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4.4 SHORTEST PATHS

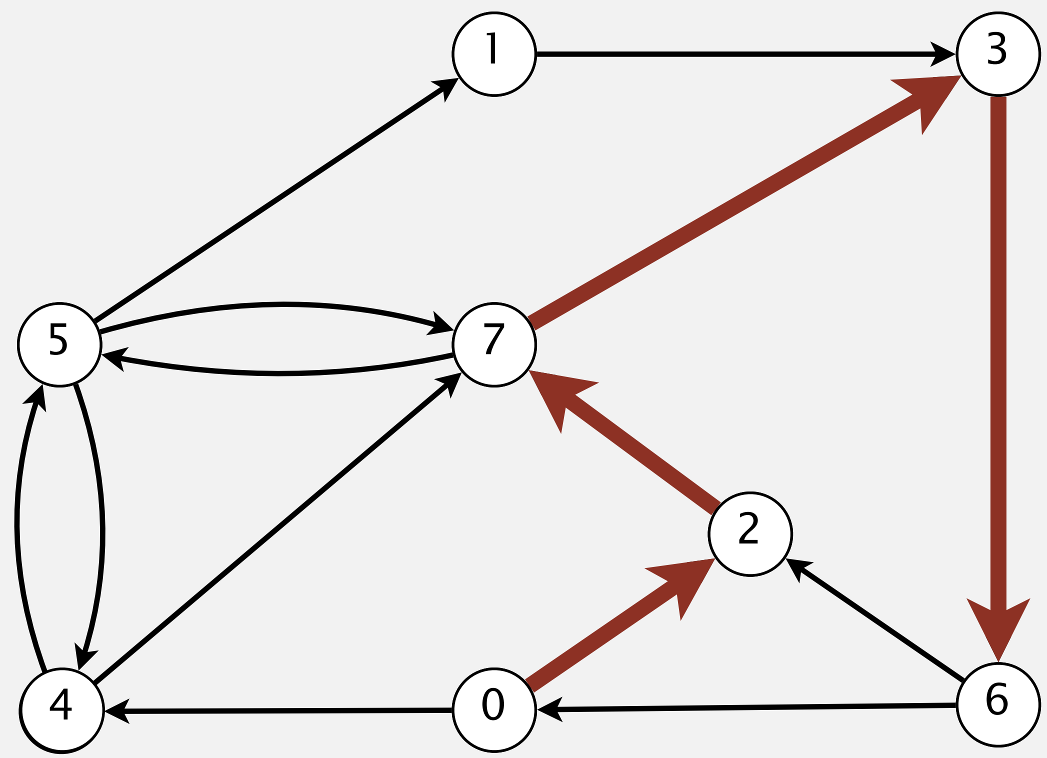
- *APIs*
- *properties*
- *Bellman–Ford algorithm*
- *Dijkstra's algorithm*
- *topological sort algorithm*
- *negative weights*

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

edge-weighted digraph

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



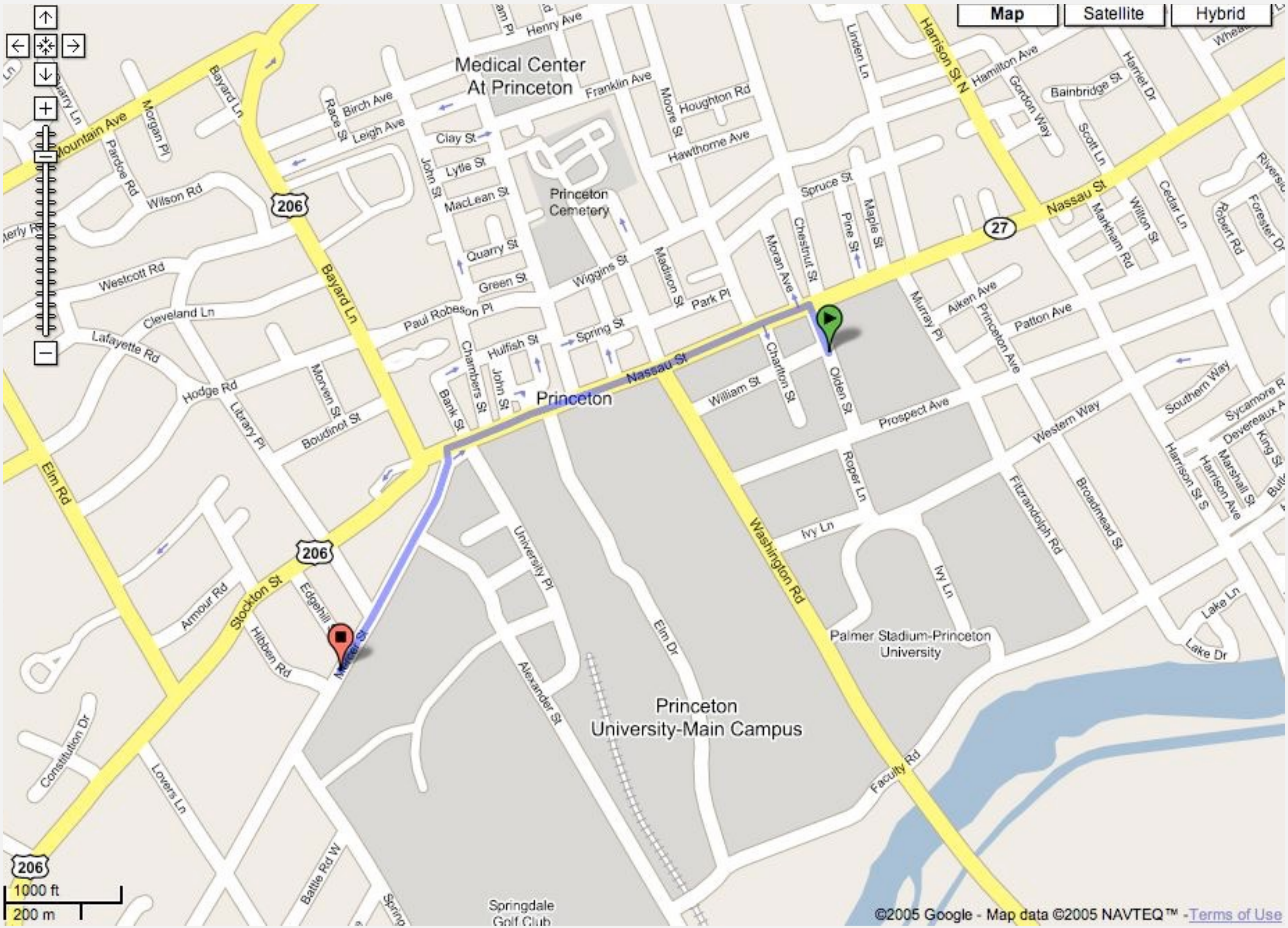
shortest path from 0 to 6

$0 \to 2 \to 7 \to 3 \to 6$

length of path = 1.51

$(0.26 + 0.34 + 0.39 + 0.52)$

Google maps



Map
Satellite
Hybrid
Medical Center
At Princeton
Princeton
Cemetery
Princeton
Princeton
University-Main Campus
Palmer Stadium-Princeton
University
Springdale
Golf Club
Nassau St
Washington Rd
Stockton St
Bayard Ln
Mountain Ave
Elm Rd
206
27
1000 ft
200 m
©2005 Google - Map data ©2005 NAVTEQ™ - Terms of Use

Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving. ← see Assignment 7
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Currency exchange.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source–sink: from one vertex s to another t.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Non-negative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

Simplifying assumption. Each vertex is reachable from s.

Shortest paths: quiz 1

Which variant in car GPS?

A. Single source: from one vertex s to every other vertex.

B. Single sink: from every vertex to one vertex t.

C. Source–sink: from one vertex s to another t.

D. All pairs: between all pairs of vertices.

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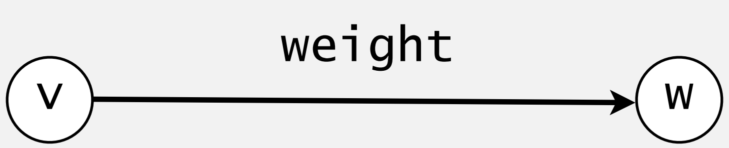
4.4 Shortest Paths

- *APIs*
- *properties*
- *Bellman–Ford algorithm*
- *Dijkstra's algorithm*
- *topological sort algorithm*

Weighted directed edge API

public class DirectedEdge		
	DirectedEdge(int v, int w, double weight)	*weighted edge v→w*
int	from()	*vertex v*
int	to()	*vertex w*
double	weight()	*weight of this edge*
String	toString()	*string representation*



Idiom for processing an edge `e`: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to `Edge` for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
     this.v = v;
     this.w = w;
     this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w; }

   public double weight()
   {  return weight;  }

}
```

from() and to() replace either() and other()

Edge-weighted digraph API

public class	EdgeWeightedDigraph	
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges adjacent from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

Conventions. Allow self-loops and parallel edges.

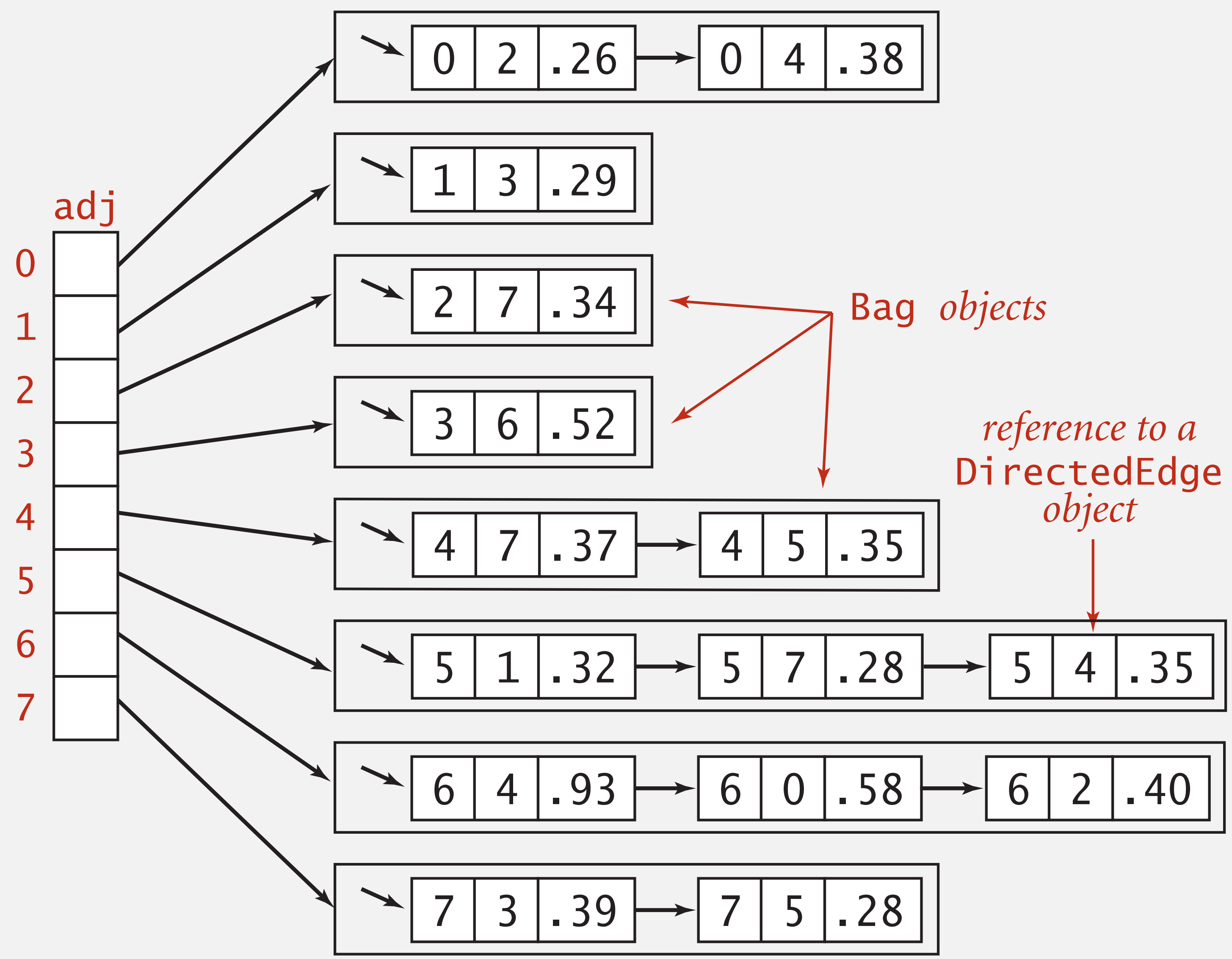
Edge-weighted digraph: adjacency-lists representation

tinyEWD.txt

```
8
15
4 5  0.35
5 4  0.35
4 7  0.37
5 7  0.28
7 5  0.28
5 1  0.32
0 4  0.38
0 2  0.26
7 3  0.39
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```

V

E



Edge-weighted digraph: adjacency-lists implementation in Java

Almost identical to `EdgeWeightedGraph`.

```
public class EdgeWeightedDigraph
{
   private final int V;
   private final Bag<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<DirectedEdge>();
   }

   public void addEdge(DirectedEdge e)
   {
      int v = e.from(), w = e.to();
      adj[v].add(e);
   }

   public Iterable<DirectedEdge> adj(int v)
   {  return adj[v];  }
}
```

add edge e = v→w to only v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

	public class SP	
	SP(EdgeWeightedDigraph G, int s)	*shortest paths from s in graph G*
double	distTo(int v)	*length of shortest path from s to v*
Iterable <DirectedEdge>	pathTo(int v)	*shortest path from s to v*
boolean	hasPathTo(int v)	*is there a path from s to v?*

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4.4 Shortest Paths

- *APIs*
- ***properties***
- *Bellman–Ford algorithm*
- *Dijkstra's algorithm*
- *topological sort algorithm*

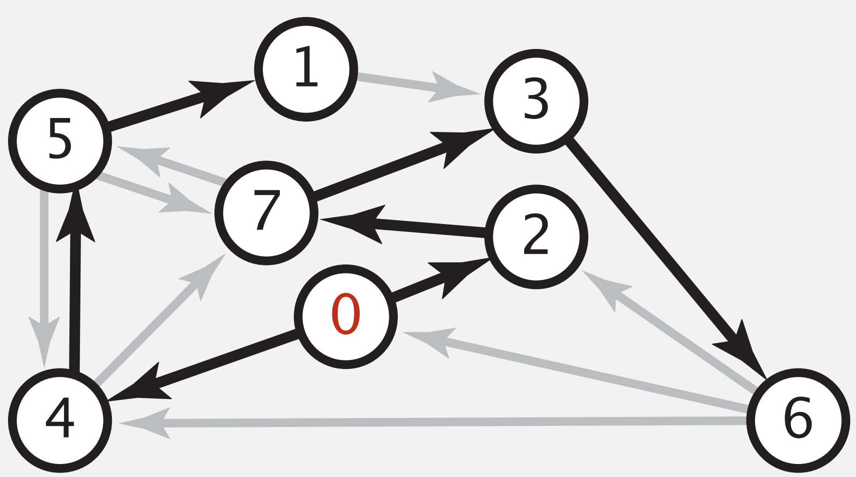
Data structures for single-source shortest paths

Goal. Find a shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of a (shortest) path from s to v.
- `edgeTo[v]` is last edge on a (shortest) path from s to v.



shortest-paths tree from 0

	distTo[]	edgeTo[]
0	0	null
1	1.05	5->1 0.32
2	0.26	0->2 0.26
3	0.97	7->3 0.37
4	0.38	0->4 0.38
5	0.73	4->5 0.35
6	1.49	3->6 0.52
7	0.60	2->7 0.34

parent-link representation

Data structures for single-source shortest paths

Goal. Find a shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of a (shortest) path from s to v.
- `edgeTo[v]` is last edge on a (shortest) path from s to v.

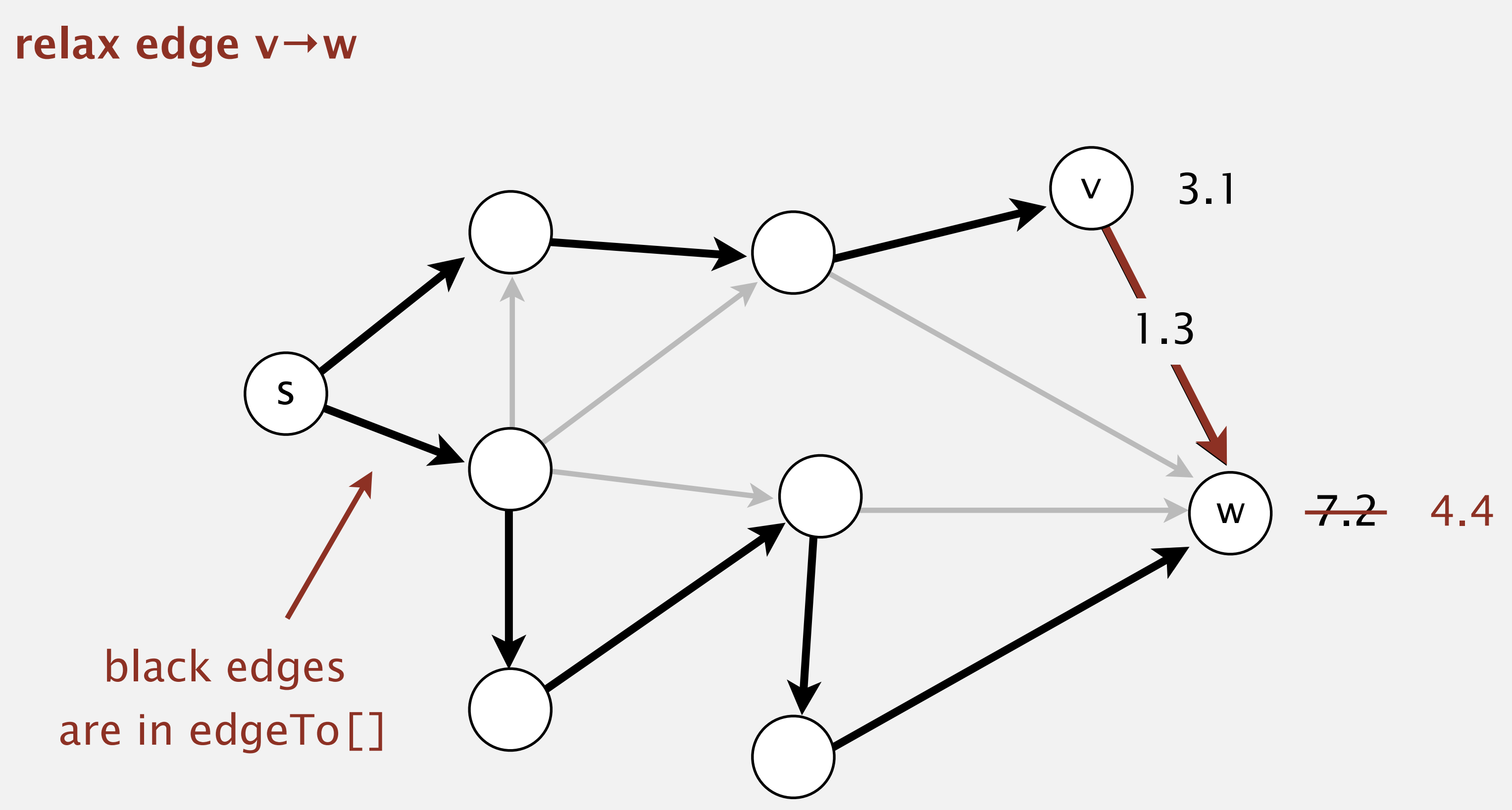
```
public double distTo(int v)
{  return distTo[v];  }

public Iterable<DirectedEdge> pathTo(int v)
{
   Stack<DirectedEdge> path = new Stack<DirectedEdge>();
   for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
      path.push(e);
   return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
- `distTo[w]` is length of shortest known path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` yields shorter path to `w`, update `distTo[w]` and `edgeTo[w]`.



Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
- `distTo[w]` is length of shortest known path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` yields shorter path to `w`, update `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```

Shortest paths: quiz 2

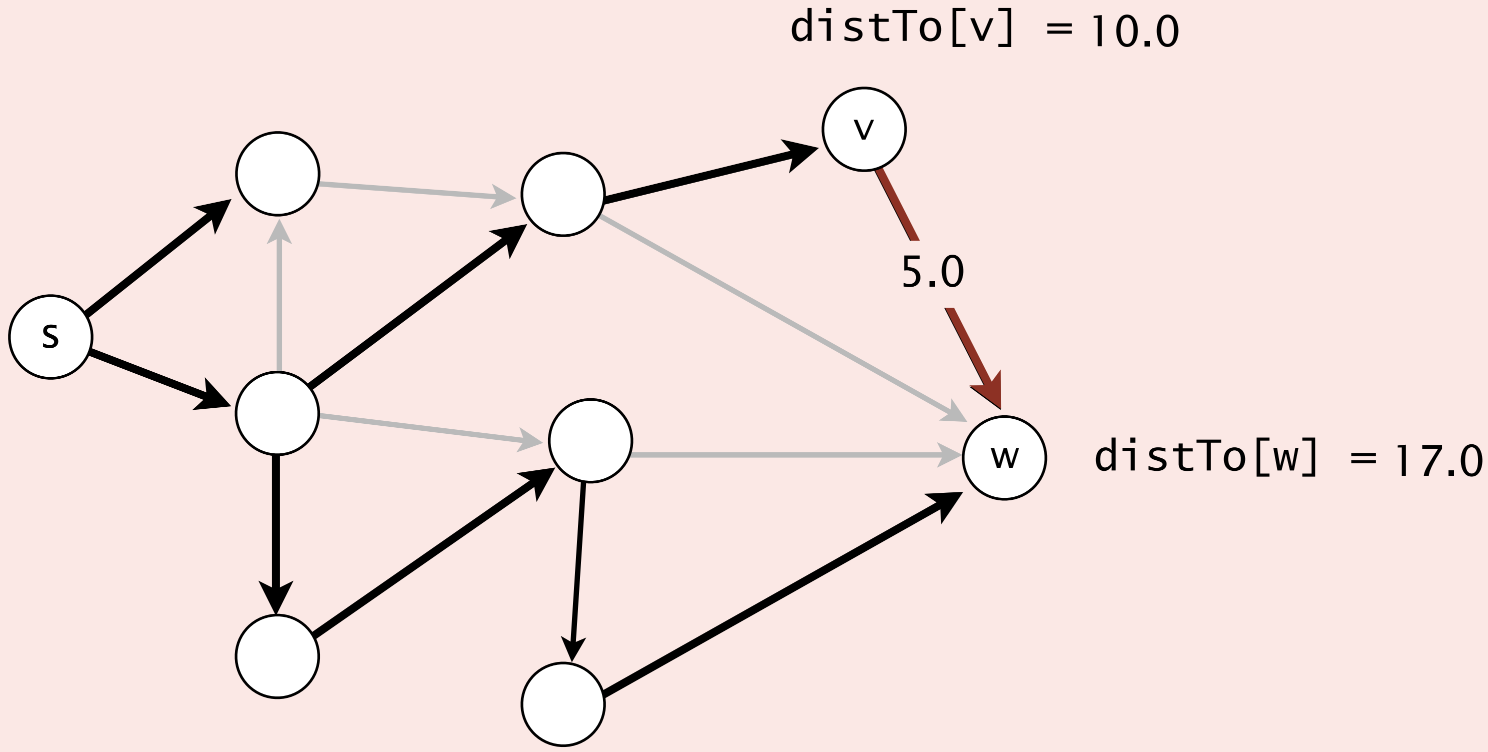
What are the values of `distTo[v]` and `distTo[w]` after relaxing `v→w` ?

A. 10.0 and 15.0

B. 10.0 and 17.0

C. 12.0 and 15.0

D. 12.0 and 17.0



Framework for shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat until done:

- **Relax any edge.**

Key properties.

- `distTo[v]` is the length of a simple path from `s` to `v`.
- `distTo[v]` cannot increase.

Framework for shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat until done:

- **Relax any edge.**

Efficient implementations.

- How to choose which edge to relax next?
- How many edge relaxations needed?

Ex 1. Dijkstra's algorithm (non-negative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman–Ford algorithm (no negative cycles).

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4.4 Shortest Paths

- *APIs*
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- *Dijkstra's algorithm*
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Bellman–Ford algorithm

Bellman–Ford algorithm

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat V–1 times:

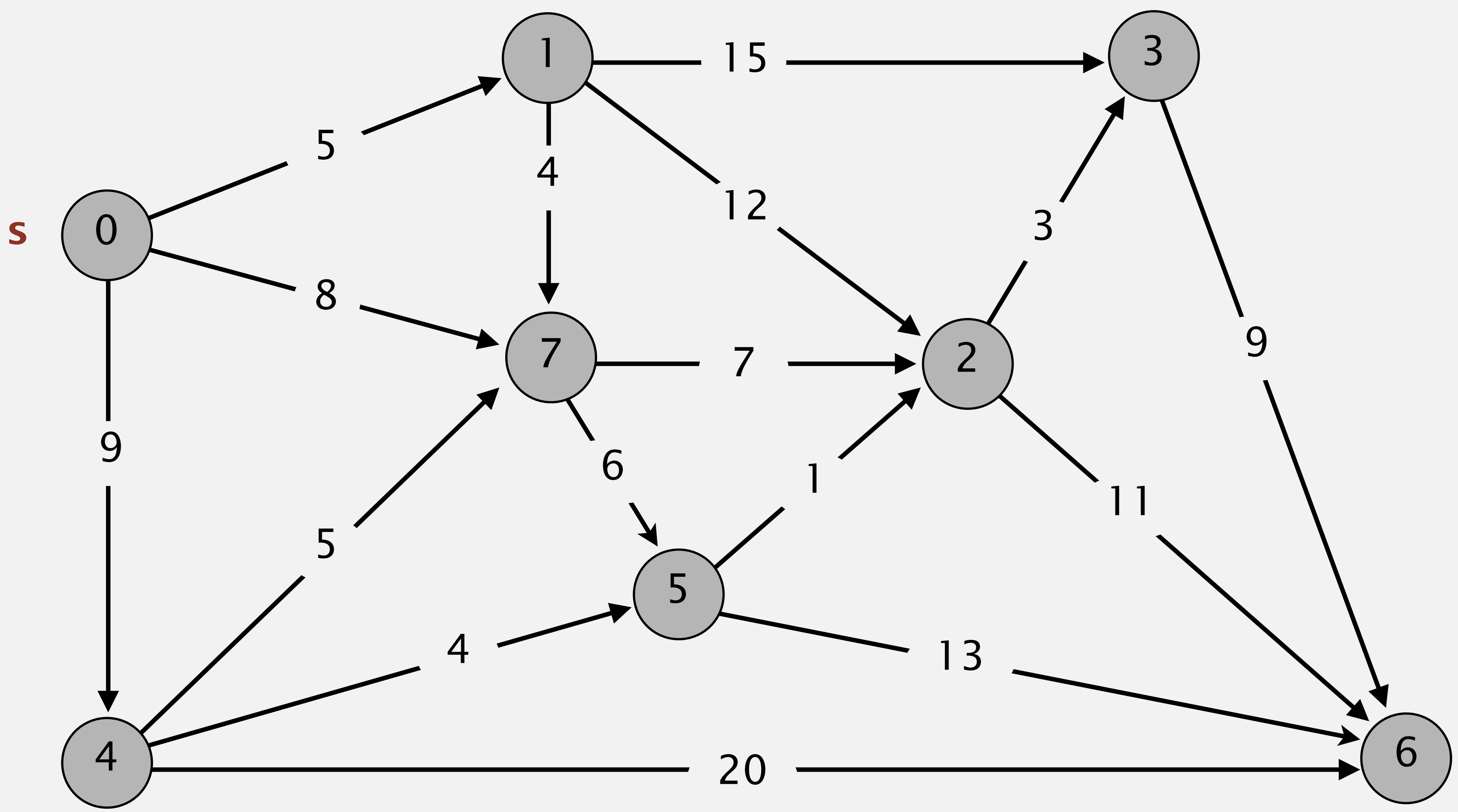
- **Relax each edge.**

```
for (int i = 1; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)

Bellman–Ford algorithm demo

Repeat $V - 1$ times: relax all E edges.

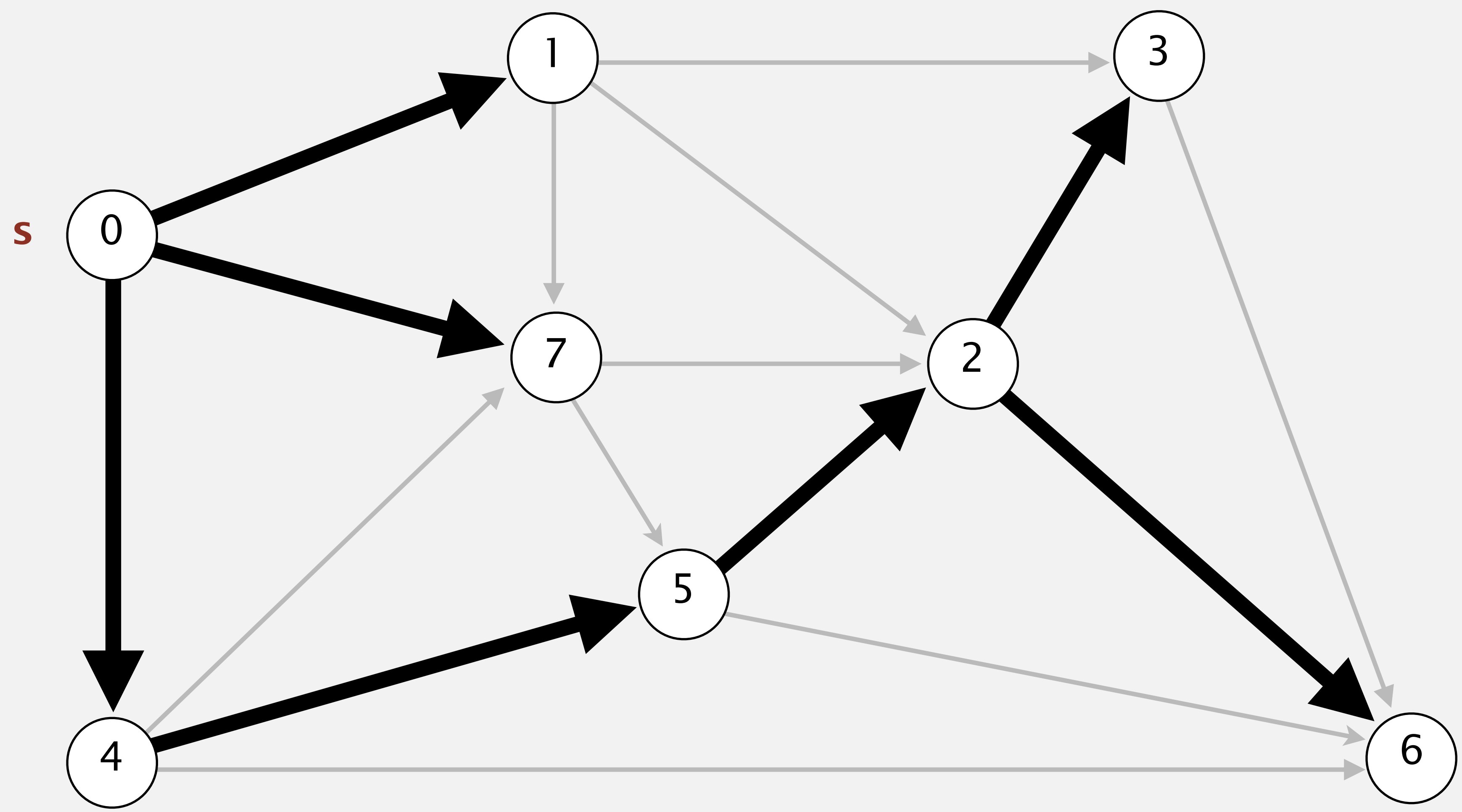


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Bellman–Ford algorithm demo

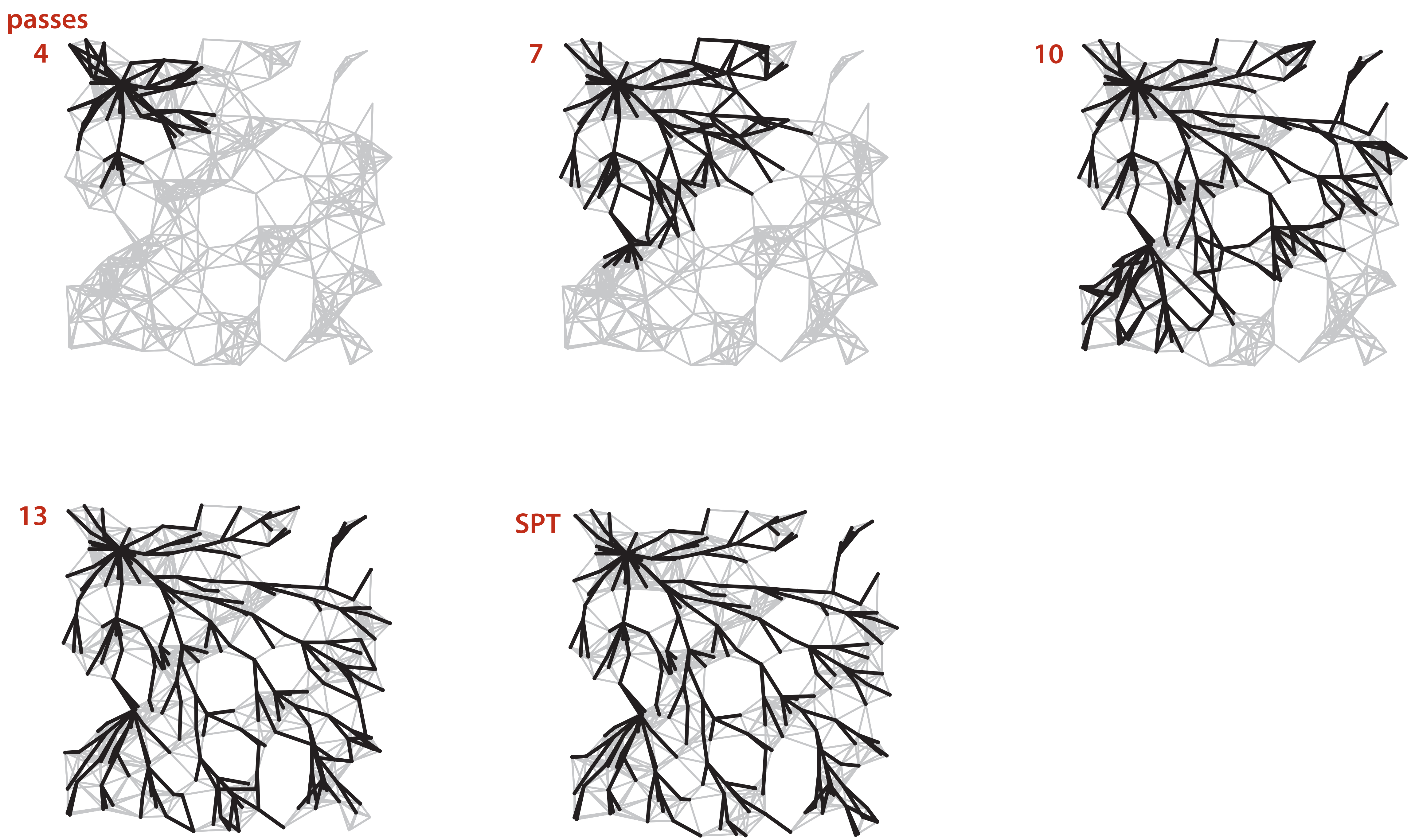
Repeat $V-1$ times: relax all E edges.



shortest-paths tree from vertex s

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

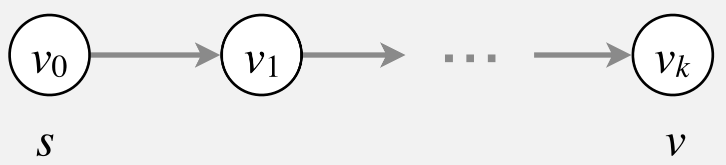
Bellman–Ford algorithm: visualization



Bellman–Ford algorithm: correctness proof

Proposition. Let $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k = v$ be a shortest path from s to v. Then, after pass i, `distTo[`v_i`]` $= d^*(v_i)$.

length of shortest path from s to v_i



Pf. [by induction on i]

- Suppose `distTo[`v_i`]` $= d^*(v_i)$ after pass i.
- Since `distTo[`v_{i+1}`]` is the length of some path from s to v_{i+1}, we must have `distTo[`v_{i+1}`]` $\ge d^*(v_{i+1})$.
- Immediately after relaxing edge $v_i \to v_{i+1}$ in pass $i+1$, we have

$$\texttt{distTo[}v_{i+1}\texttt{]} \le \texttt{distTo[}v_i\texttt{]} + weight(v_i, v_{i+1})$$
$$= d^*(v_i) + weight(v_i, v_{i+1})$$
$$= d^*(v_{i+1}).$$

- Thus, at the end of pass $i+1$, `distTo[`v_{i+1}`]` $= d^*(v_{i+1})$. ▪

Corollary. Bellman–Ford computes shortest path distances.

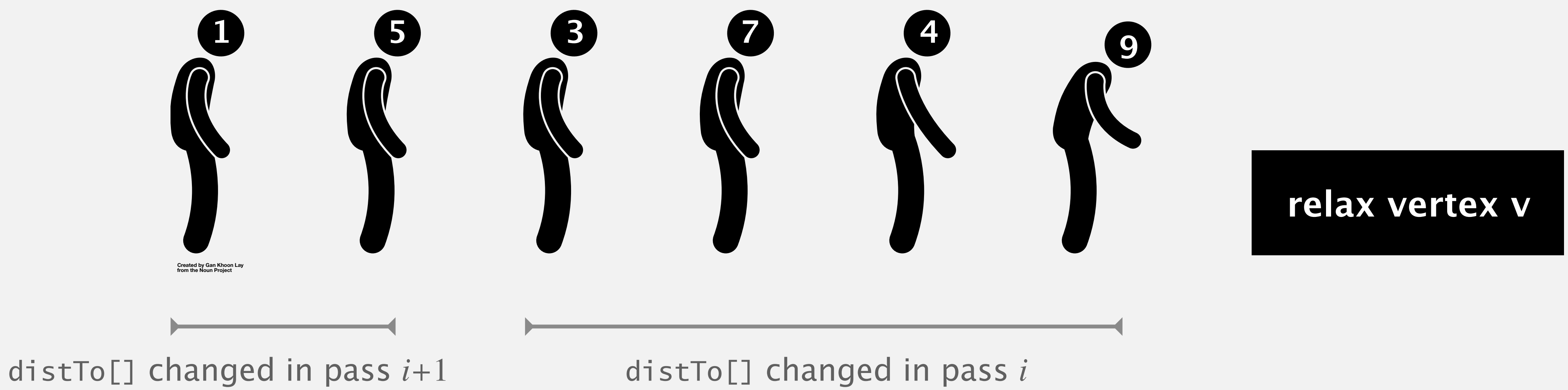
Pf. There exists a shortest path from s to v with at most $V - 1$ edges.

$\Rightarrow \;\le\; V - 1$ passes.

Bellman–Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass i, no need to relax any edge pointing from v in pass $i + 1$.

FIFO implementation. Maintain queue of vertices whose `distTo[]` values needs updating.



Impact.

- In the worst case, the running time is still proportional to $E \times V$.
- But much faster in practice.

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4.4 Shortest Paths

- APIs
- properties
- Bellman–Ford algorithm
- Dijkstra's algorithm
- topological sort algorithm

Edsger W. Dijkstra: select quotes

" *Do only what only you can do.* "

" *The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.* "

" *It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.* "

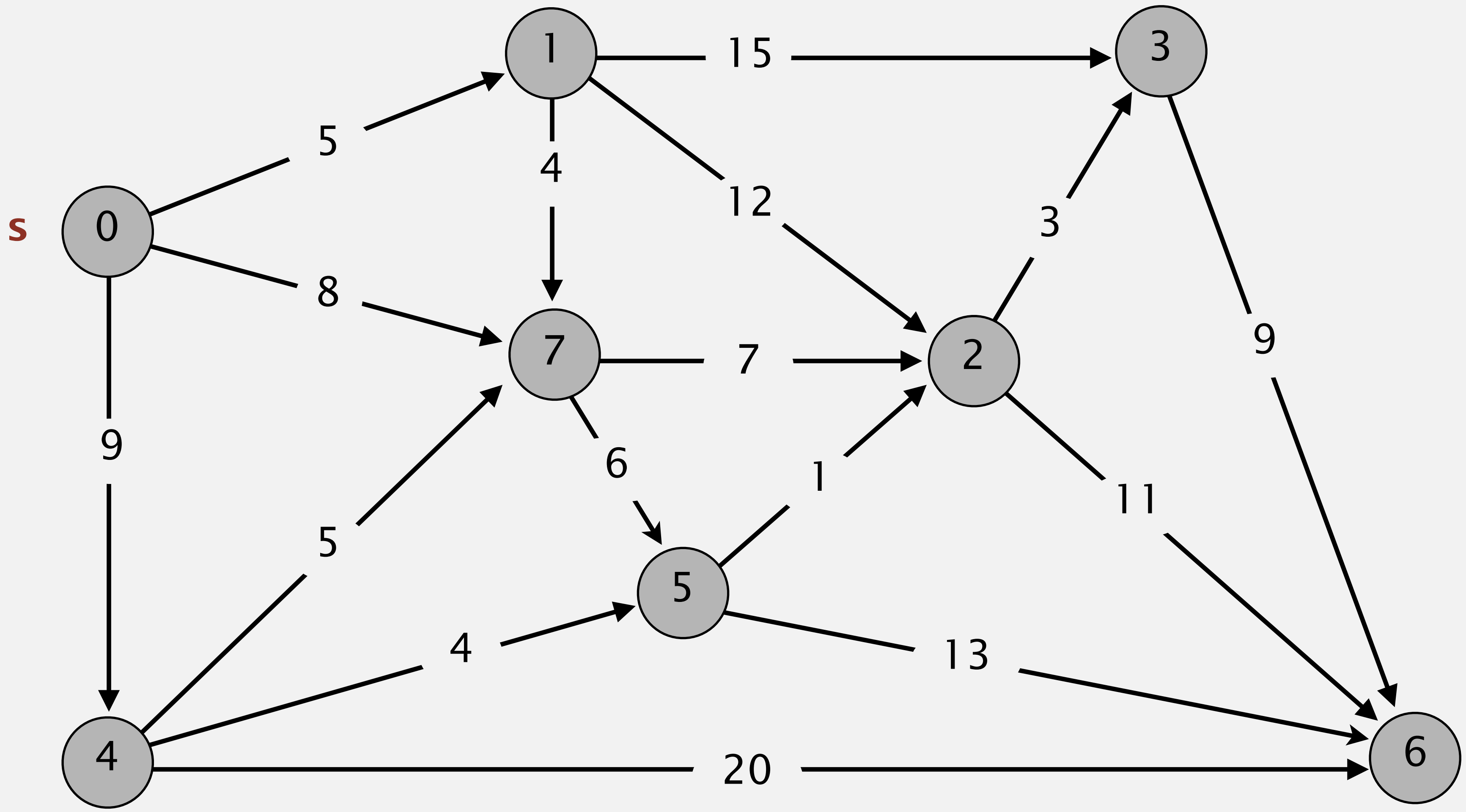
" *APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.* "

```
Φ'□',∈Nρ⊂S←'←□←(3=T)∨M∧2=T←⊃+/(VΦ''⊂M),(VΘ''⊂M),(V,ΦV)Φ''(V,V←1¯1)Θ''⊂M'
```

Edsger W. Dijkstra
Turing award 1972

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.

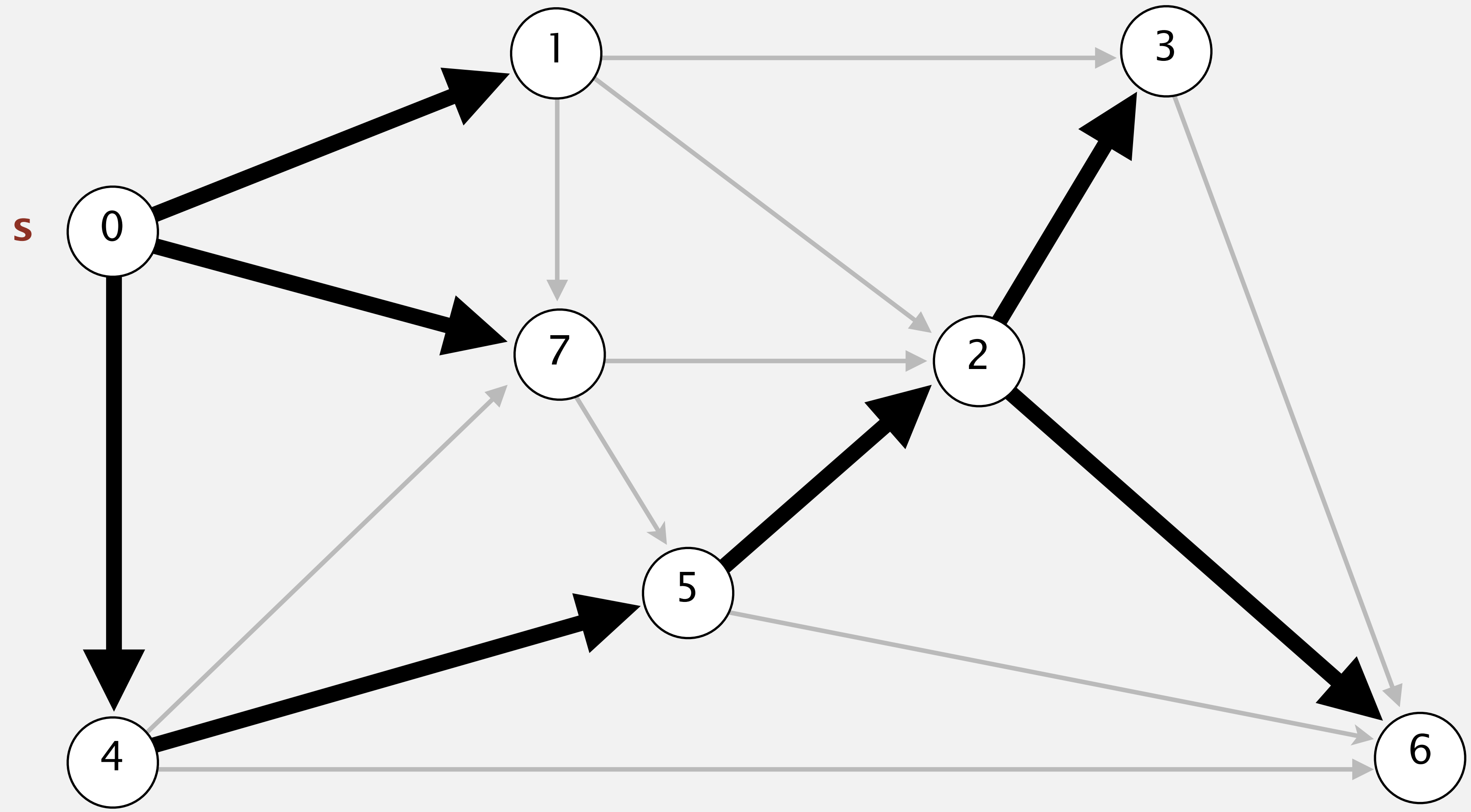


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

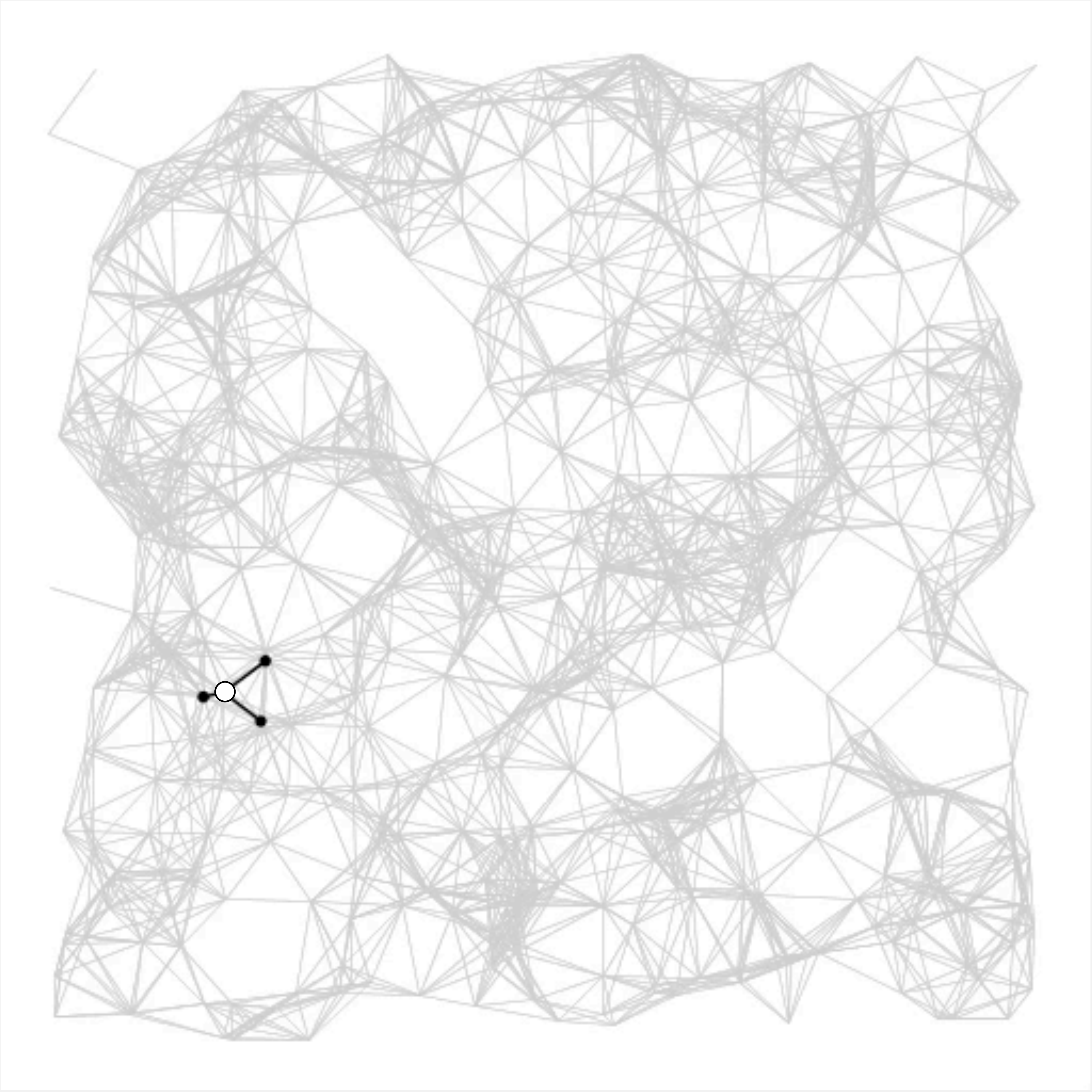
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



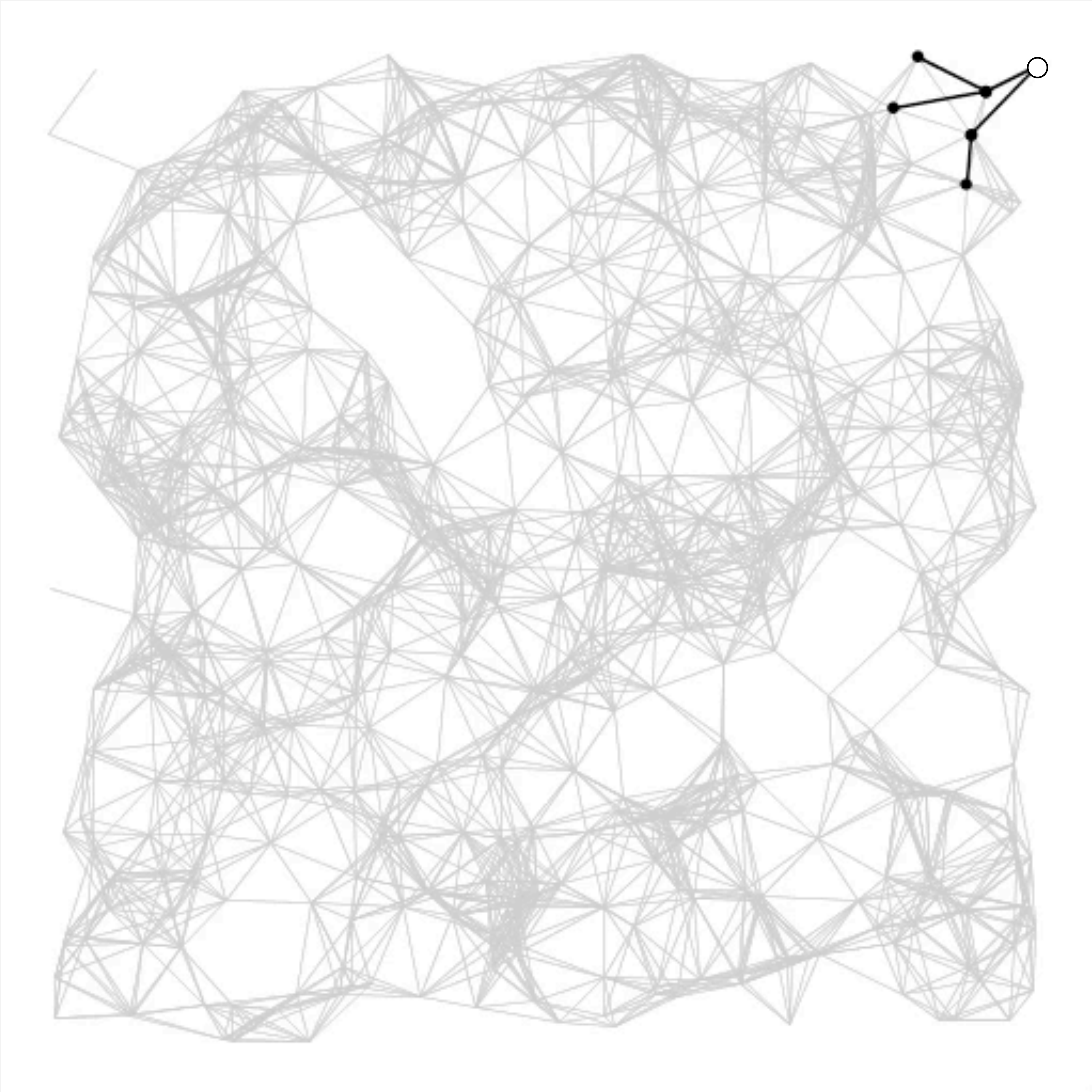
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



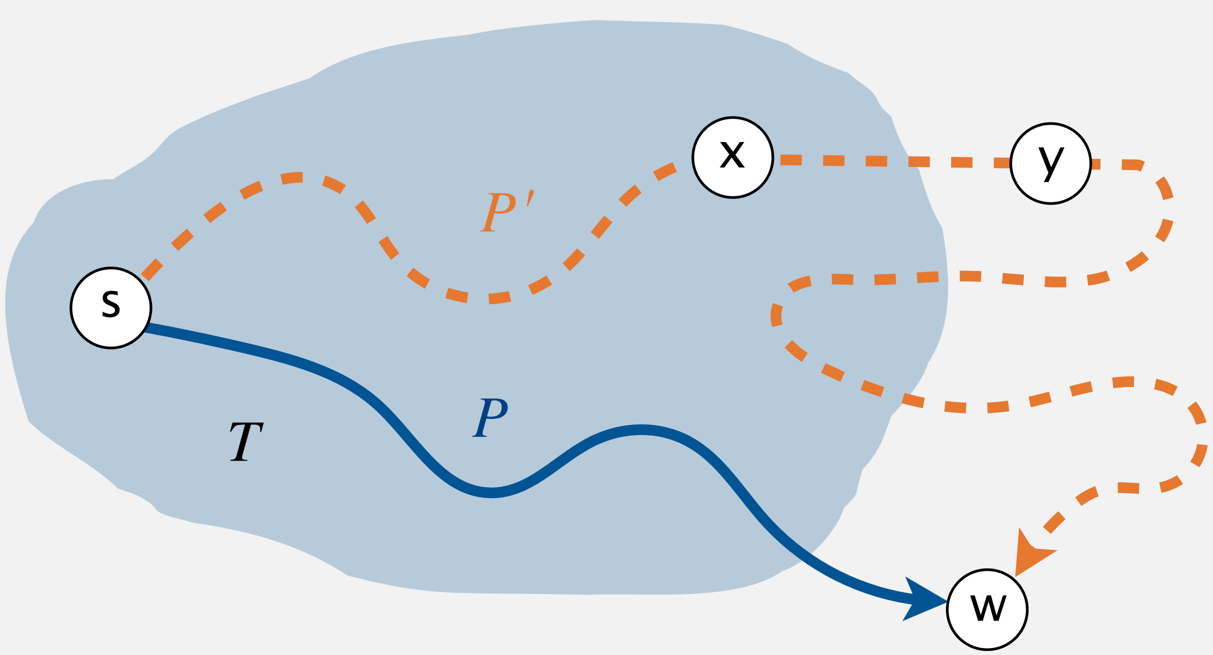
Dijkstra's algorithm: correctness proof

Invariant. For each vertex v in T, `distTo[v]` $= d^*(v)$.

length of shortest $s \to v$ path

Pf. [by induction on $|T|$]

- Let w be next vertex added to T.
- Let P be the $s \to w$ path of length `distTo[w]`.
- Consider any other $s \to w$ path P'.
- Let $x \to y$ be first edge in P' that leaves T.
- P' is no shorter than P :



by construction

length(P) = `distTo[w]`

Dijkstra chose w instead of y → ≤ `distTo[y]`

relax vertex x → ≤ `distTo[x]` + $weight(x, y)$

induction → = $d^*(x)$ + $weight(x, y)$

weight are non-negative → ≤ length(P') ▪

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;
   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];
      pq = new IndexMinPQ<Double>(G.V());

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty())
      {
         int v = pq.delMin();
         for (DirectedEdge e : G.adj(v))
            relax(e);
      }
   }
}
```

relax vertices in order of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
       else                pq.insert     (w, distTo[w]);
   }
}
```

update PQ

Shortest paths: quiz 3

What is the order of growth of the running time of Dijkstra's algorithm when using a binary heap for the priority queue?

A. $V + E$

B. $V \log E$

C. $E \log V$

D. $E \log E$

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V INSERT, V DELETE-MIN, $\le E$ DECREASE-KEY.

PQ implementation	INSERT	DELETE-MIN	DECREASE-KEY	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	1 †	$\log V$ †	1 †	$E + V \log V$

† amortized

Bottom line.

- Array implementation optimal for complete graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

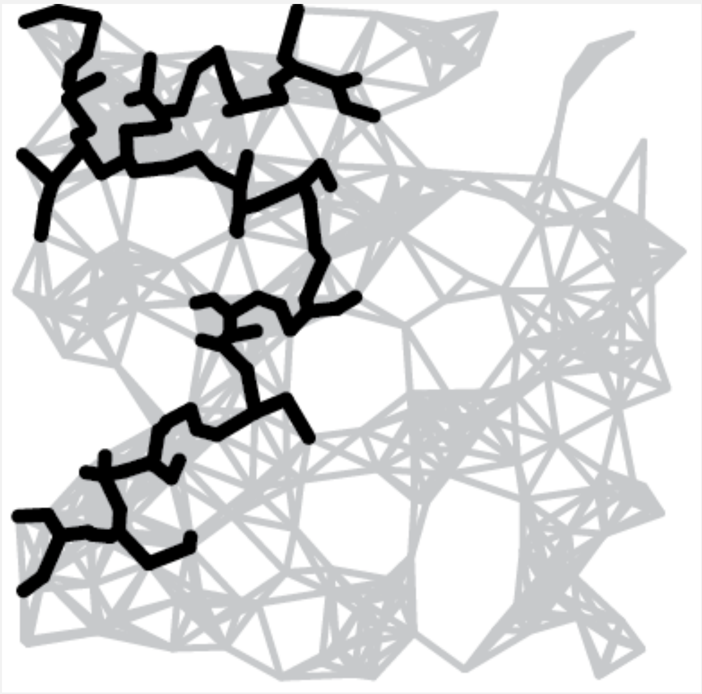
Priority-first search

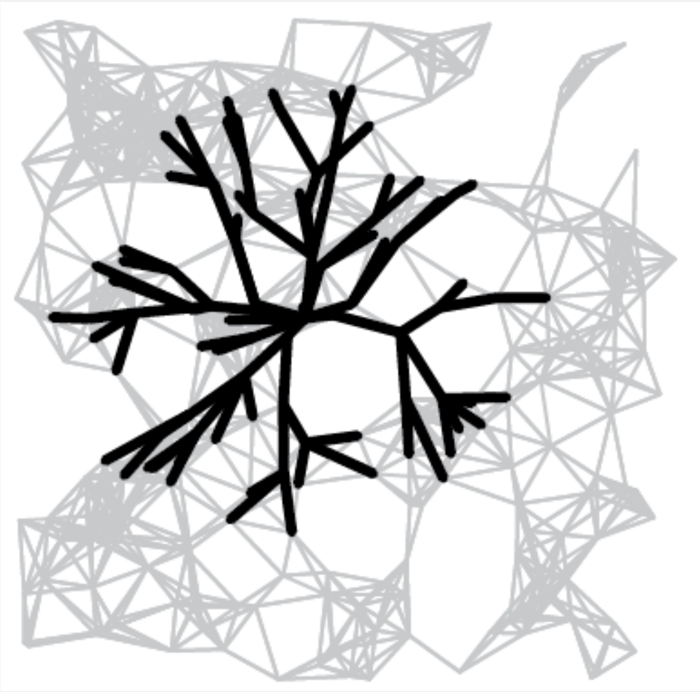
Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both in same family of algorithms.

Main distinction: rule used to choose next vertex for the tree.

- Prim: Closest vertex to the tree (via an undirected edge).
- Dijkstra: Closest vertex to the source (via a directed path).

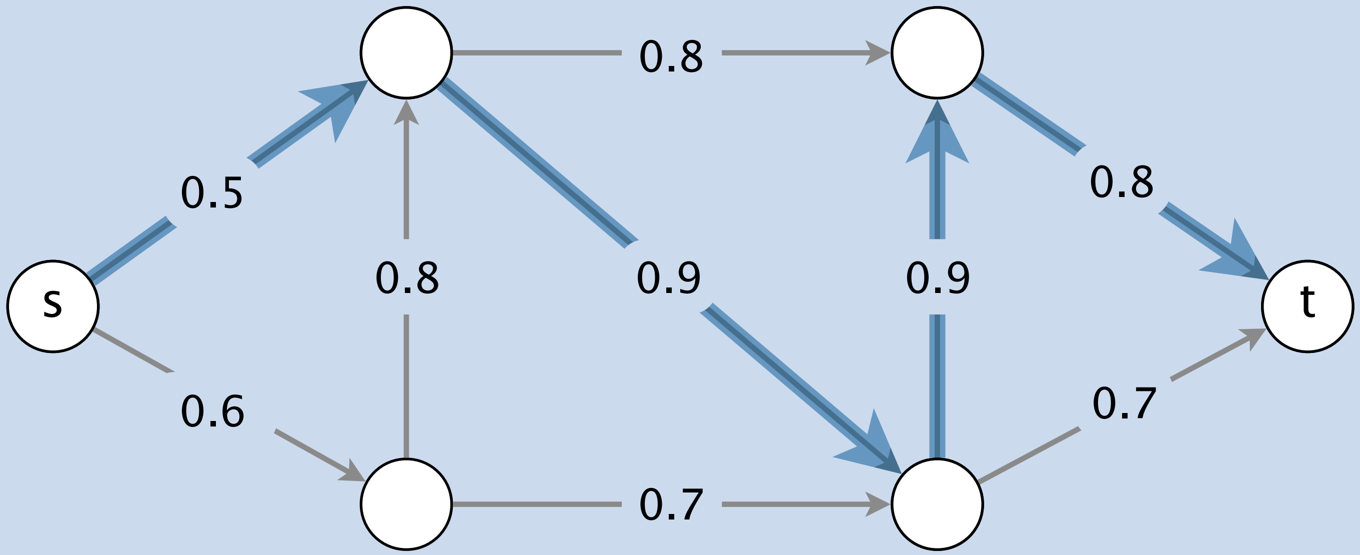




Note: DFS and BFS are also in same family.

Most Reliable Path

Goal. Given a digraph G, let $p(e)$ be the probability that edge e succeeds. Find a path from s to t that maximizes the probability of success (assuming edge failures are independent).



probability = 0.5 × 0.9 × 0.9 × 0.8 = 0.324

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4.4 Shortest Paths

- *APIs*
- *properties*
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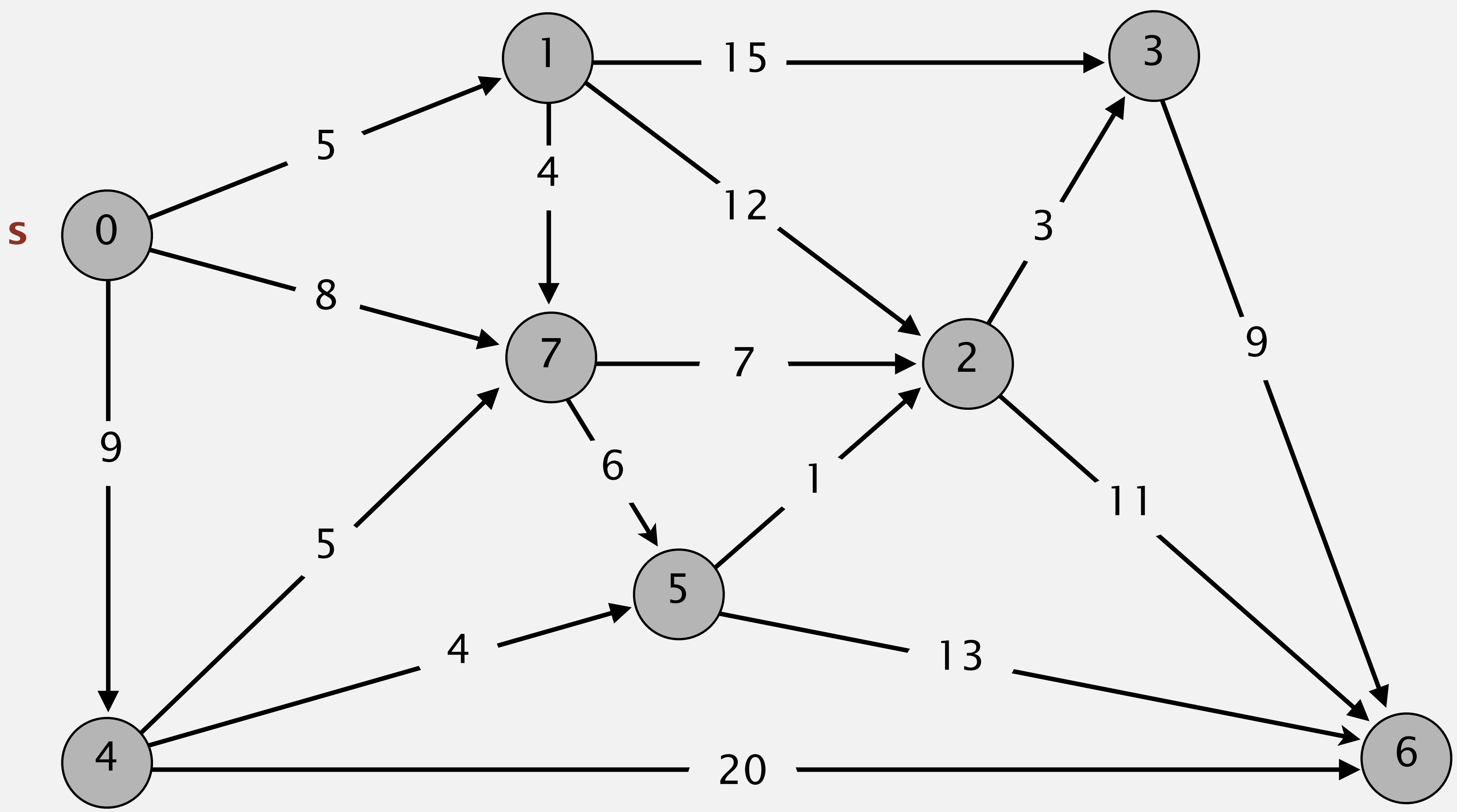
- *Dijkstra's algorithm*
- *topological sort algorithm*

Edge-weighted DAGs

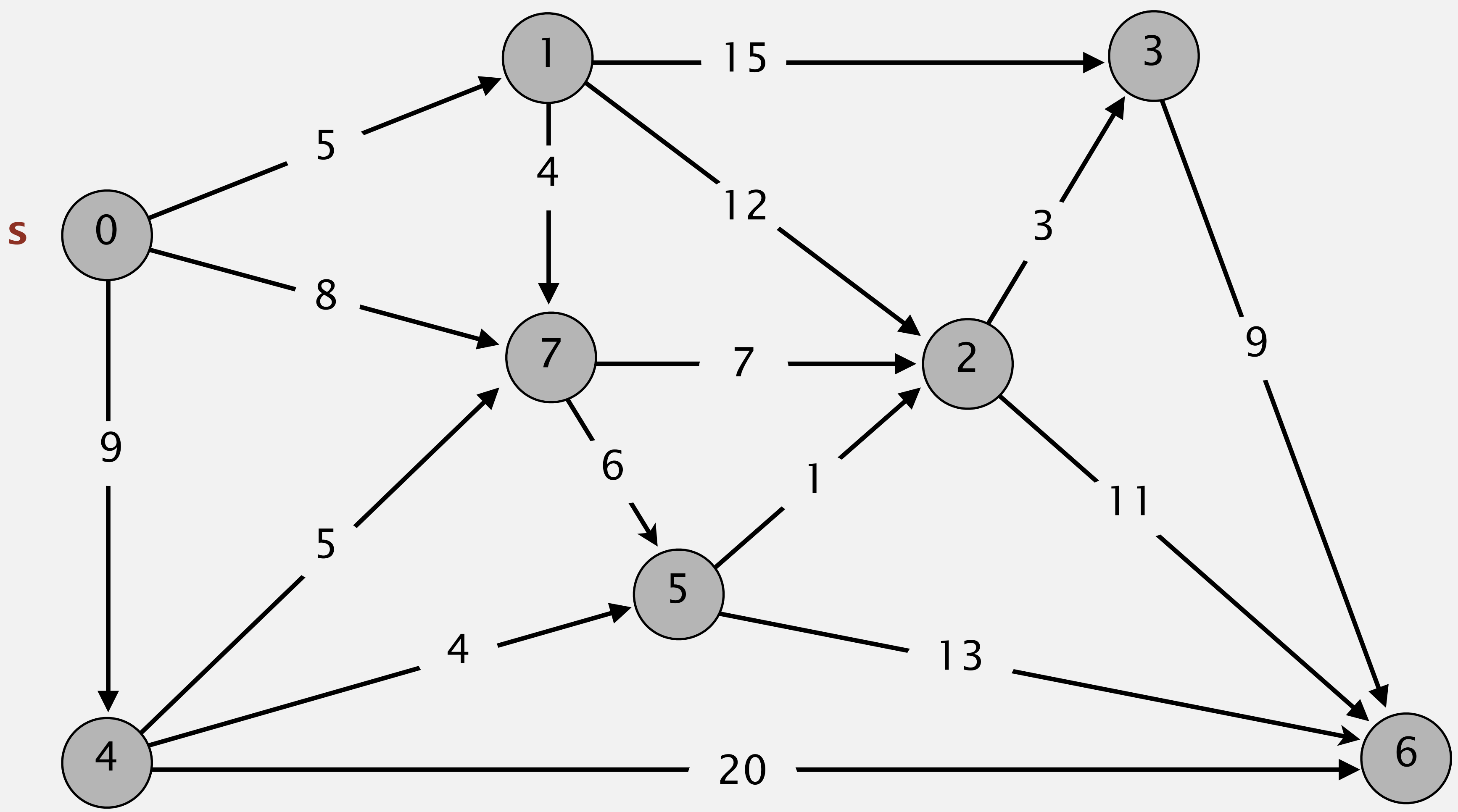
Q. Suppose that an edge-weighted digraph has no directed cycles.
Is it easier to find shortest paths than in a general digraph?



A. Yes!

Topological sort algorithm demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.



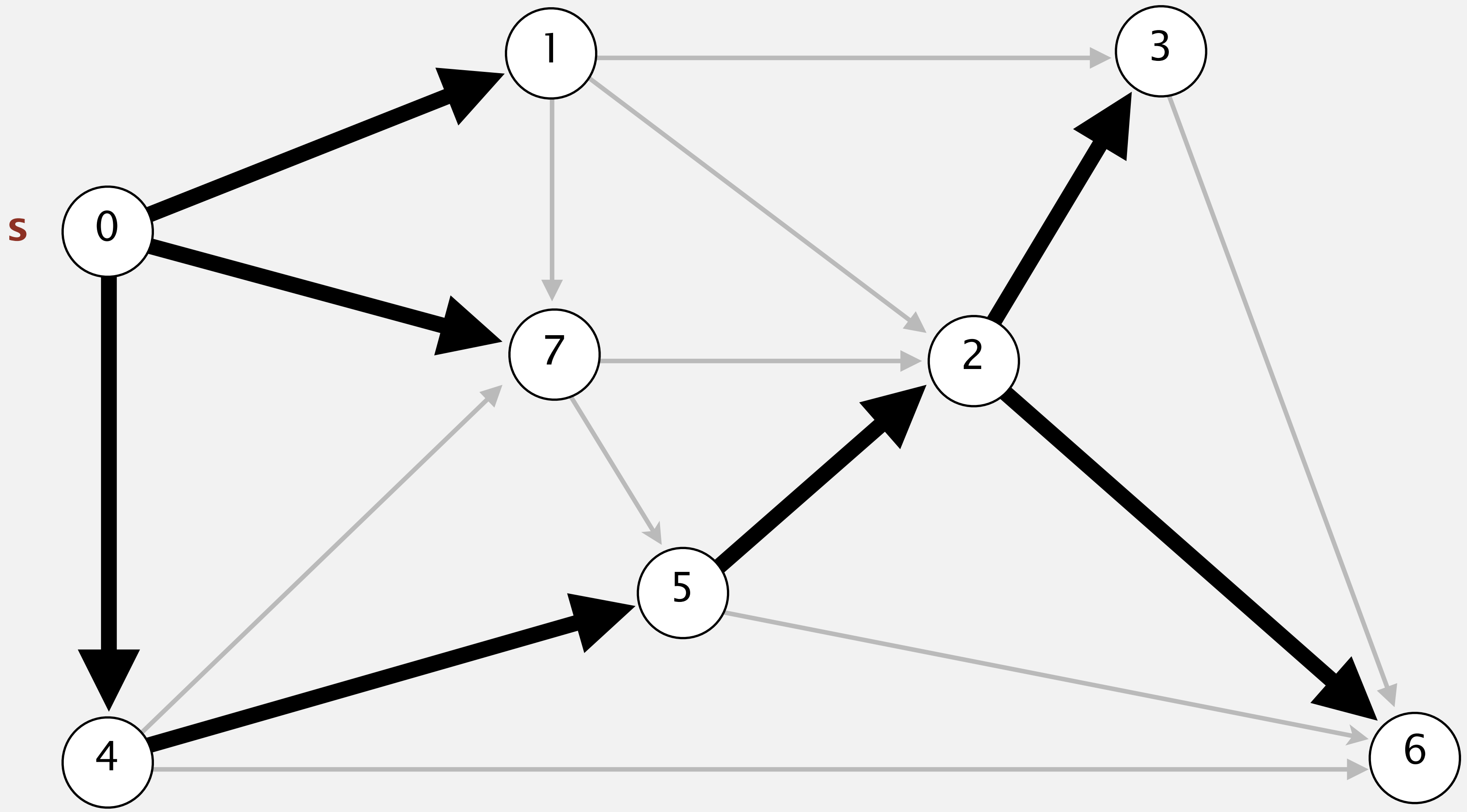
an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Topological sort algorithm demo

- Consider vertices in topological order.
- Relax all edges adjacent from that vertex.

0 1 4 7 5 2 3 6



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   public AcyclicSP(EdgeWeightedDigraph G, int s)
   {
      edgeTo = new DirectedEdge[G.V()];
      distTo = new double[G.V()];

      for (int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      Topological topological = new Topological(G);
      for (int v : topological.order())
         for (DirectedEdge e : G.adj(v))
            relax(e);

   }
}
```

topological order

What is the order of the running time of the topological sort algorithm for computing a SPT in an edge-weighted DAG?

A. V

B. E

C. $V + E$

D. $V \log E$

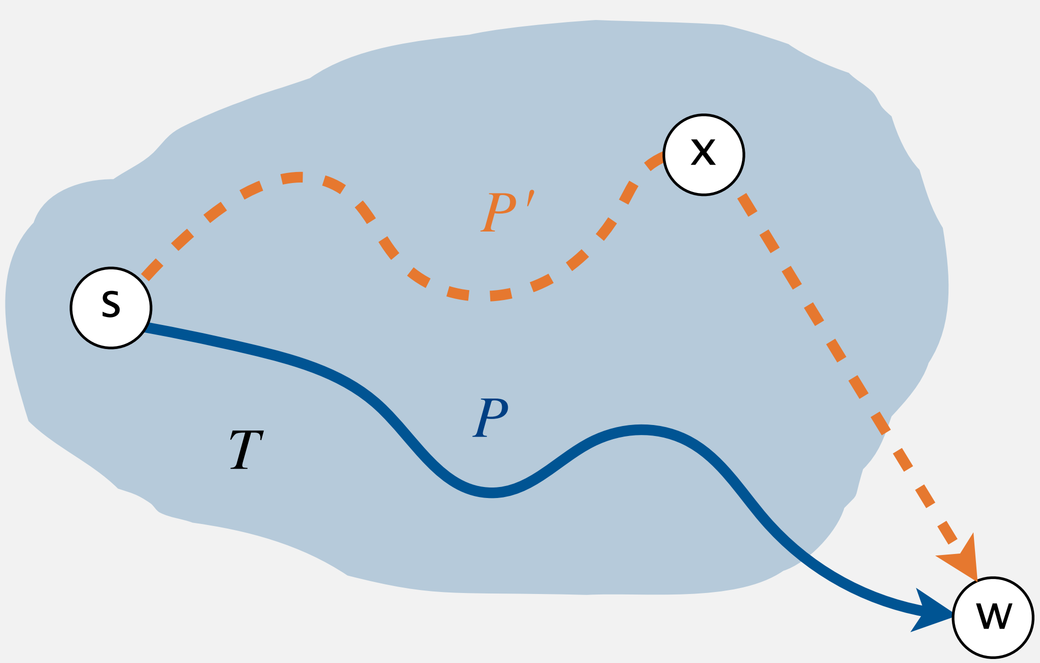
Topological sort algorithm: correctness proof

Invariant. For each vertex v in T, `distTo[v]` $= d^*(v)$.

length of shortest path from s to v

Pf. [by induction on $|T|$]

- Let w be next vertex (in topological order) added to T.
- Let P be the $s \to w$ path of length `distTo[w]`.
- Consider any other $s \to w$ path P'.
- P' must be a path to a vertex in T plus one extra edge, say $x \to w$. Why?
- P' is no shorter than P :



by construction

$$\text{length}(P) = \texttt{distTo[w]}$$

relax vertex x → $\leq \texttt{distTo[x]} + \mathit{weight}(x, w)$

induction → $= d^*(x) + \mathit{weight}(x, w)$

P' is some path to x plus edge $x \to w$ → $\leq \text{length}(P')$ ▪

Content-aware resizing

Seam carving. [Avidan–Shamir] Resize an image without distortion for display on cell phones and web browsers.

http://www.youtube.com/watch?v=vIFCV2spKtg

Content-aware resizing

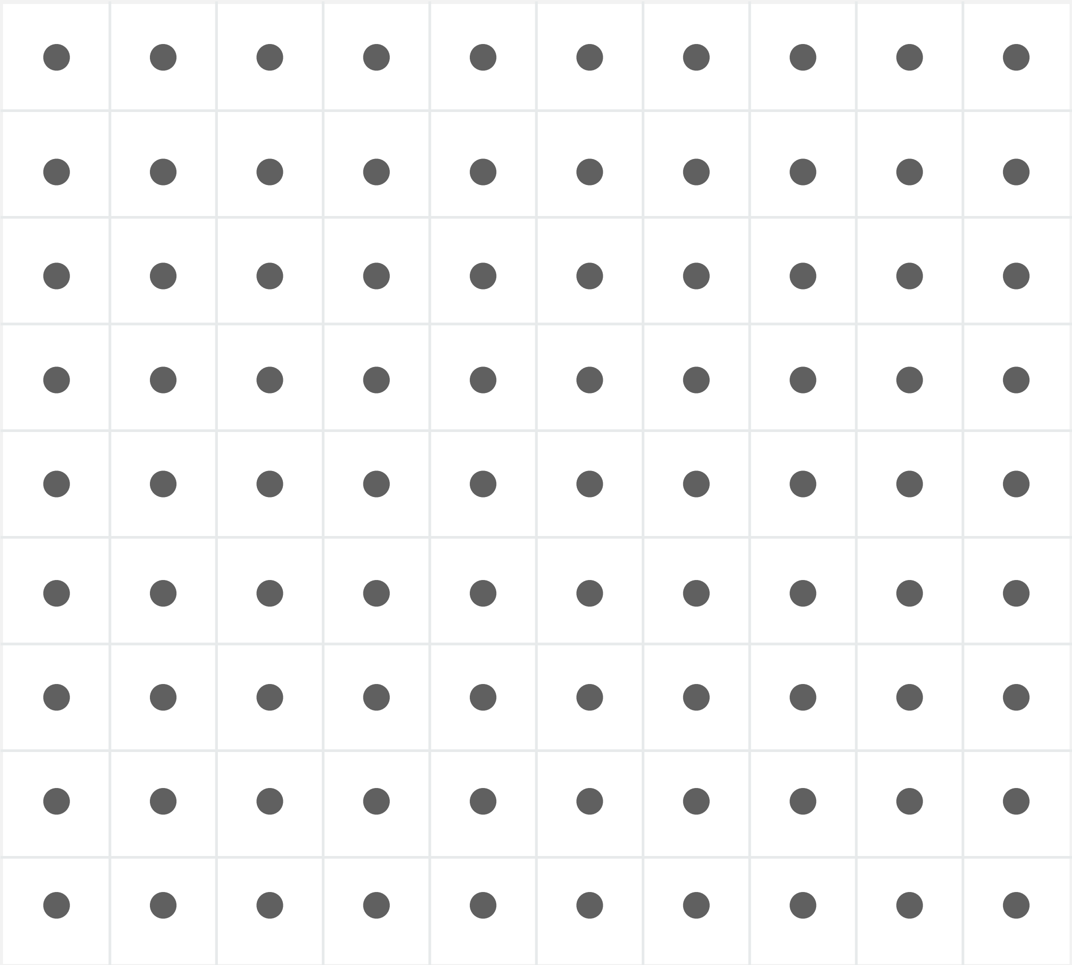
Seam carving. [Avidan–Shamir] Resize an image without distortion for display on cell phones and web browsers.

In the wild. Photoshop, Imagemagick, GIMP, ...

Content-aware resizing

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = "energy function" of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

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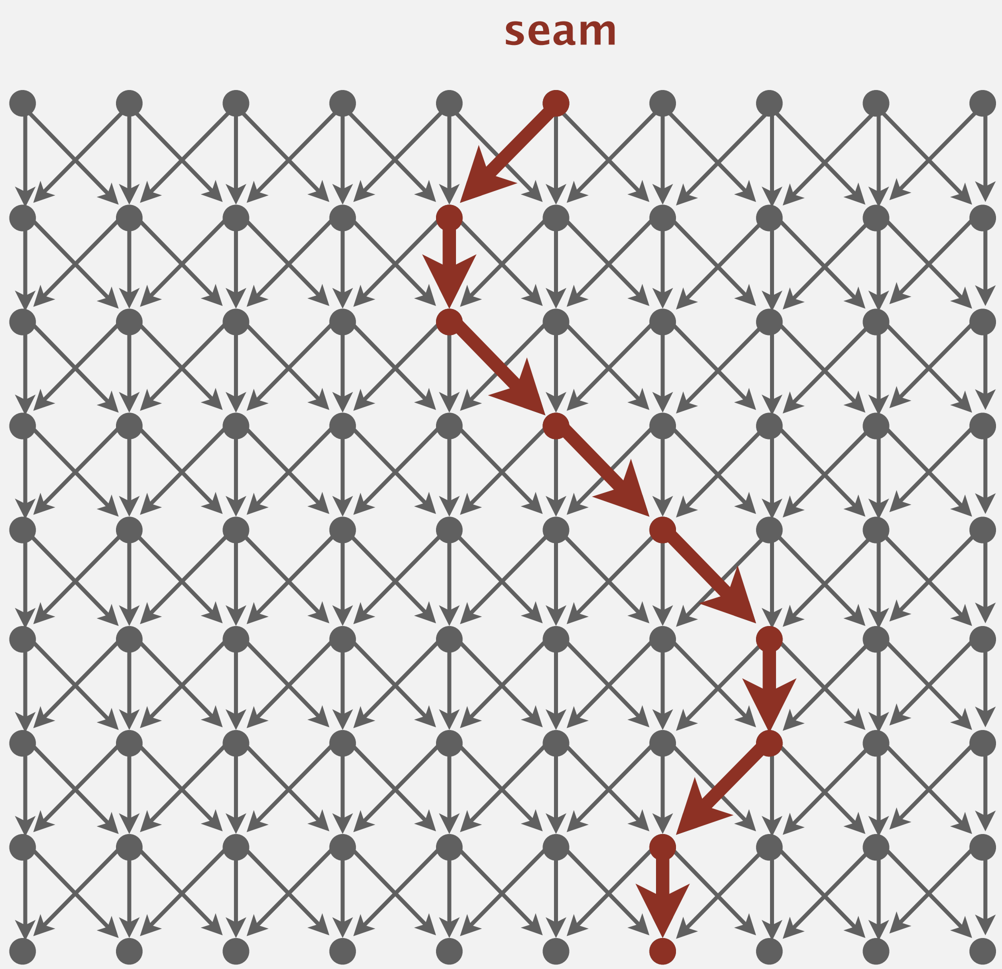
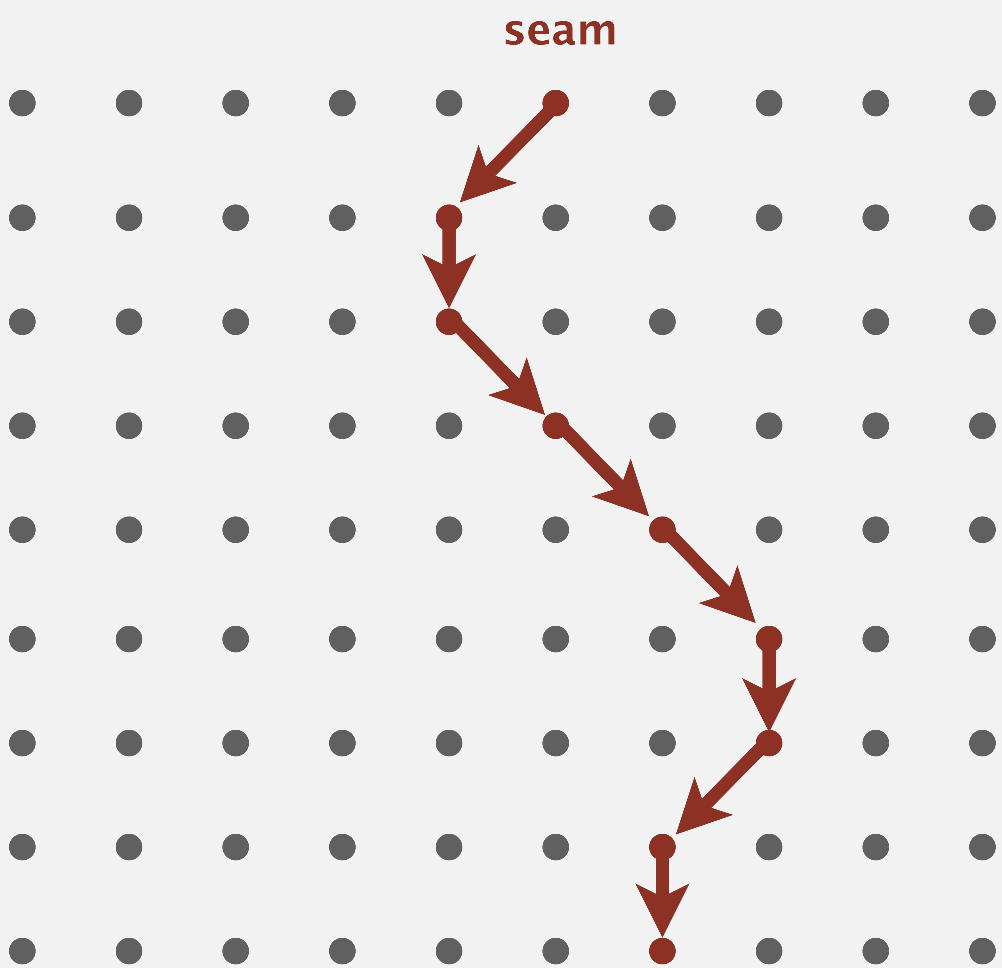


Content-aware resizing

To remove vertical seam:

- Delete pixels on seam (one in each row).



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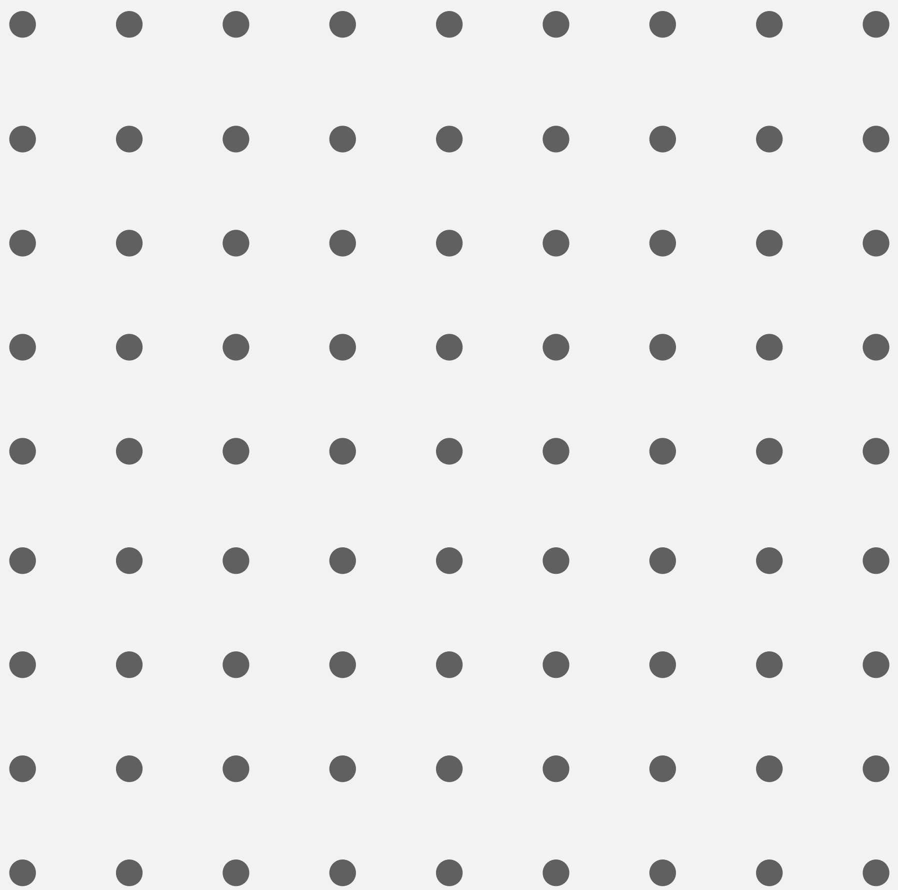
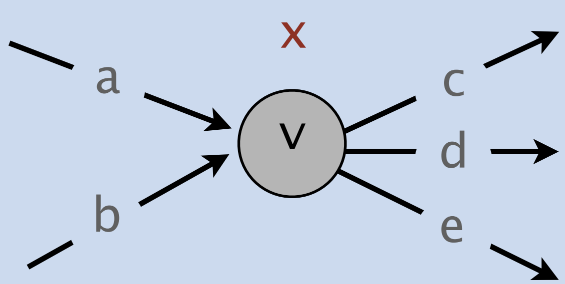
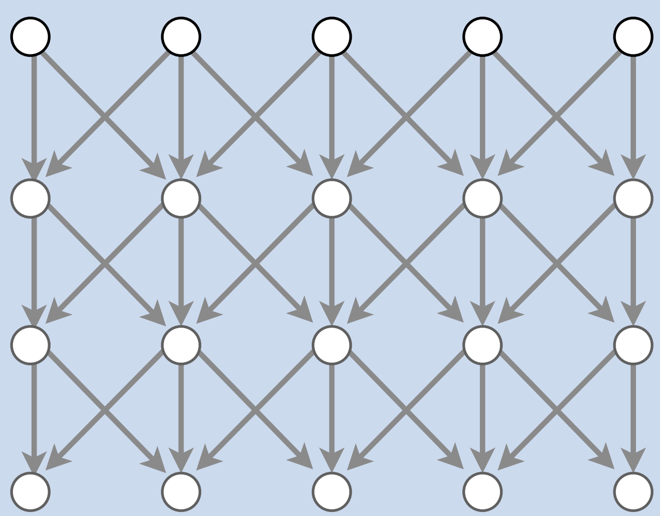


Shortest Path Variants in a Digraph

Q1. How to model vertex weights (along with edge weights)?



Q2. How to model multiple sources and sinks?

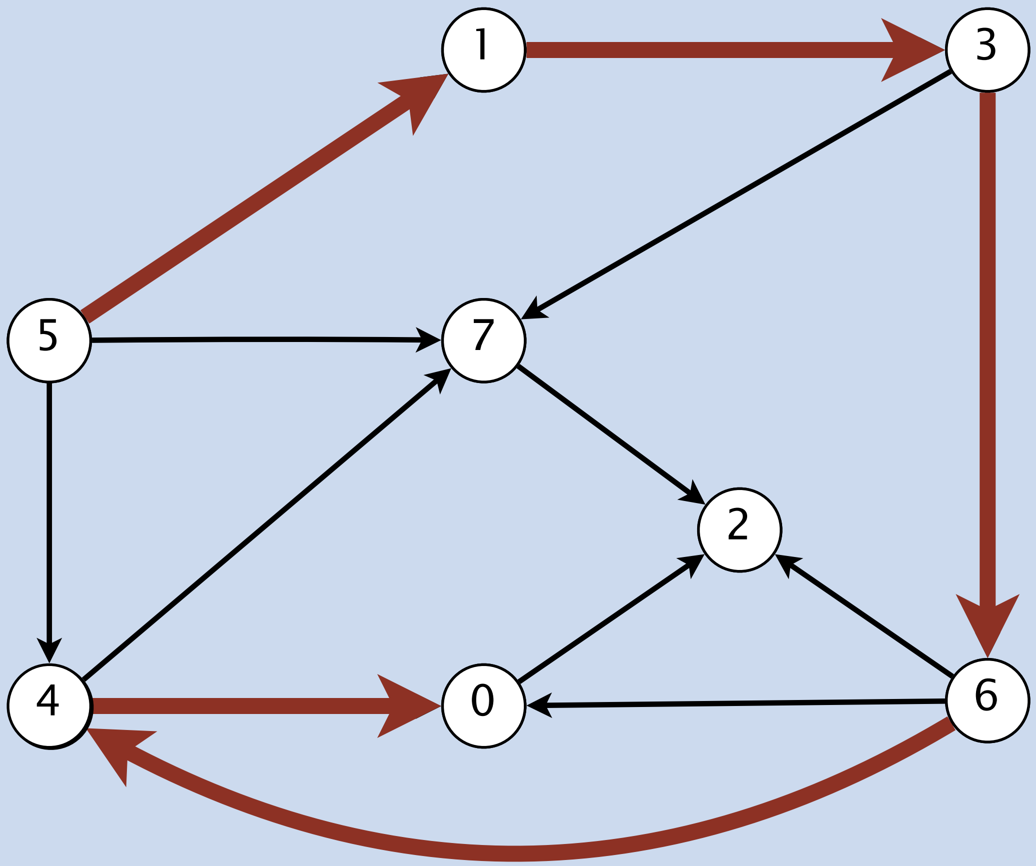


LONGEST PATH IN A DAG

Challenge. Given an edge-weighted DAG, find the longest path from s to every other vertex.

longest paths input

```
5->4  0.35
4->7  0.37
5->7  0.28
5->1  0.32
4->0  0.38
0->2  0.26
3->7  0.39
1->3  0.29
7->2  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



longest path from 5 to 0

(0.32 + 0.29 + 0.52 + 0.93 + 0.38 = 2.44)

Algorithm for shortest paths

Variations on a theme: vertex relaxations.

- Bellman–Ford: relax all vertices; repeat $V - 1$ times.
- Dijkstra: relax vertices in order of distance from s.
- Topological sort: relax vertices in topological order.

algorithm	worst-case running time	negative weights †	directed cycles
Bellman-Ford	$E\ V$	✔	✔
Dijkstra	$E \log V$		✔
topological sort	E	✔	

† no negative cycles