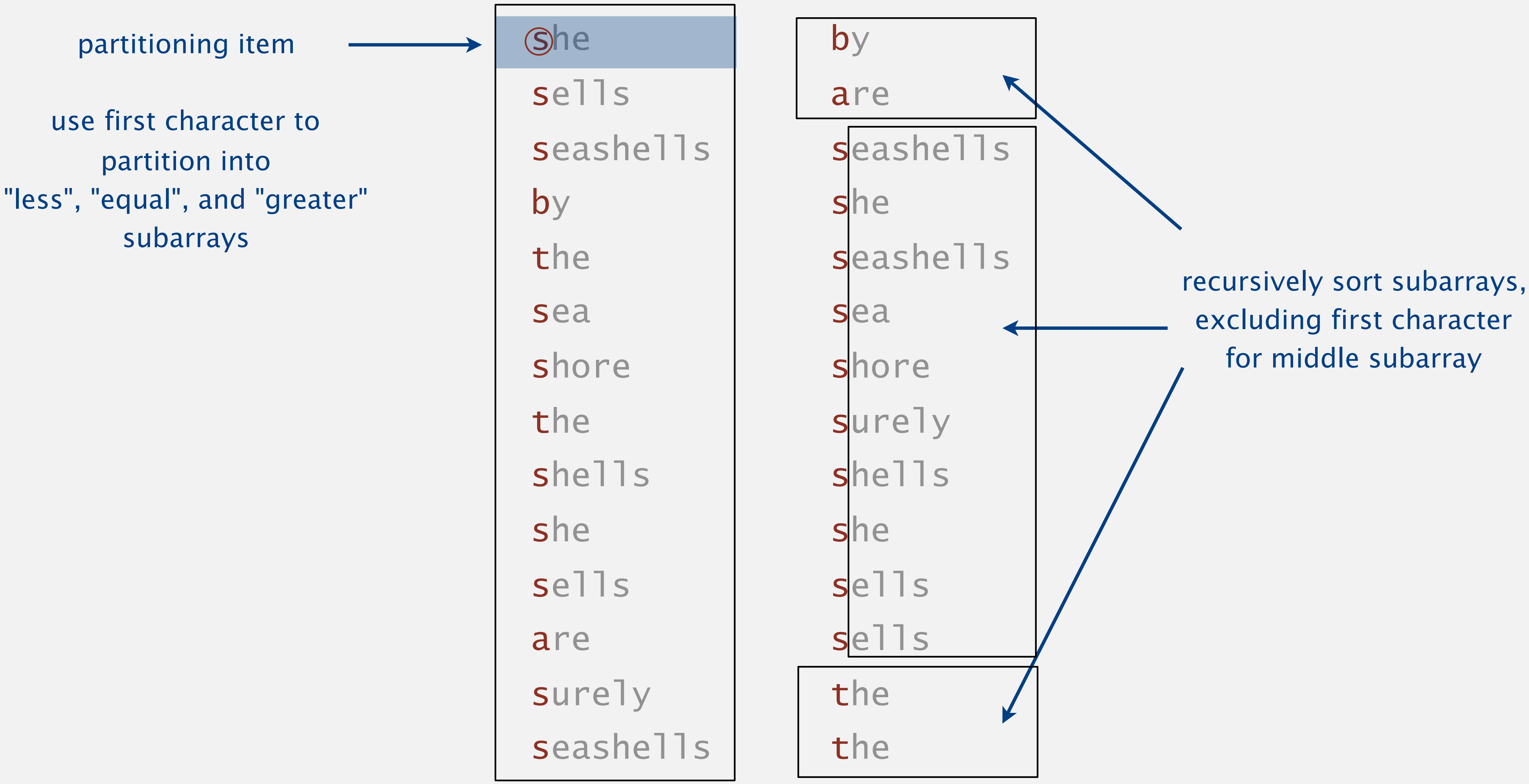

http://algs4.cs.princeton.edu

# 5.1 String Sorts

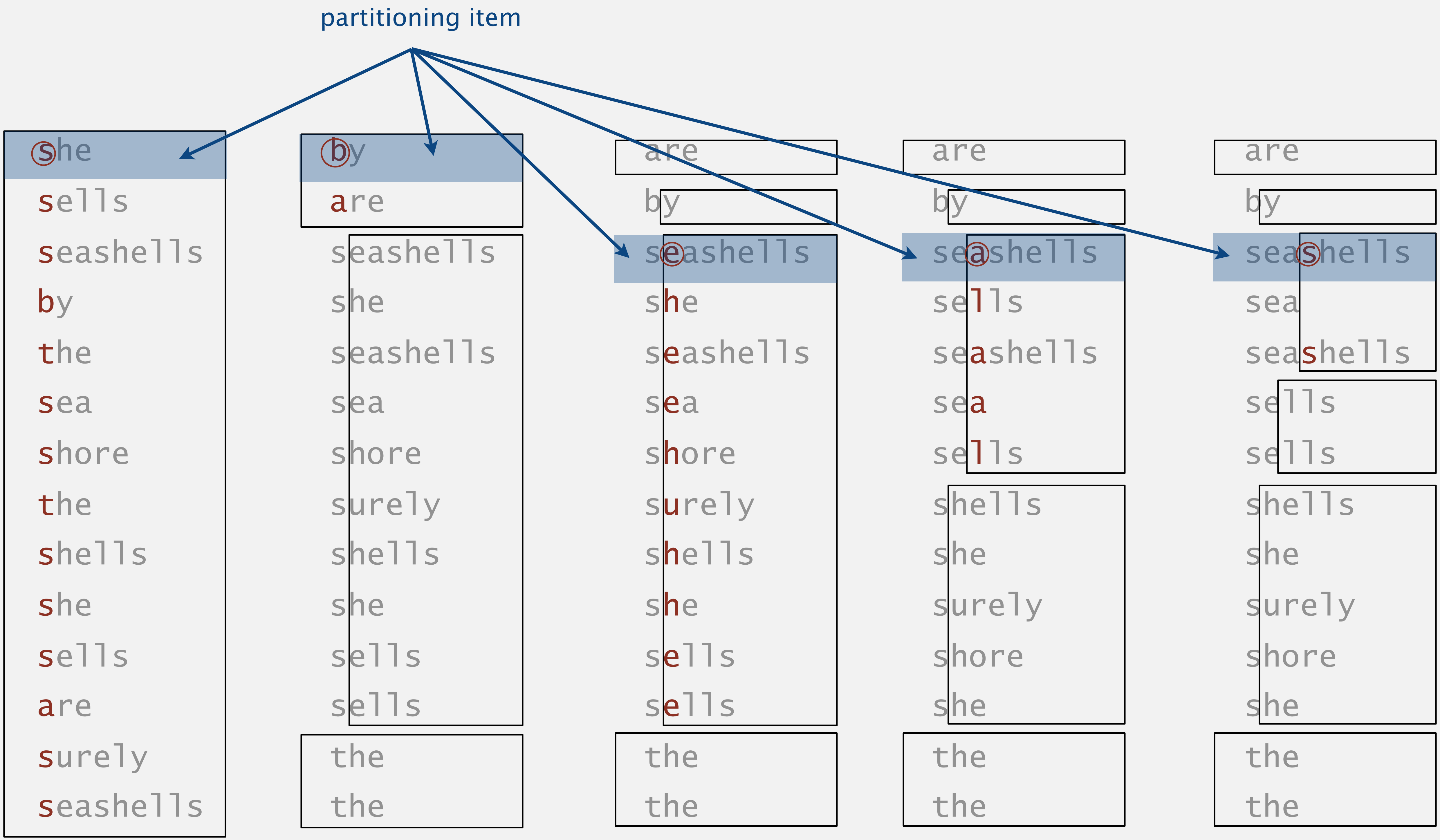
# 3-way string quicksort

Overview. Do 3-way partitioning on the $d^{th}$ character.

- Less overhead than $R$-way partitioning in MSD radix sort.
- Does not re-examine characters equal to the partitioning char.
  (but does re-examine characters not equal to the partitioning char)



# 3-way string quicksort: trace of recursive calls




**Trace of first few recursive calls for 3-way string quicksort (subarrays of size 1 not shown)**

# 3-way string quicksort: Java implementation

```
private static void sort(String[] a)
{  sort(a, 0, a.length - 1, 0);  }

private static void sort(String[] a, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int v = charAt(a[lo], d);
   int lt = lo, gt = hi;
   int i = lo + 1;
   while (i <= gt)
   {
      int t = charAt(a[i], d);
      if      (t < v) exch(a, lt++, i++);
      else if (t > v) exch(a, i, gt--);
      else            i++;
   }

   sort(a, lo, lt-1, d);
   if (v != -1) sort(a, lt, gt, d+1);
   sort(a, gt+1, hi, d);

}
```

3-way partitioning (using $d^{th}$ character)

sort 3 subarrays recursively

# 3-way string quicksort vs. standard quicksort

## Standard quicksort.

- Uses $\sim 2\,n \ln n$ string compares on average.
- Costly for keys with long common prefixes (and this is a common case!)

## 3-way string (radix) quicksort.

- Uses $\sim 2\,n \ln n$ character compares on average for random strings.
- Avoids re-comparing long common prefixes.

### Fast Algorithms for Sorting and Searching Strings

Jon L. Bentley*  Robert Sedgewick#

**Abstract**

We present theoretical algorithms for sorting and searching multikey data, and derive from them practical C implementations for applications in which keys are character strings. The sorting algorithm blends Quicksort and radix sort; it is competitive with the best known C sort codes. The searching algorithm blends tries and binary search trees; the new program is a symbol table implementation that is competitive with the most efficient string sorting programs known. The second program is a symbol table implementation that is faster than hashing, which is commonly regarded as the fastest symbol table implementation. The symbol table implementation is much more space-efficient than multiway trees, and supports more advanced searches.

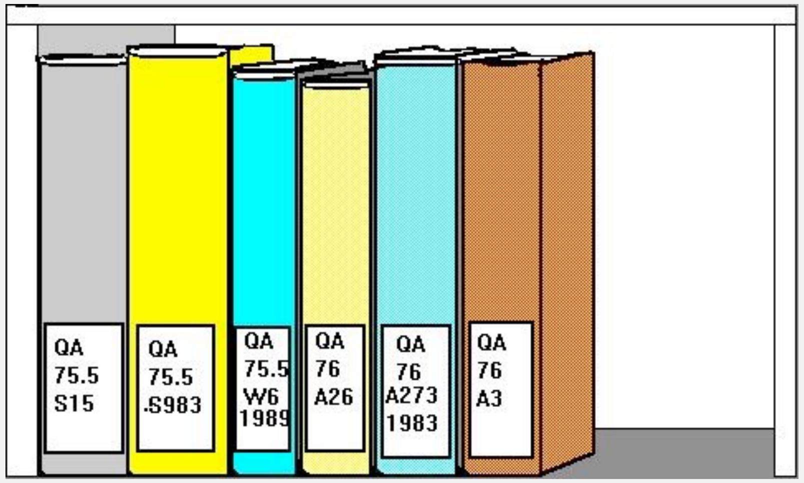
# 3-way string quicksort vs. MSD string sort

MSD string sort.

- Is cache-inefficient.
- Too much memory storing `count[]`.
- Too much overhead reinitializing `count[]` and `aux[]`.

3-way string quicksort.

- Is in-place.
- Is cache-friendly.
- Has a short inner loop.
- But not stable.




library of Congress call numbers

Bottom line. 3-way string quicksort is method of choice for sorting strings.

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| **insertion sort** | $\frac{1}{2} n^2$ | $\frac{1}{4} n^2$ | $1$ | ✔ | `compareTo()` |
| **mergesort** | $n \lg n$ | $n \lg n$ | $n$ | ✔ | `compareTo()` |
| **quicksort** | $1.39\, n \lg n$ * | $1.39\, n \lg n$ | $c \lg n$ * | | `compareTo()` |
| **heapsort** | $2\, n \lg n$ | $2\, n \lg n$ | $1$ | | `compareTo()` |
| **LSD sort** † | $2\, W (n + R)$ | $2\, W (n + R)$ | $n + R$ | ✔ | `charAt()` |
| **MSD sort** ‡ | $2\, W (n + R)$ | $n \log_R n$ | $n + D R$ | ✔ | `charAt()` |
| **3-way string quicksort** | $1.39\, W n \lg R$ * | $1.39\, n \lg n$ | $\log n + W$ * | | `charAt()` |

* probabilistic
† fixed-length W keys
‡ average-length W keys


http://algs4.cs.princeton.edu

# 5.1 String Sorts

- strings in Java
- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

# Keyword-in-context search

Given a text of $n$ characters, preprocess it to enable fast substring search (find all occurrences of query string context).

```
% more tale.txt
it was the best of times
it was the worst of times
it was the age of wisdom
it was the age of foolishness
it was the epoch of belief
it was the epoch of incredulity
it was the season of light
it was the season of darkness
it was the spring of hope
it was the winter of despair
    ⋮
```

# Keyword-in-context search

Given a text of $n$ characters, preprocess it to enable fast substring search (find all occurrences of query string context).

```
% java KWIC tale.txt 15
search
o st giless to search for contraband
her unavailing search for your fathe
le and gone in search of her husband
t provinces in search of impoverishe
 dispersing in search of other carri
n that bed and search the straw hold

better thing
t is a far far better thing that i do than
 some sense of better things else forgotte
was capable of better things mr carton ent
```

← number of characters of surrounding context

**Applications.** Linguistics, databases, web search, word processing, ….

# Suffix sort

**input string**

| i | t | w | a | s | b | e | s | t | i | t | w | a | s | w |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

**form suffixes**

| | |
|---|---|
| 0 | i t w a s b e s t i t w a s w |
| 1 | t w a s b e s t i t w a s w |
| 2 | w a s b e s t i t w a s w |
| 3 | a s b e s t i t w a s w |
| 4 | s b e s t i t w a s w |
| 5 | b e s t i t w a s w |
| 6 | e s t i t w a s w |
| 7 | s t i t w a s w |
| 8 | t i t w a s w |
| 9 | i t w a s w |
| 10 | t w a s w |
| 11 | w a s w |
| 12 | a s w |
| 13 | s w |
| 14 | w |

**sort suffixes to bring query strings together**

| | |
|---|---|
| 3 | a s b e s t i t w a s w |
| 12 | a s w |
| 5 | b e s t i t w a s w |
| 6 | e s t i t w a s w |
| 0 | i t w a s b e s t i t w a s w |
| 9 | i t w a s w |
| 4 | s b e s t i t w a s w |
| 7 | s t i t w a s w |
| 13 | s w |
| 8 | t i t w a s w |
| 1 | t w a s b e s t i t w a s w |
| 10 | t w a s w |
| 14 | w |
| 2 | w a s b e s t i t w a s w |
| 11 | w a s w |

array of suffix indices in sorted order

# Keyword-in-context search: suffix-sorting solution

- Preprocess: suffix sort the text.
- Query: binary search for query; scan until mismatch.

**KWIC search for "search" in Tale of Two Cities**

| | |
|---|---|
| | ⋮ |
| 632698 | s e a l e d _ m y _ l e t t e r _ a n d _ … |
| 713727 | s e a m s t r e s s _ i s _ l i f t e d _ … |
| 660598 | s e a m s t r e s s _ o f _ t w e n t y _ … |
| 67610 | s e a m s t r e s s _ w h o _ w a s _ w i … |
| → 4430 | s e a r c h _ f o r _ c o n t r a b a n d … |
| 42705 | s e a r c h _ f o r _ y o u r _ f a t h e … |
| 499797 | s e a r c h _ o f _ h e r _ h u s b a n d … |
| 182045 | s e a r c h _ o f _ i m p o v e r i s h e … |
| 143399 | s e a r c h _ o f _ o t h e r _ c a r r i … |
| 411801 | s e a r c h _ t h e _ s t r a w _ h o l d … |
| 158410 | s e a r e d _ m a r k i n g _ a b o u t _ … |
| 691536 | s e a s _ a n d _ m a d a m e _ d e f a r … |
| 536569 | s e a s e _ a _ t e r r i b l e _ p a s s … |
| 484763 | s e a s e _ t h a t _ h a d _ b r o u g h … |
| | ⋮ |

# War story

Q. How to efficiently form (and sort) suffixes?

```
String[] suffixes = new String[n];
for (int i = 0; i < n; i++)
    suffixes[i] = s.substring(i, n);

Arrays.sort(suffixes);
```


**3rd printing (2012)**

| input file | characters | Java 7u5 | Java 7u6 |
|---|---|---|---|
| amendments.txt | 18 thousand | 0.25 sec | 2.0 sec |
| aesop.txt | 192 thousand | 1.0 sec | *out of memory* |
| mobydick.txt | 1.2 million | 7.6 sec | *out of memory* |
| chromosome11.txt | 7.1 million | 61 sec | *out of memory* |

# Radix sorting:  quiz 3

**How much memory as a function of n?**

```
String[] suffixes = new String[n];
for (int i = 0; i < n; i++)
    suffixes[i] = s.substring(i, n);

Arrays.sort(suffixes);
```


**3rd printing (2012)**

**A.** 1

**B.** $n$

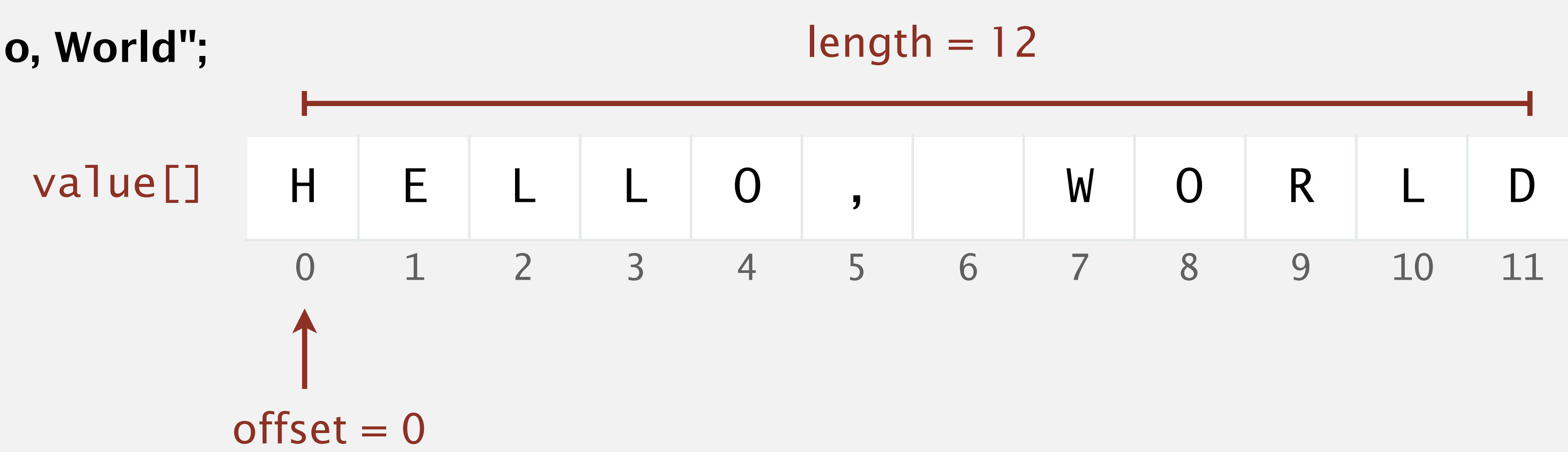
**C.** $n \log n$

**D.** $n^2$

# The String data type: Java 7u5 implementation

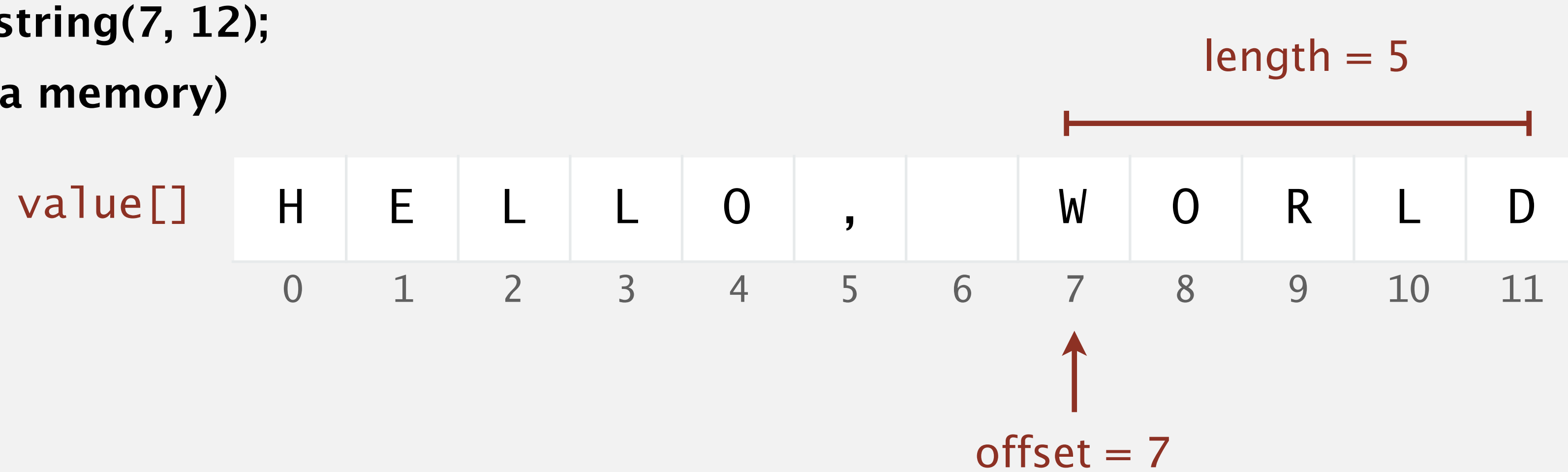
```
public final class String implements Comparable<String>
{
   private char[] value;   // characters
   private int offset;     // index of first char in array
   private int length;     // length of string
   private int hash;       // cache of hashCode()
   …
```

**String s = "Hello, World";**




**String t = s.substring(7, 12);**

**(constant extra memory)**



# The String data type: Java 7u6 implementation

```
public final class String implements Comparable<String>
{
   private char[] value;  // characters
   private int hash;      // cache of hashCode()
   …
```

**String s = "Hello, World";**

value[]

| H | E | L | L | O | , |  | W | O | R | L | D |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

**String t = s.substring(7, 12);**

**(linear extra memory)**

value[]

| W | O | R | L | D |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |

# The String data type: performance

String data type (in Java). Sequence of characters (immutable).

Java 7u5. Immutable `char[]` array, offset, length, hash cache.

Java 7u6. Immutable `char[]` array, hash cache.

| operation | Java 7u5 | Java 7u6 |
|---|---|---|
| **length** | 1 | 1 |
| **indexing** | 1 | 1 |
| **substring extraction** | 1 | $n$ |
| **concatenation** | $m + n$ | $m + n$ |
| **immutable?** | ✔ | ✔ |
| **memory** | $64 + 2n$ | $56 + 2n$ |

# A Reddit exchange

I'm the author of the substring() change. As has been suggested in the analysis here there were two motivations for the change

- Reduce the size of String instances. Strings are typically 20-40% of common apps footprint.
- Avoid memory leakage caused by retained substrings holding the entire character array.

**bondolo**

Changing this function, in a bugfix release no less, was totally irresponsible. It broke backwards compatibility for numerous applications with errors that didn't even produce a message, just freezing and timeouts... All pain, no gain. Your work was not just vain, it was thoroughly destructive, even beyond its immediate effect.

**cypherpunks**

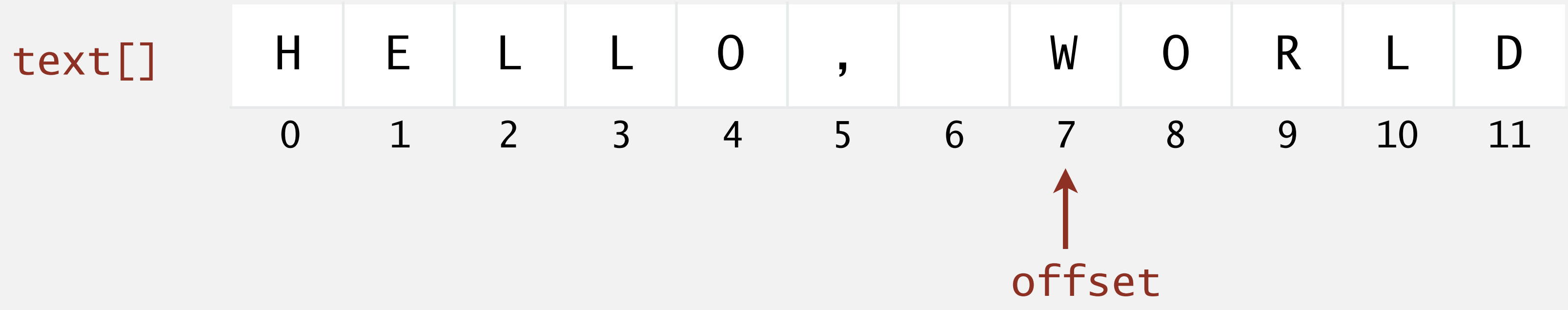
http://www.reddit.com/r/programming/comments/1qw73v/til_oracle_changed_the_internal_string

# Suffix sort

Q. How to efficiently form (and sort) suffixes in Java 7u6?

A. Define `Suffix` class ala Java 7u5 `String`.

```
public class Suffix implements Comparable<Suffix>
{
   private final String text;
   private final int offset;
   public Suffix(String text, int offset)
   {
      this.text = text;
      this.offset = offset;
   }
   public int length()                 { return text.length() - offset;   }
   public char charAt(int i)           { return text.charAt(offset + i);  }
   public int compareTo(Suffix that) { /* see textbook */               }
}
```



# Suffix sort

Q. How to efficiently form (and sort) suffixes in Java 7u6?

A. Define `Suffix` class ala Java 7u5 `String`.

```
Suffix[] suffixes = new Suffix[n];
for (int i = 0; i < n; i++)
    suffixes[i] = new Suffix(s, i);

Arrays.sort(suffixes);
```


**4th printing (2013)**

# Lessons learned

Lesson 1. Put performance guarantees in API.

Lesson 2. If API has no performance guarantees, can't rely upon them!

Corollary. May want to avoid `String` data type when performance matters.

- Are you sure `charAt()` and `length()` take constant time?
- If lots of calls to `charAt()`, overhead for function calls is large.
- If lots of small strings, memory overhead of `String` is large.

Ex. Our optimized algorithm for suffix arrays is 5× faster than our original Java 7u5 solution (and reduces memory footprint by 32×).

- 3-way radix quicksort.
- Avoid `Suffix` objects.

# Suffix arrays: theory

**Conjecture.** [Knuth 1970] No linear-time algorithm.

**Proposition.** [Weiner 1973] Linear-time algorithms (suffix trees).

**" has no practical virtue... but a historic monument in the area of string processing. "**

LINEAR PATTERN MATCHING ALGORITHMS

Peter Weiner

The Rand Corporation, Santa Monica, California*

Abstract

In 1970, Knuth, Pratt, and Morris [1] showed how to do basic pattern matching in linear time. Related problems, such as those discussed in [4], have previously been solved by efficient but sub-optimal algorithms. In this paper, we introduce an interesting data structure called a bi-tree. A linear time algorithm for obtaining a compacted version of a bi-tree associated with a given string is presented. With this construction as the basic tool, we indicate how to solve several pattern matching problems, including some from [4], in linear time.

**A Space-Economical Suffix Tree Construction Algorithm**

EDWARD M. McCREIGHT

*Xerox Palo Alto Research Center, Palo Alto, California*

ABSTRACT. A new algorithm is presented for constructing auxiliary digital search trees to aid in exact-match substring searching. This algorithm has the same asymptotic running time bound as previously published algorithms, but is more economical in space. Some implementation considerations are discussed, and new work on the modification of these search trees in response to incremental changes in the strings they index (the update problem) is presented.

On–line construction of suffix trees [1]

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# Suffix arrays: practice

**Applications.** Bioinformatics, information retrieval, data compression, ...

**Many ingenious algorithms.**

- Constants and memory footprint very important.
- State-of-the art still changing.

| year | algorithm | worst case | memory | |
|---|---|---|---|---|
| **1991** | **Manber–Myers** | $n \log n$ | $8\,n$ | ← see lecture videos |
| **1999** | **Larsson–Sadakane** | $n \log n$ | $8\,n$ | ← about 10× faster than Manber–Myers |
| **2003** | **Kärkkäinen–Sanders** | $n$ | $13\,n$ | |
| **2003** | **Ko–Aluru** | $n$ | $10\,n$ | |
| **2008** | **divsufsort2** | $n \log n$ | $5\,n$ | ← good choices (libdivsufsort) |
| **2010** | **sais** | $n$ | $6\,n$ | ← good choices (libdivsufsort) |

# String sorting summary

We can develop linear-time sorts.

- Key compares not necessary for string keys.
- Use characters as index in an array.

We can develop sublinear-time sorts.

- Input size is amount of data in keys (not number of keys).
- Not all of the data has to be examined.

Long strings are rarely random in practice.

- Goal is often to learn the structure!
- May need specialized algorithms.