http://algs4.cs.princeton.edu

3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees (see book or videos)*

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
goal	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

Challenge. Guarantee performance.

optimized for teaching and coding;
introduced to the world in this course!

This lecture. 2–3 trees and left-leaning red–black BSTs.

http://algs4.cs.princeton.edu

3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

2–3 tree

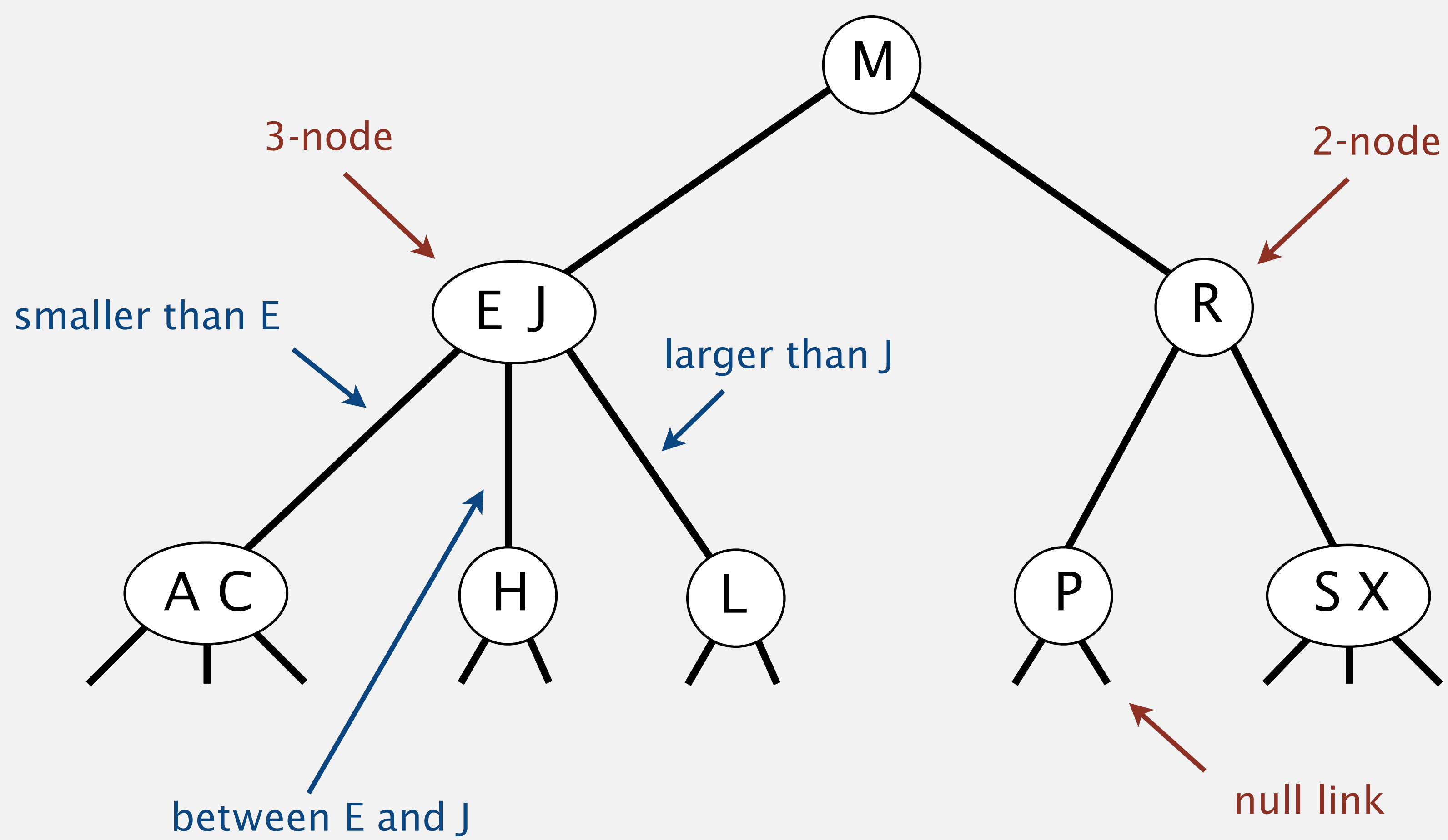
Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.

how to maintain?

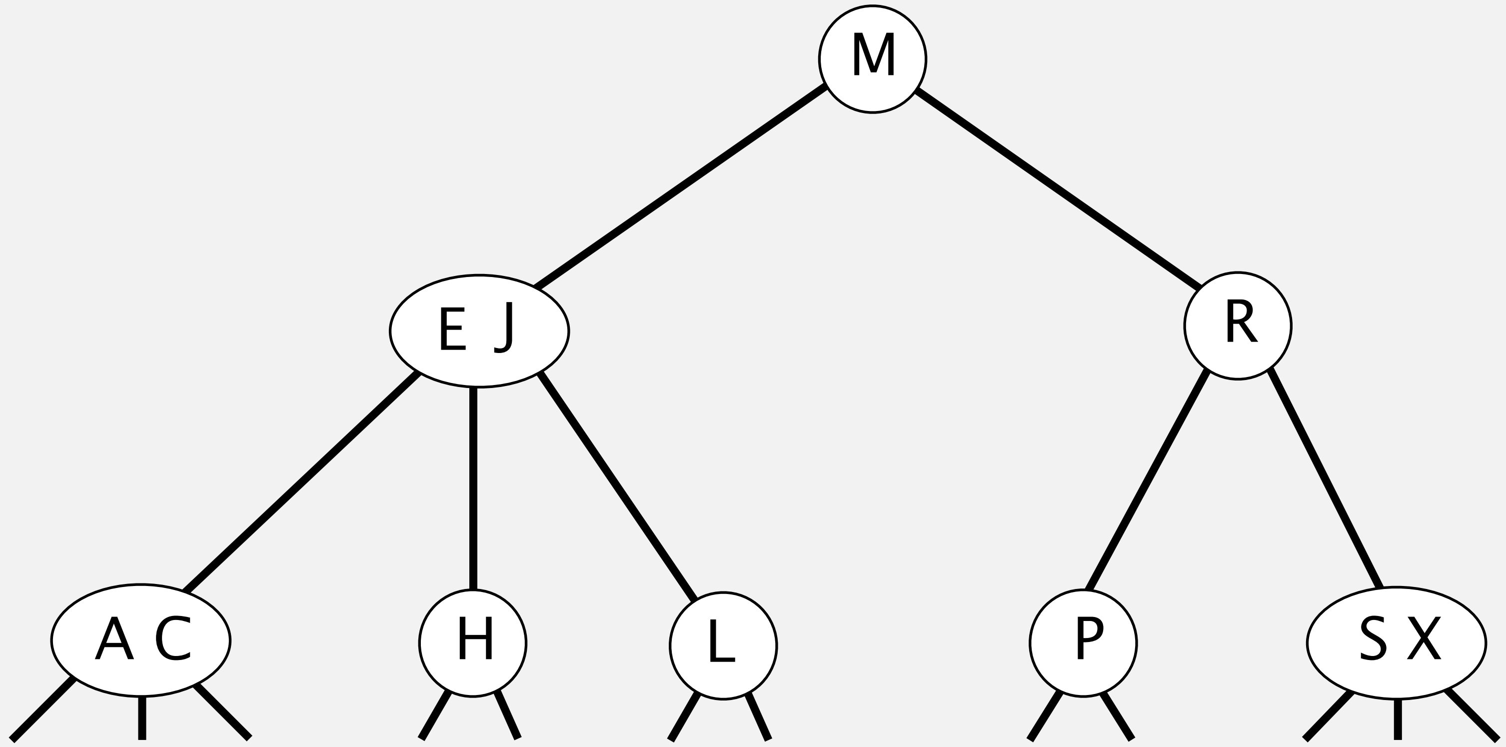


2-3 tree demo

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

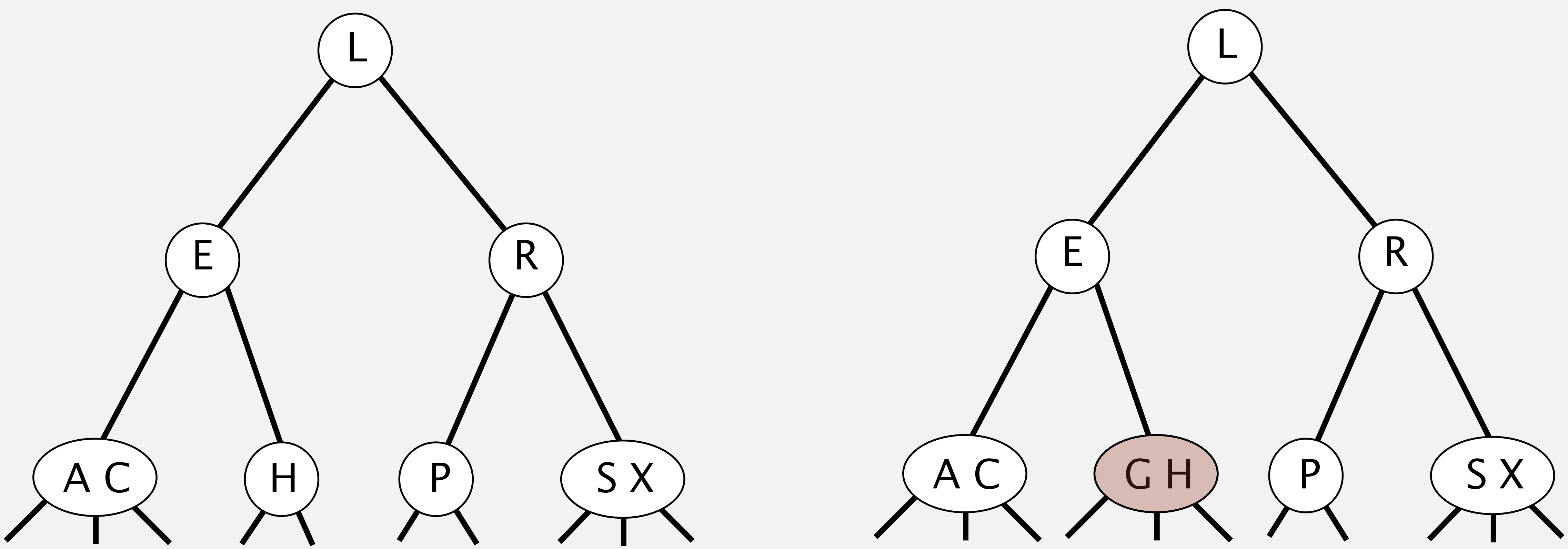


2–3 tree: insertion

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G

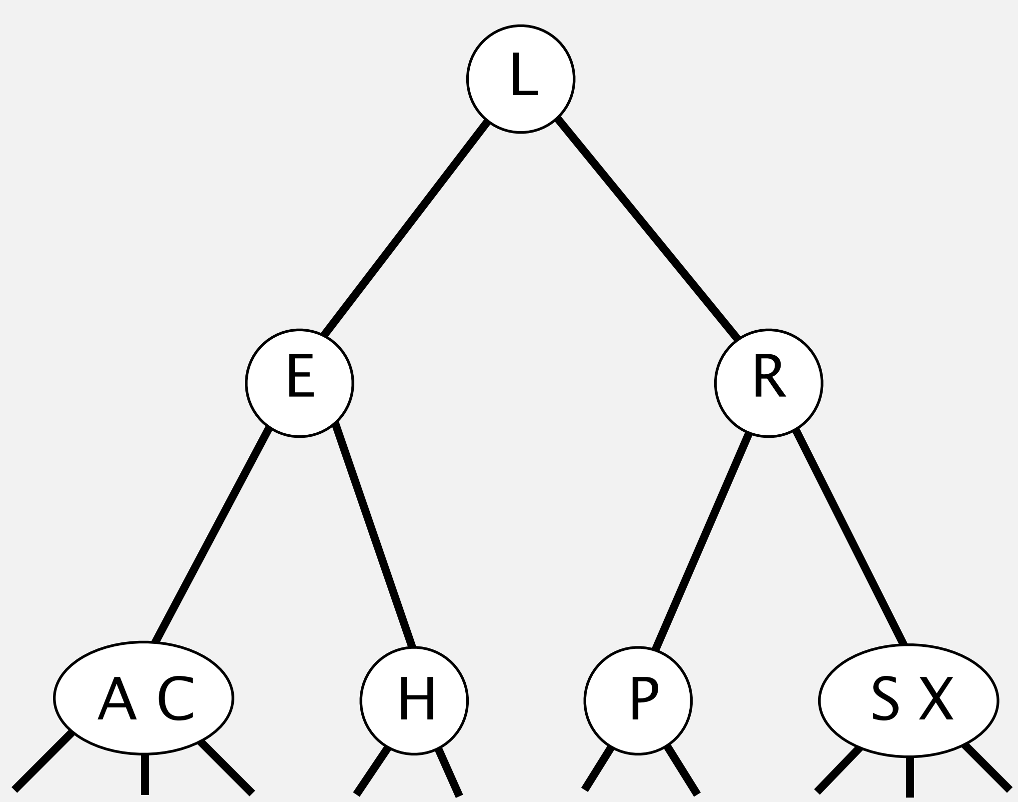


2–3 tree: insertion

Insertion into a 3-node at bottom.

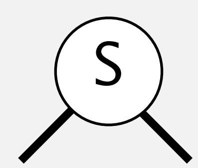
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert Z



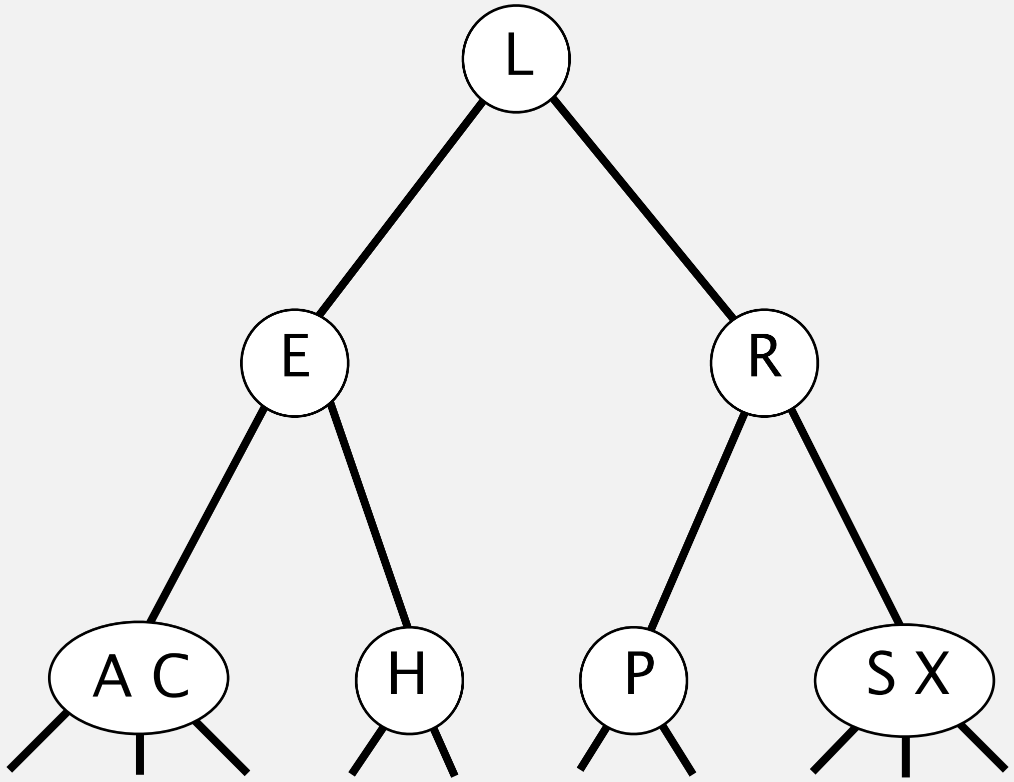
2–3 tree construction demo

insert S



2–3 tree construction demo

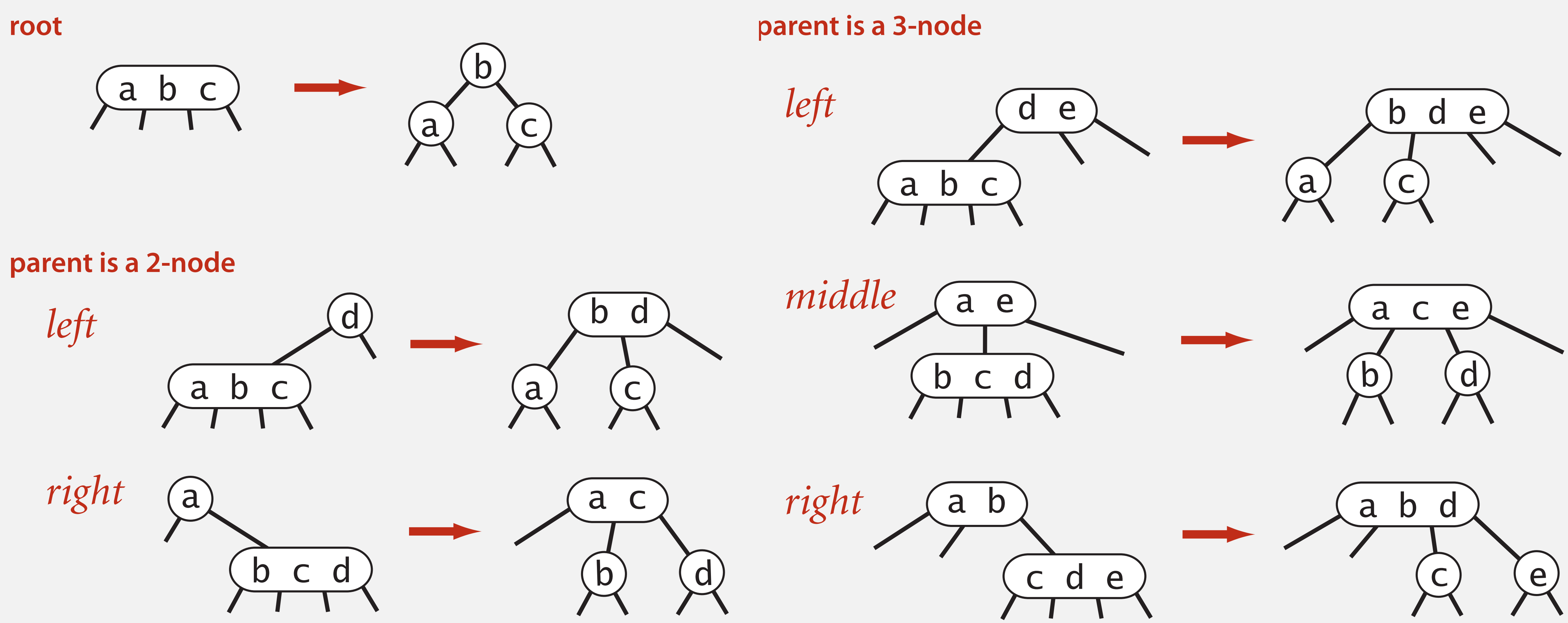
2–3 tree



2-3 tree: global properties

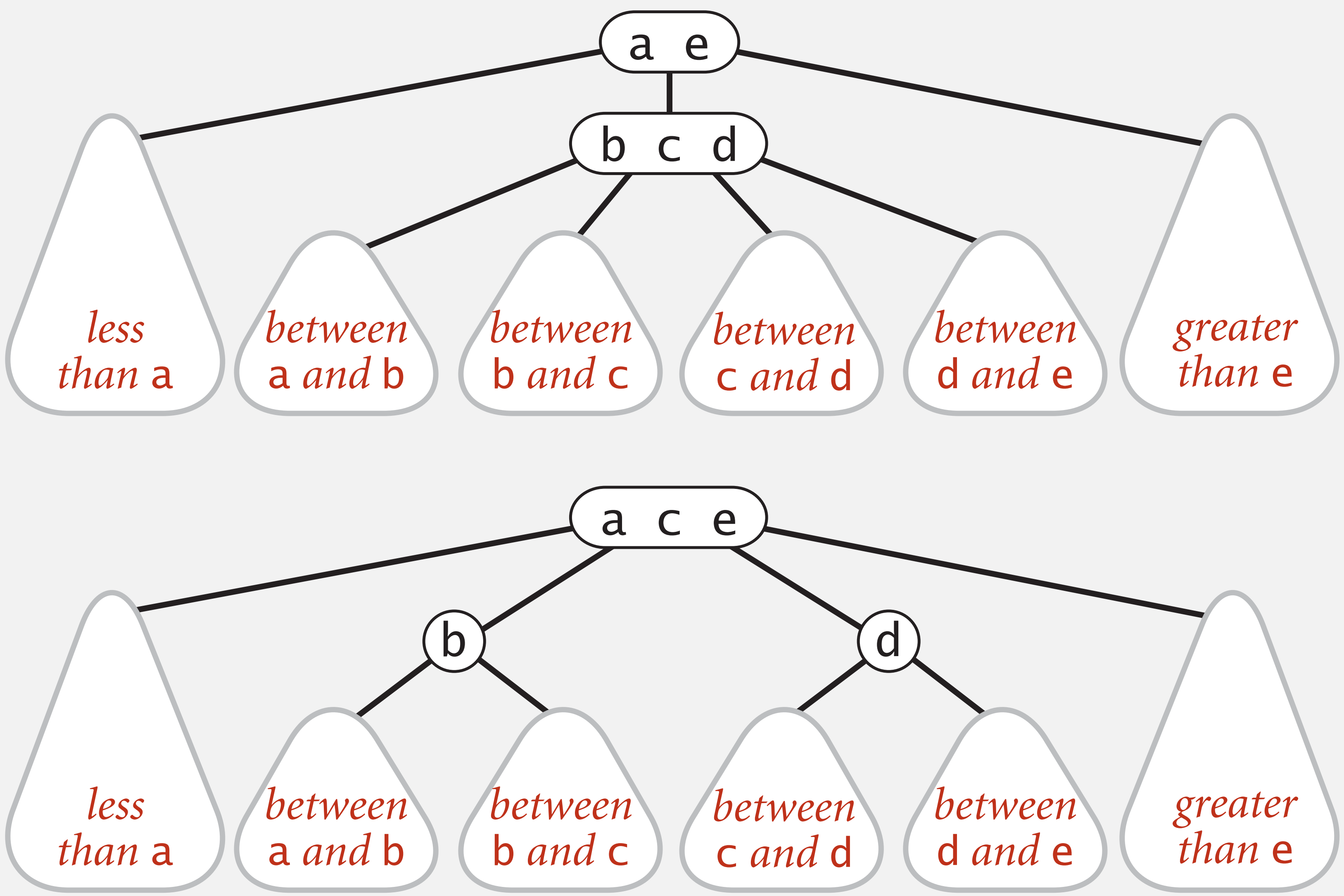
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.



2-3 tree: performance

Splitting a 4-node is a local transformation: constant number of operations.



Balanced search trees: quiz 1

What is the maximum height of a 2–3 tree with n keys?

A. $\sim \log_3 n$

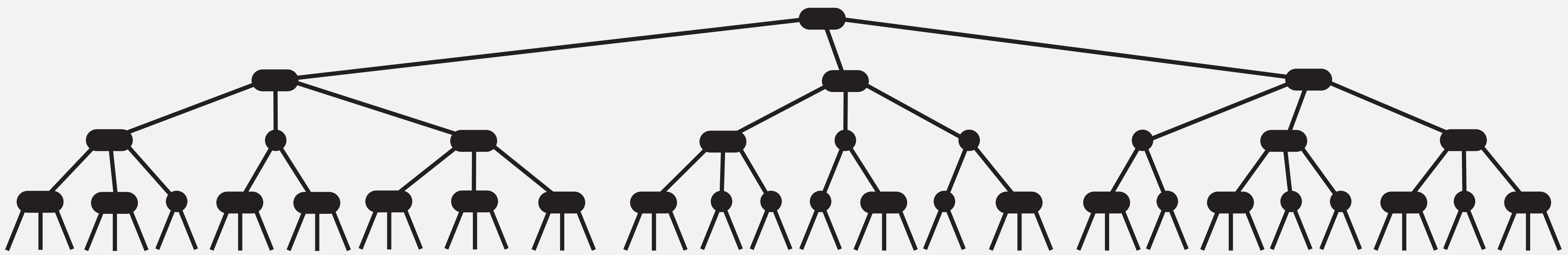
B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

D. $\sim n$

2–3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg n$. [all 2-nodes]
- Best case: $\log_3 n \approx .631 \lg n$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

but hidden constant c is large
(depends upon implementation)

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

fantasy code

```
public void put(Key key, Value val)
{
   Node x = root;
   while (x.getTheCorrectChild(key) != null)
   {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it, but there's a better way.

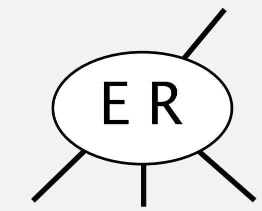
http://algs4.cs.princeton.edu

3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

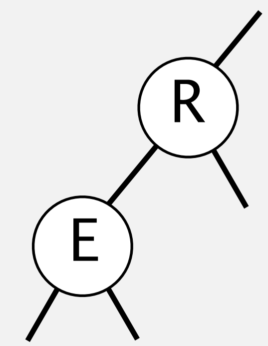
How to implement 2–3 trees with binary trees?

Challenge. How to represent a 3 node?



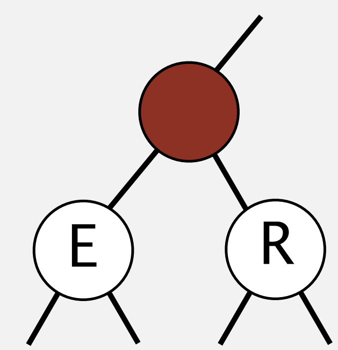
Approach 1. Regular BST.

- No way to tell a 3-node from a 2-node.
- Cannot map from BST back to 2–3 tree.



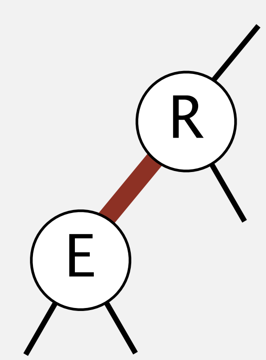
Approach 2. Regular BST with red "glue" nodes.

- Wastes space for extra node.
- Code probably messy.



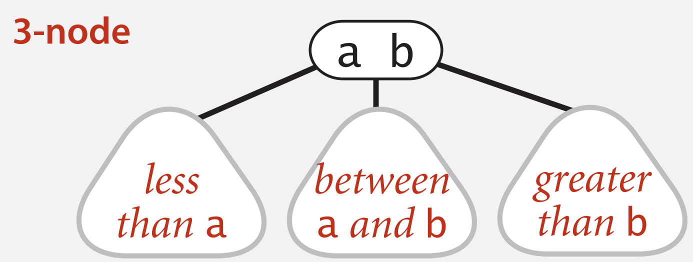
Approach 3. Regular BST with red "glue" links.

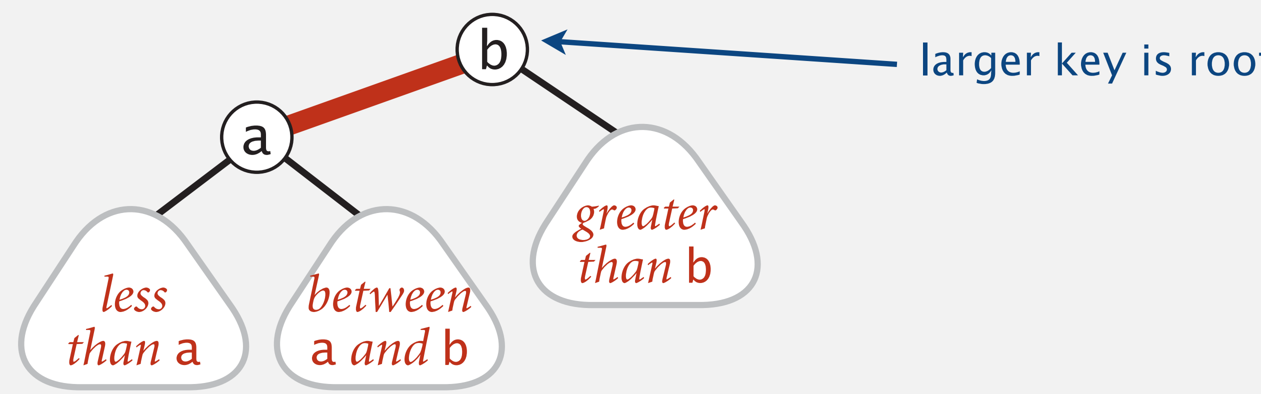
- Widely used in practice.
- Arbitrary restriction: red links lean left.

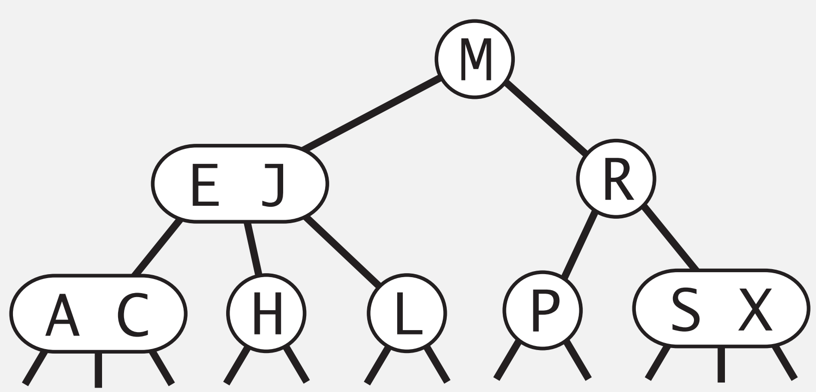


Left-leaning red–black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

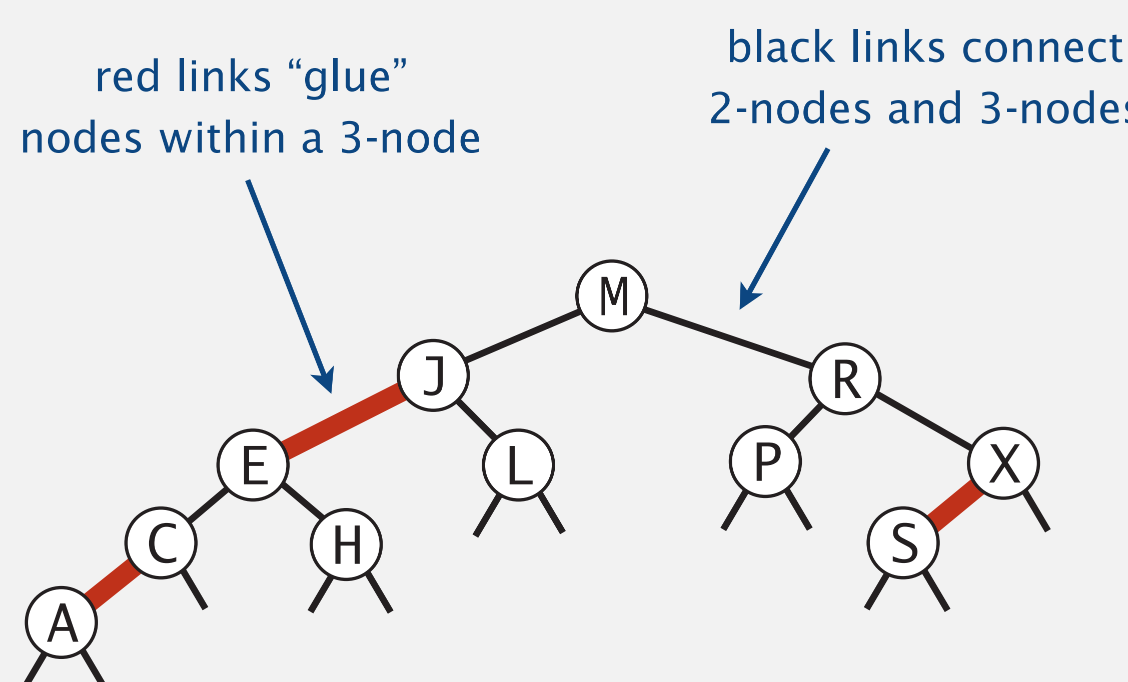
1. Represent 2–3 tree as a BST.
2. Use “internal” left-leaning links as “glue” for 3–nodes.







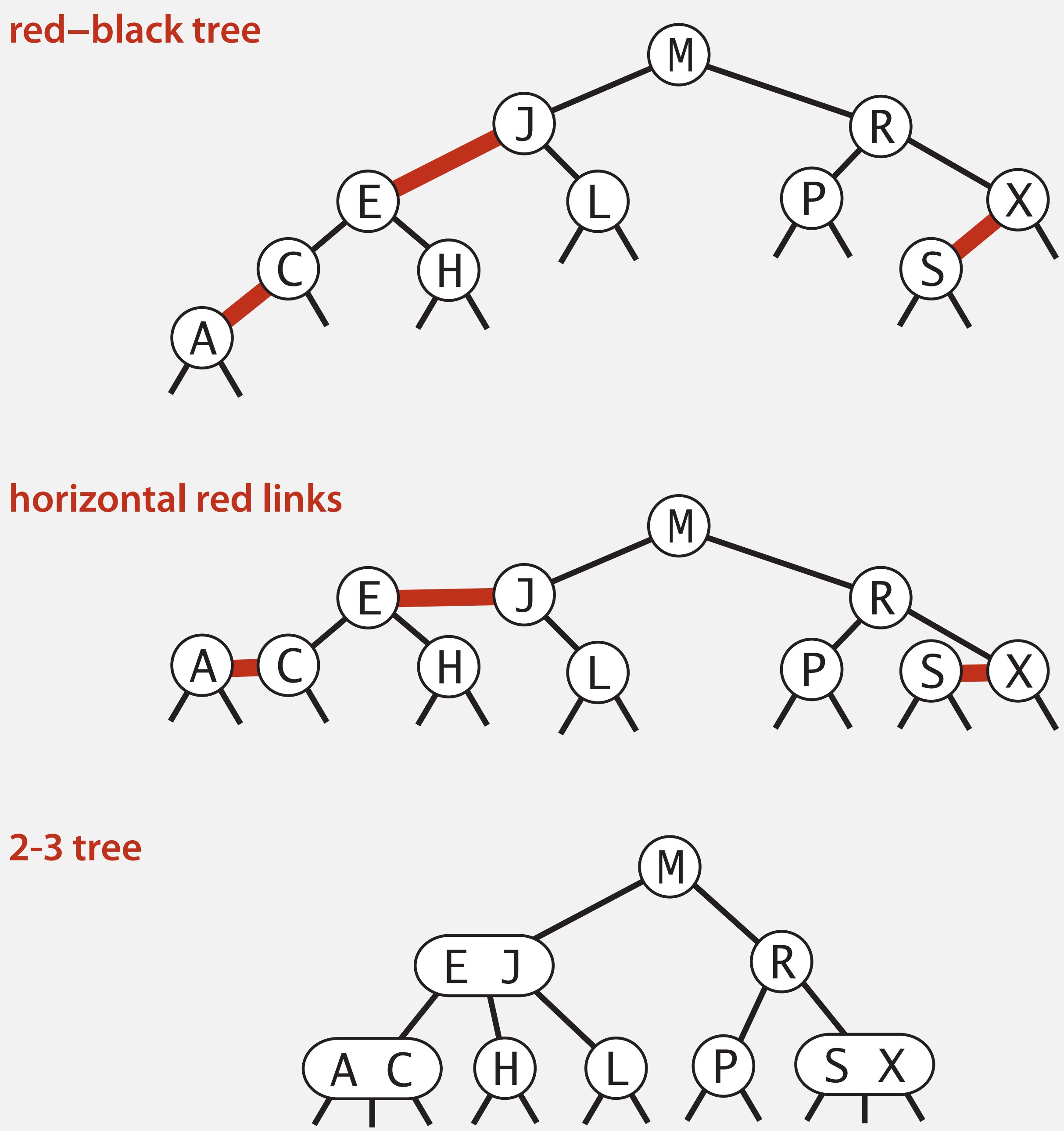
2–3 tree



corresponding red–black BST

Left-leaning red–black BSTs: 1–1 correspondence with 2–3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

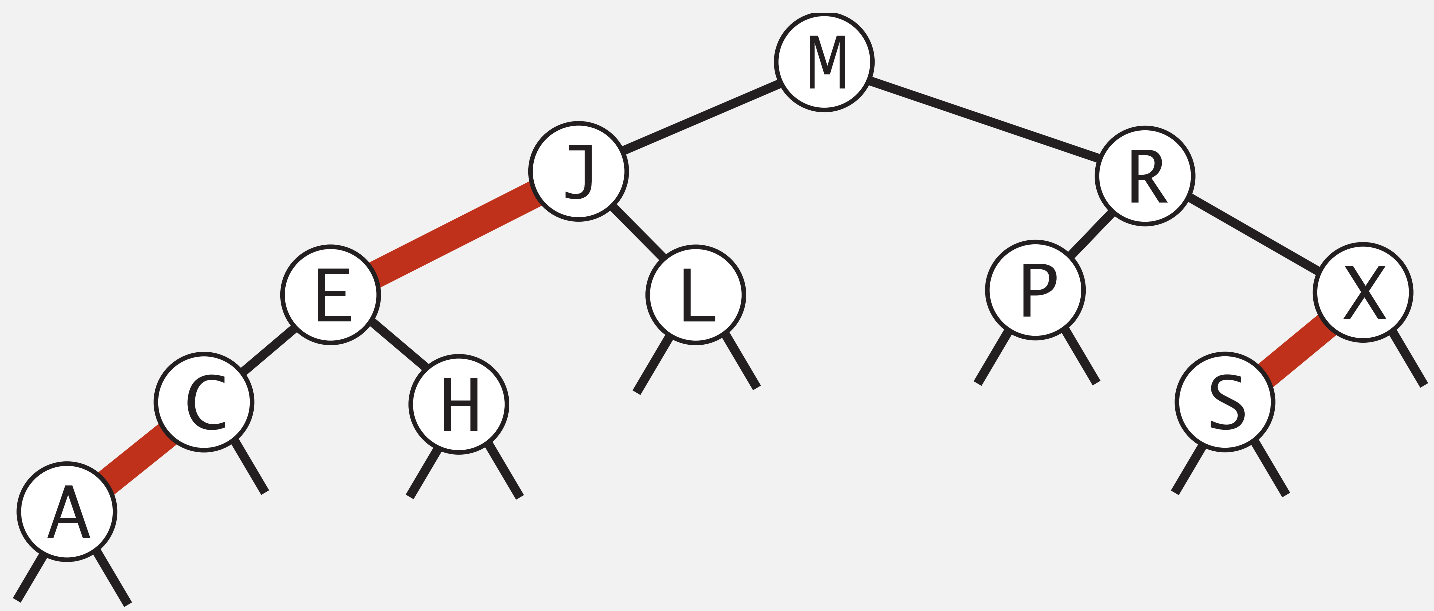


An equivalent definition

A BST such that:

- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

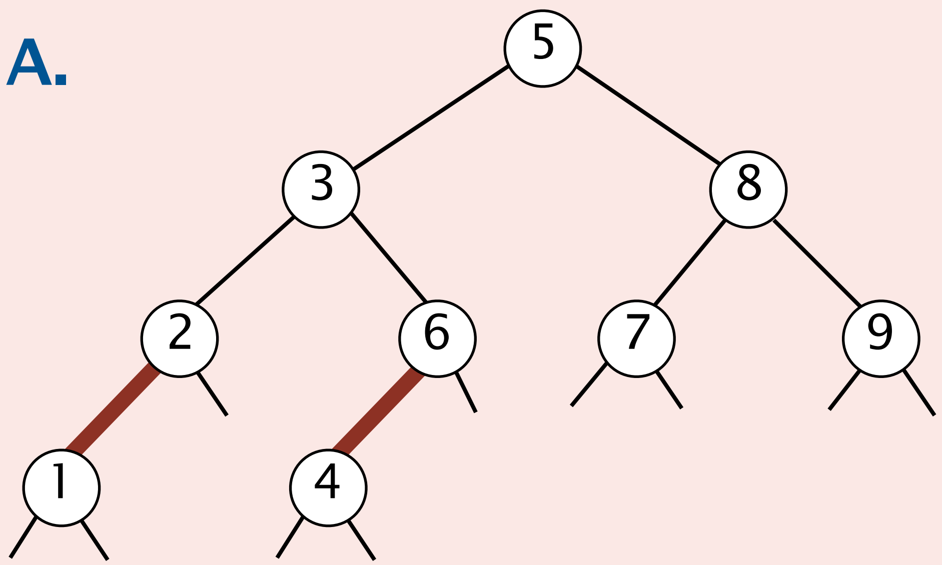
"perfect black balance"



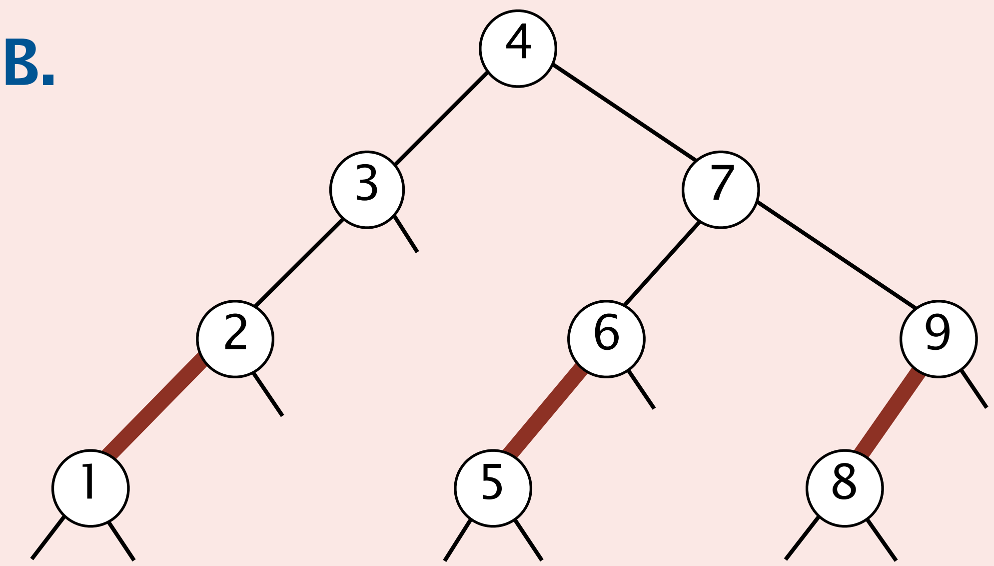
Balanced search trees: quiz 2

Which is a red–black BST?

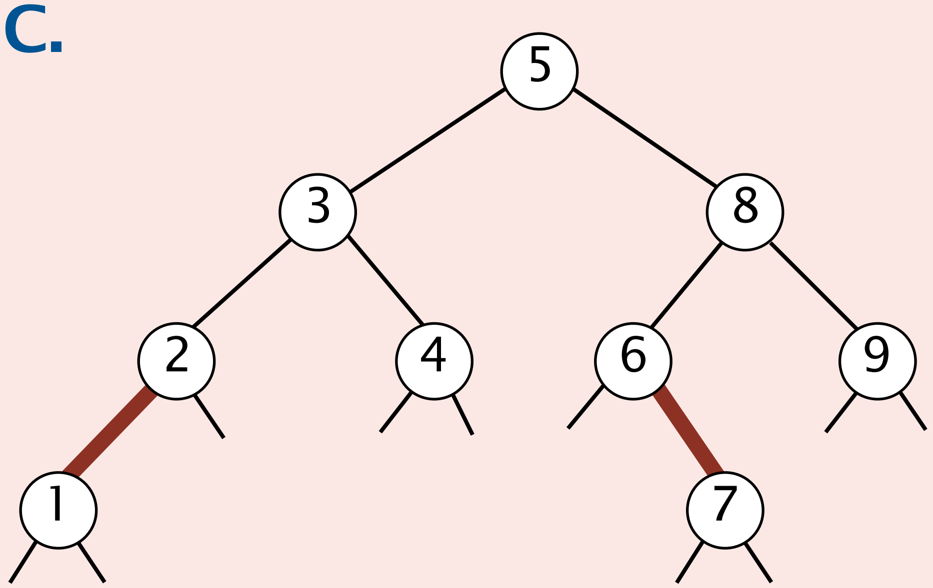
A.



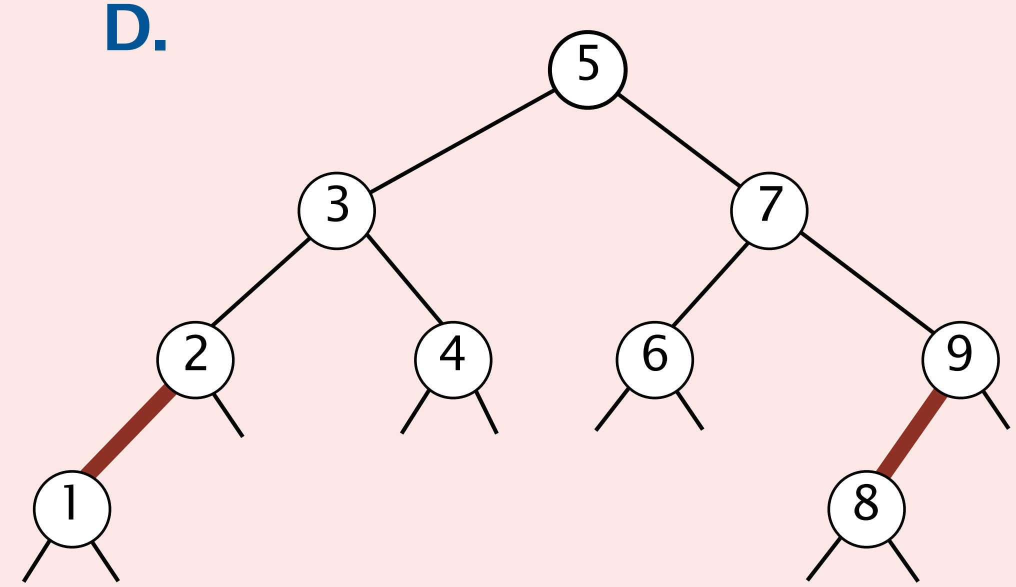
B.



C.



D.

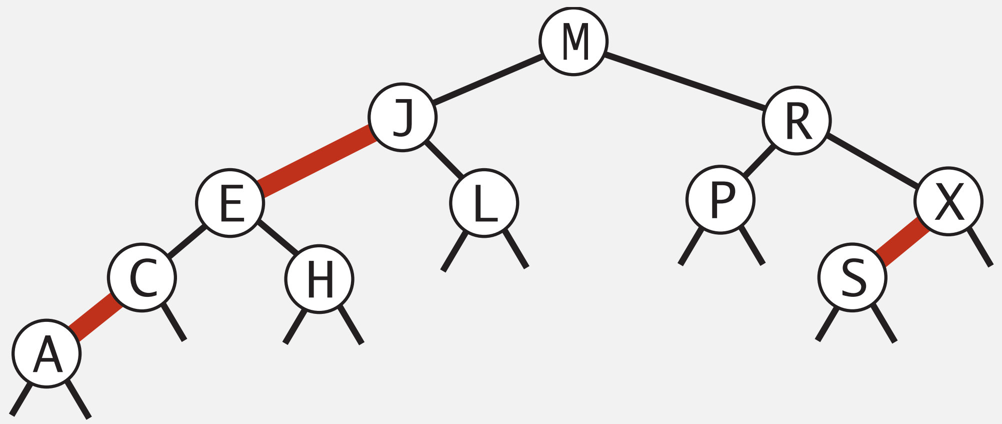


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
(because of better balance)

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Many other ops (floor, iteration, rank, selection) are also identical.

Red-black BST representation

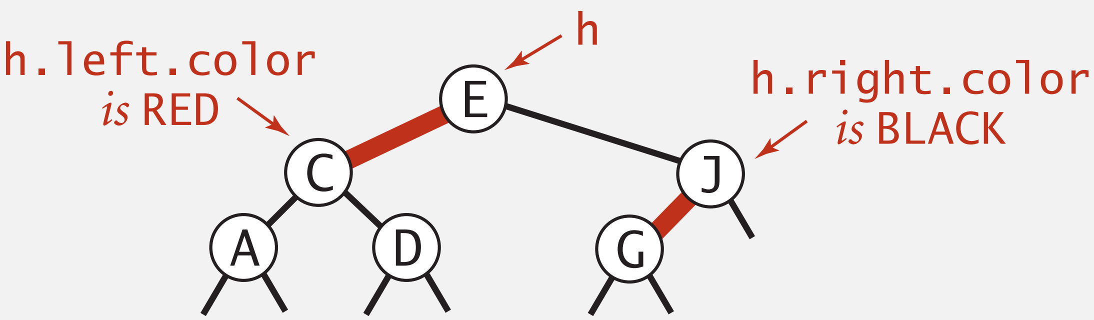
Each node is pointed to by precisely one link (from its parent) ⇒
can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



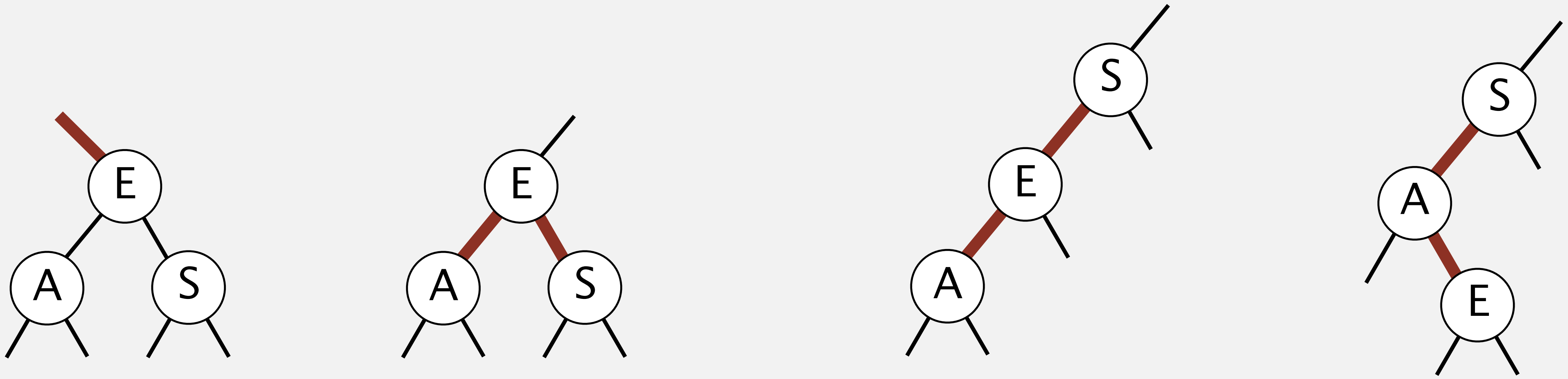
Insertion into a LLRB tree: overview

Basic strategy. Maintain 1–1 correspondence with 2–3 trees.

During internal operations, maintain:

- Symmetric order.
- Perfect black balance.

  [but not necessarily color invariants]

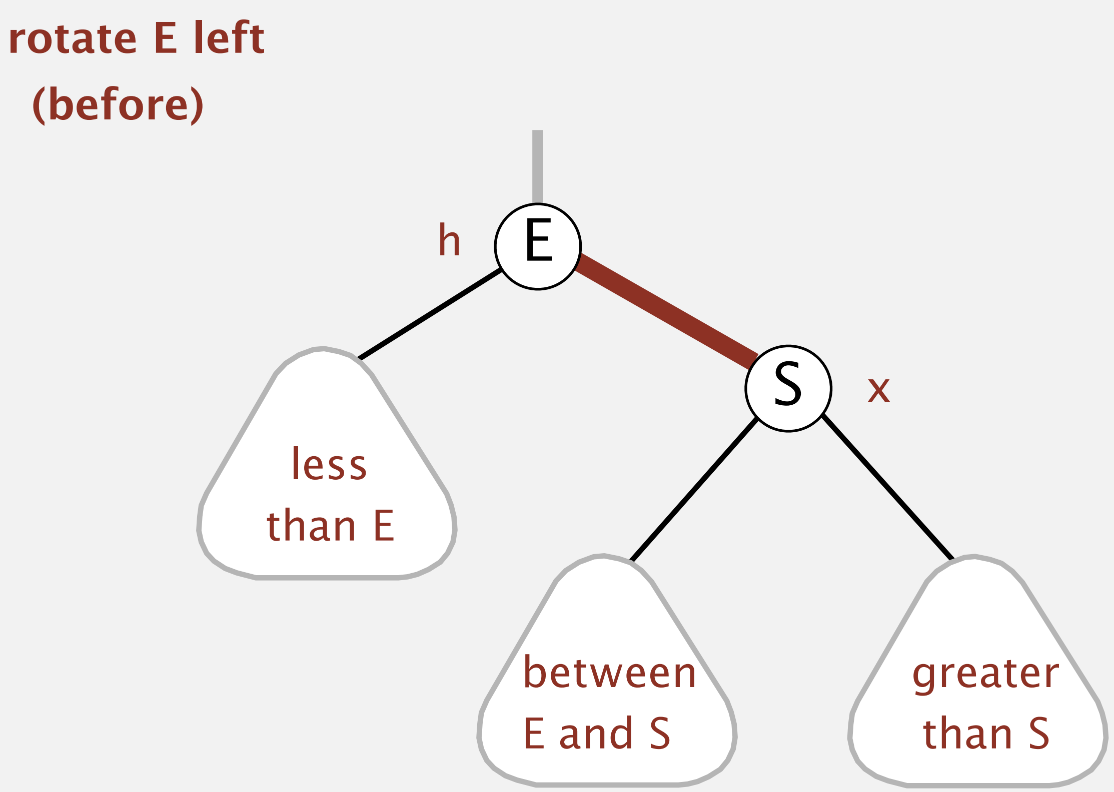


right–leaning red link | two red children (a temporary 4–node) | left–left red (a temporary 4–node) | left–right red (a temporary 4–node)

To restore color invariant: apply elementary ops (rotations and color flips).

Elementary red–black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

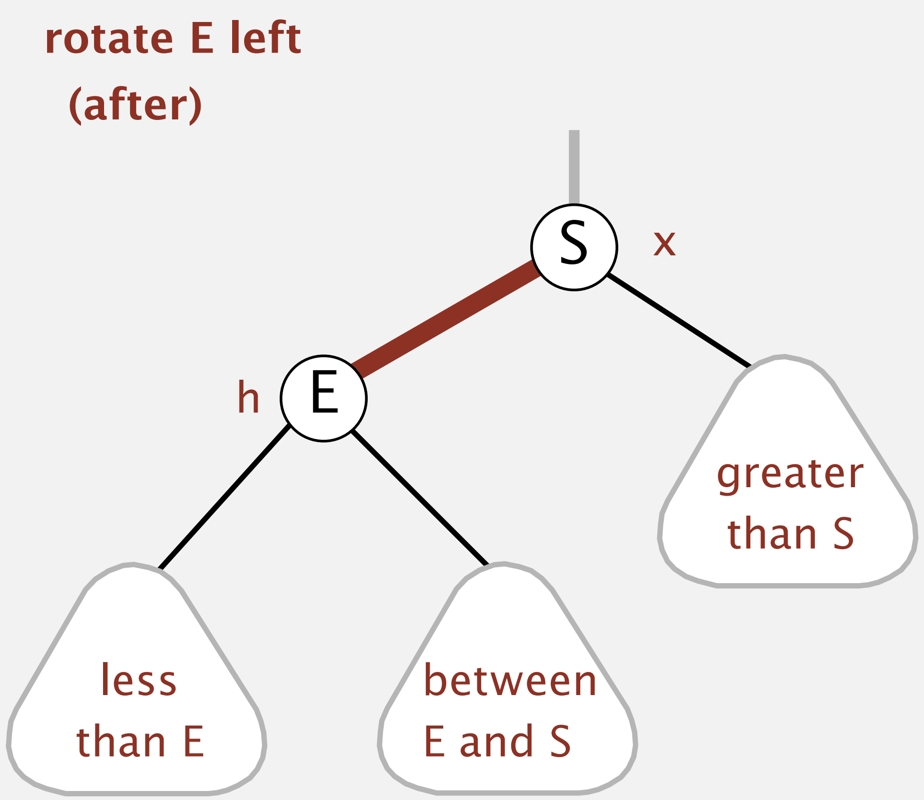



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

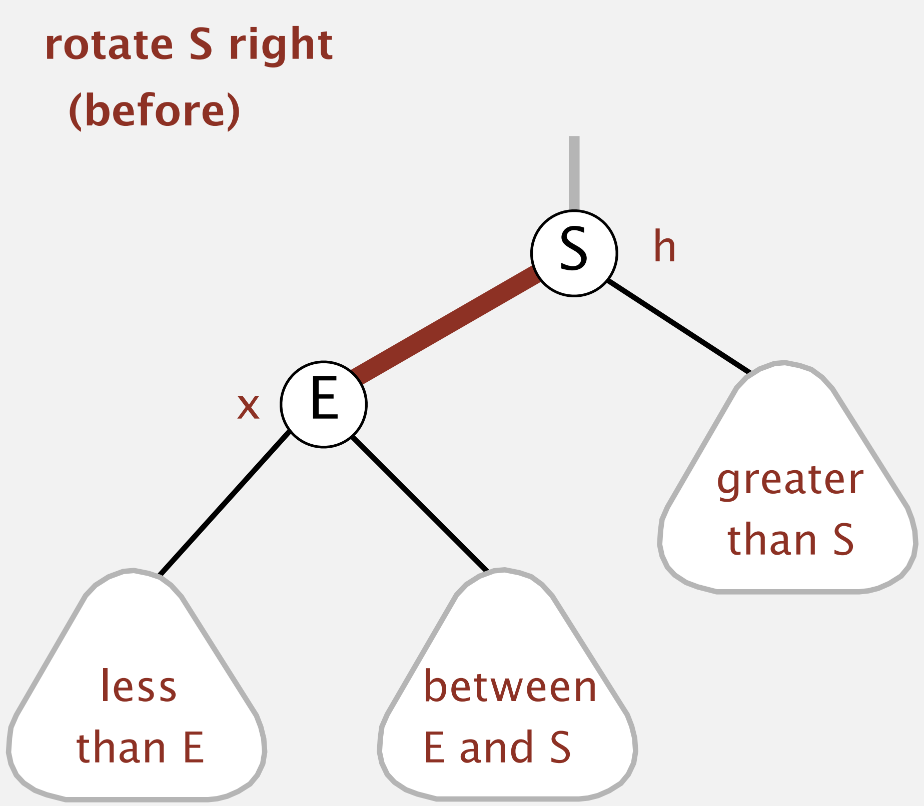



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.



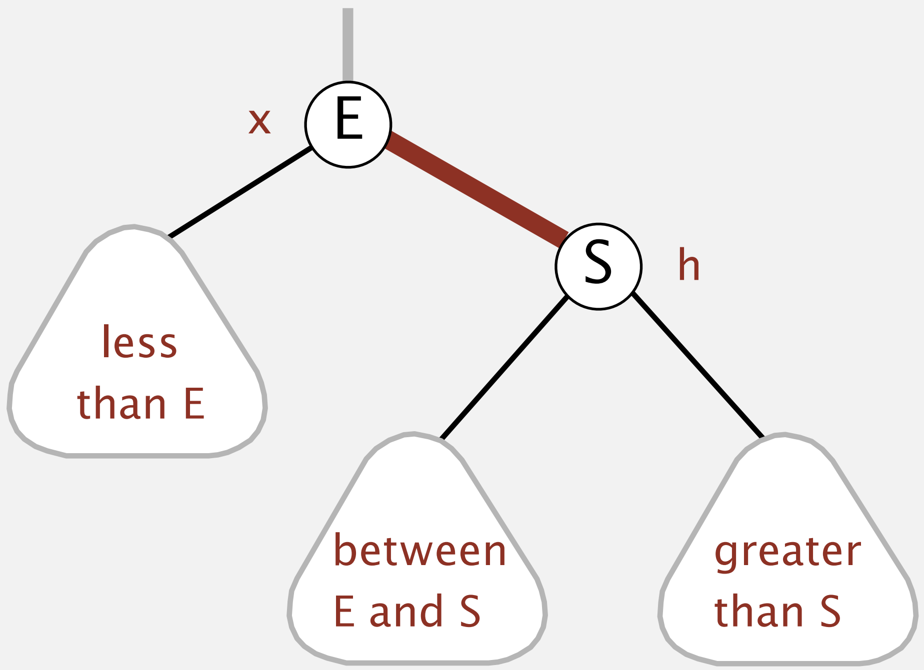

```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

rotate S right
(after)

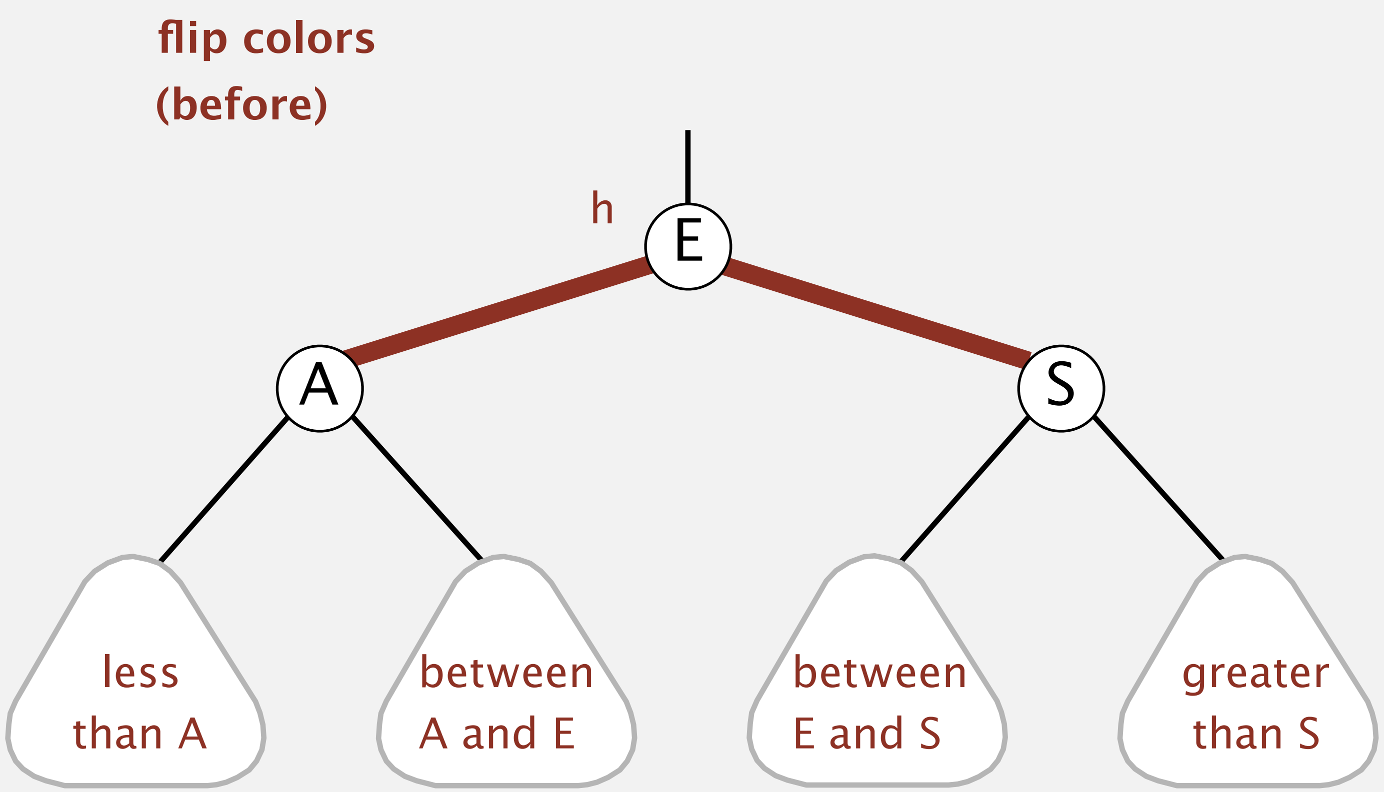



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

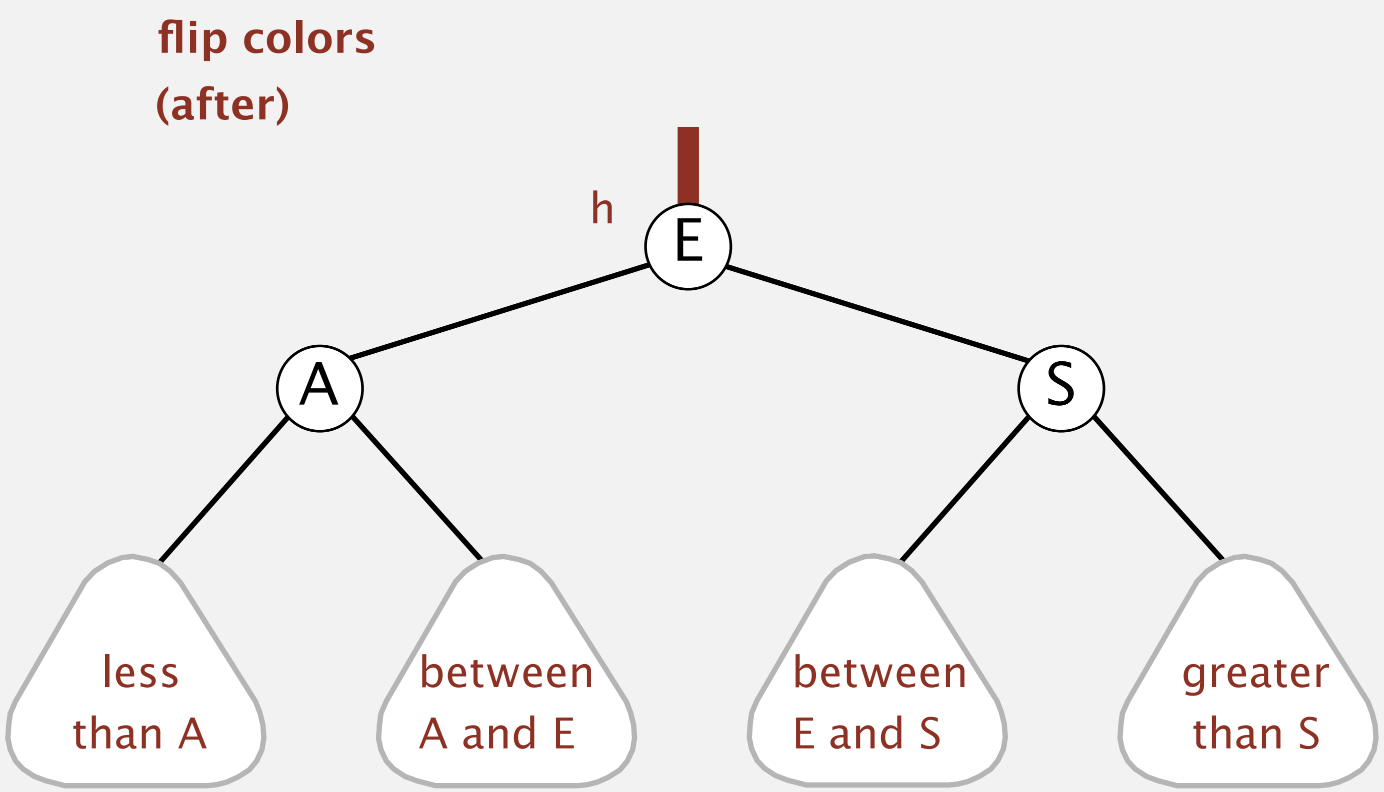



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

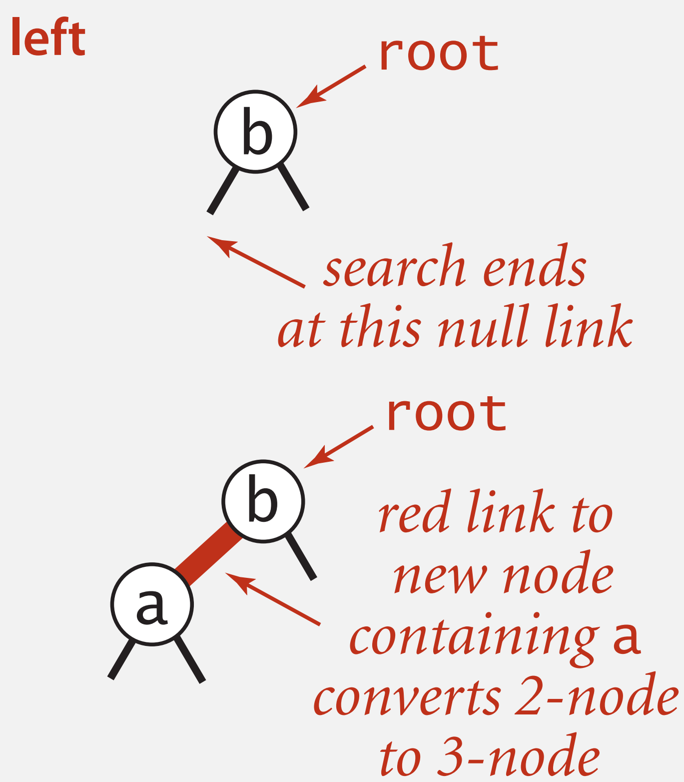


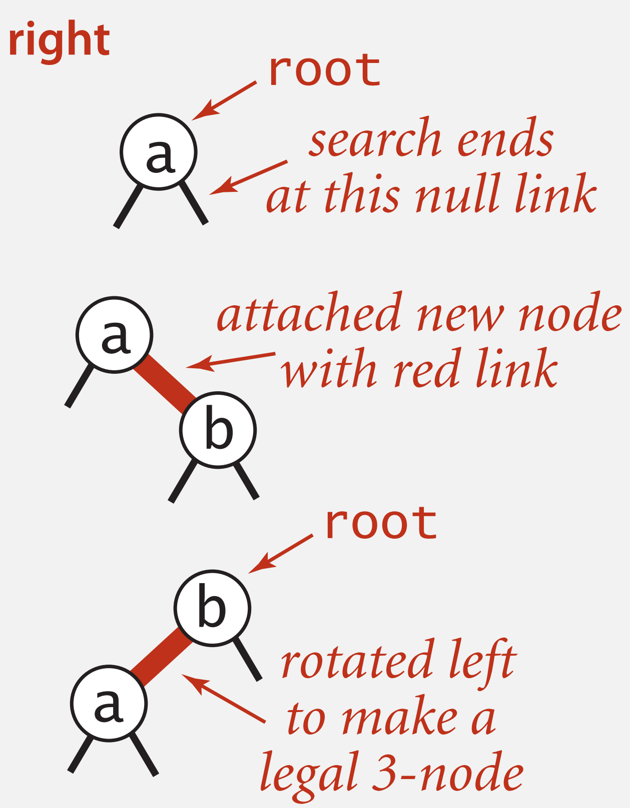

```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Insertion into a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.

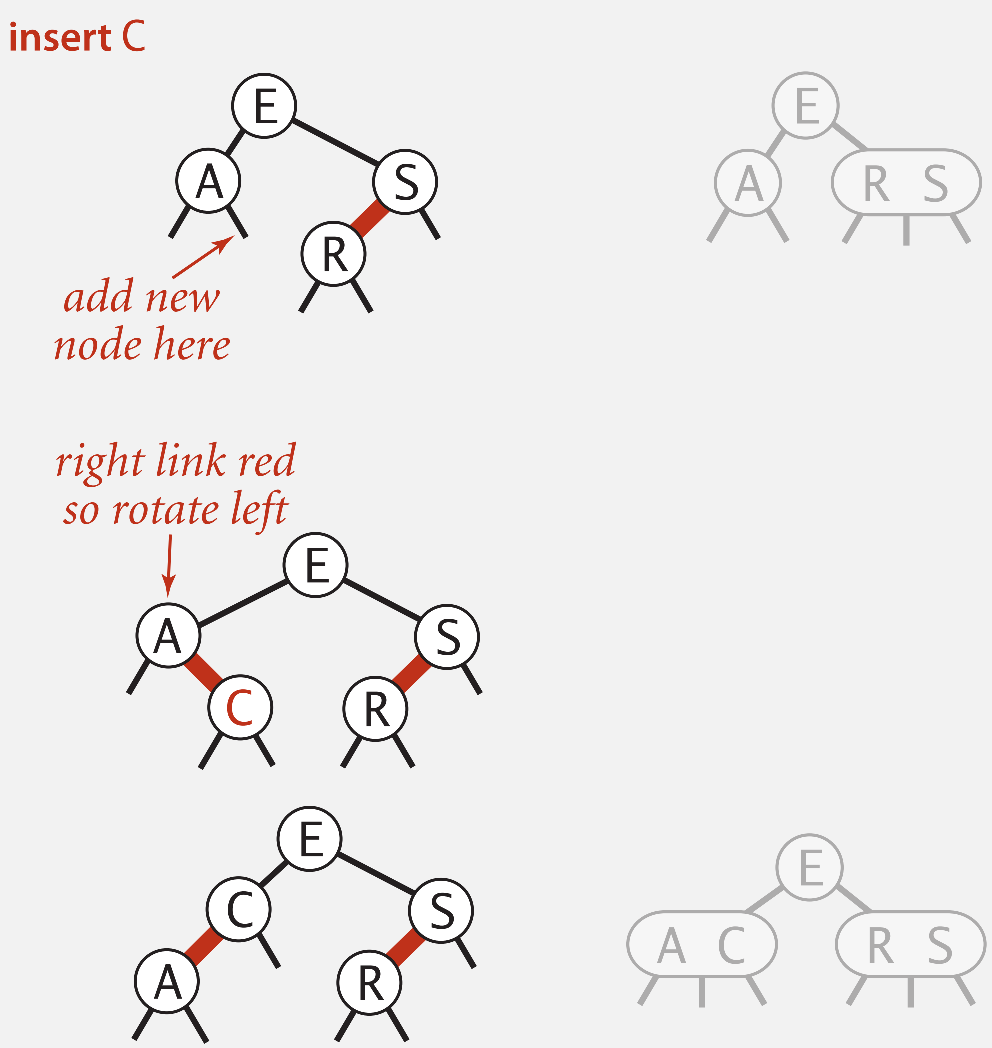




Insertion into a LLRB tree

Case 1. Insert into a 2-node at the bottom.

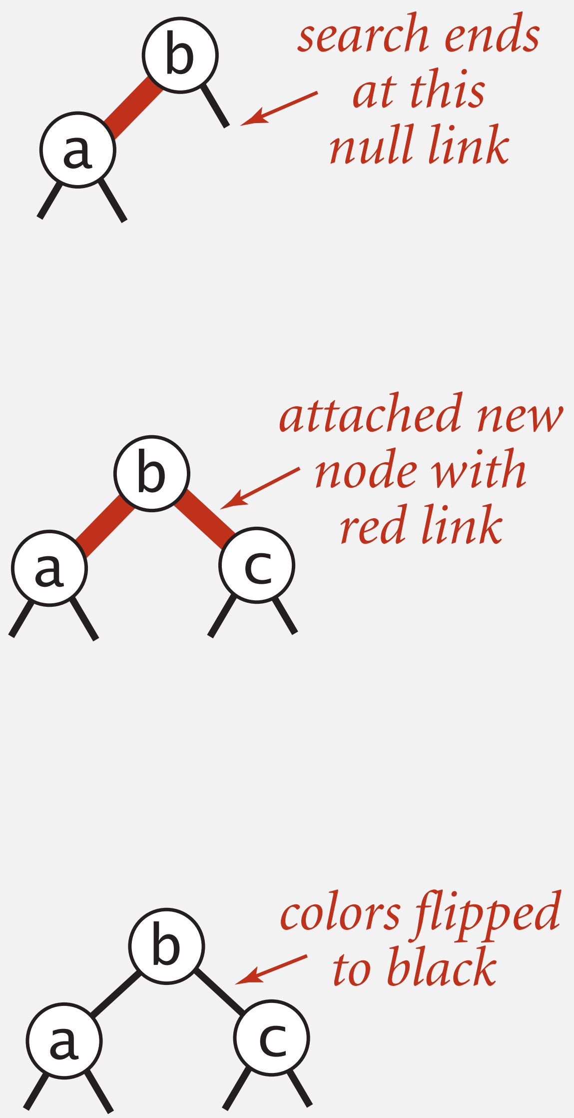
- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- If new red link is a right link, rotate left. ← to restore color invariants



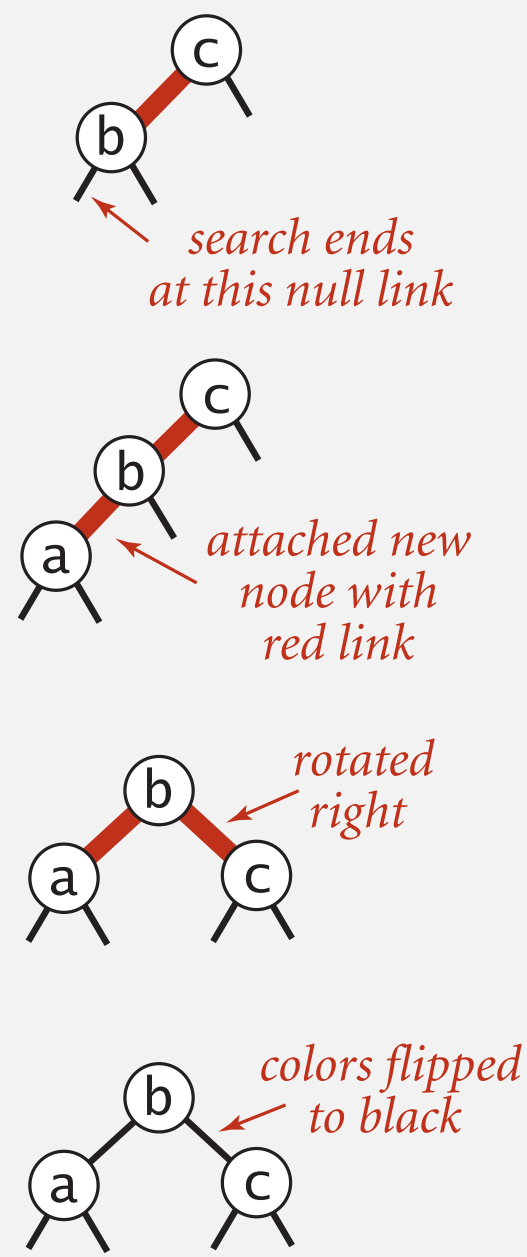
Insertion into a LLRB tree

Warmup 2. Insert into a tree with exactly 2 nodes.

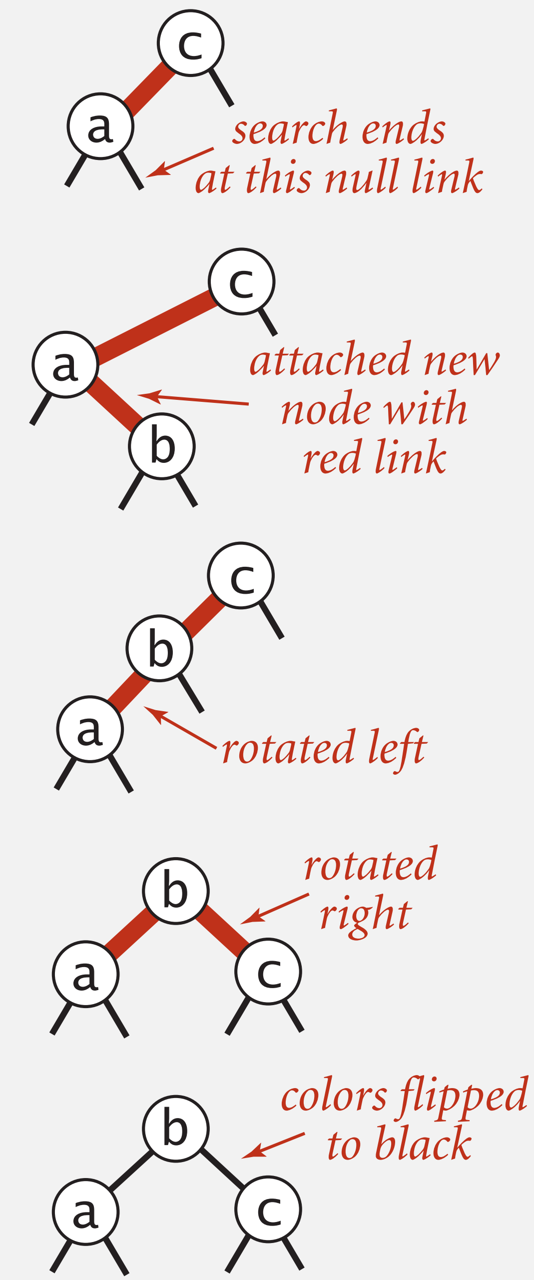
larger



smaller



between



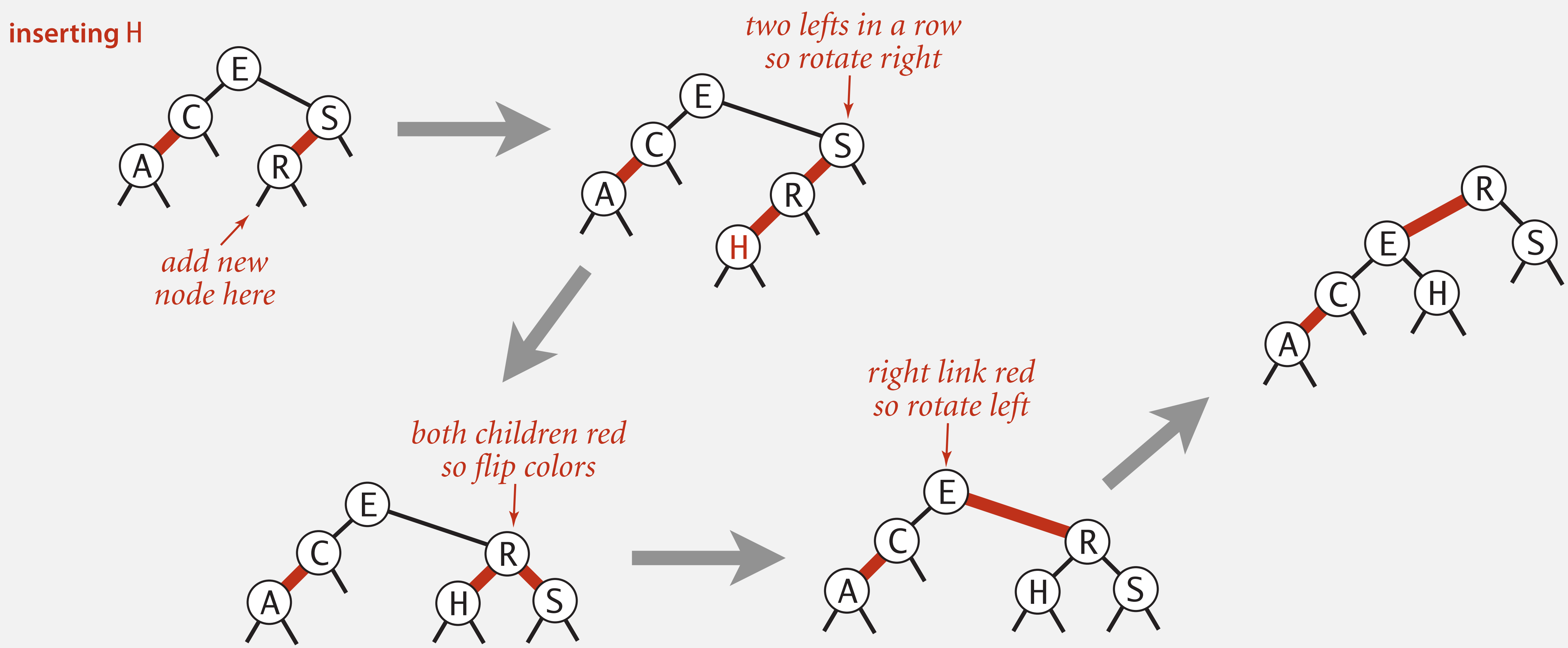
Insertion into a LLRB tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).

to maintain symmetric order and perfect black balance

to restore color invariants

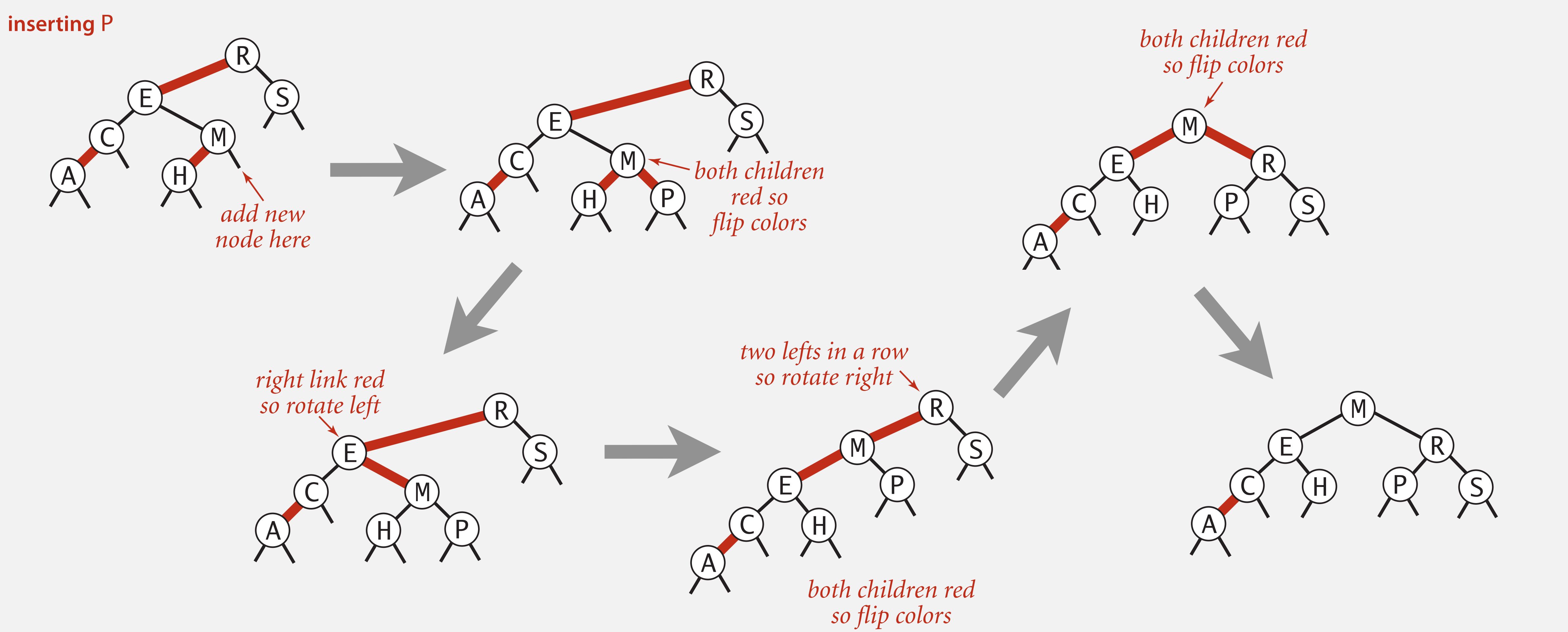


Insertion into a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

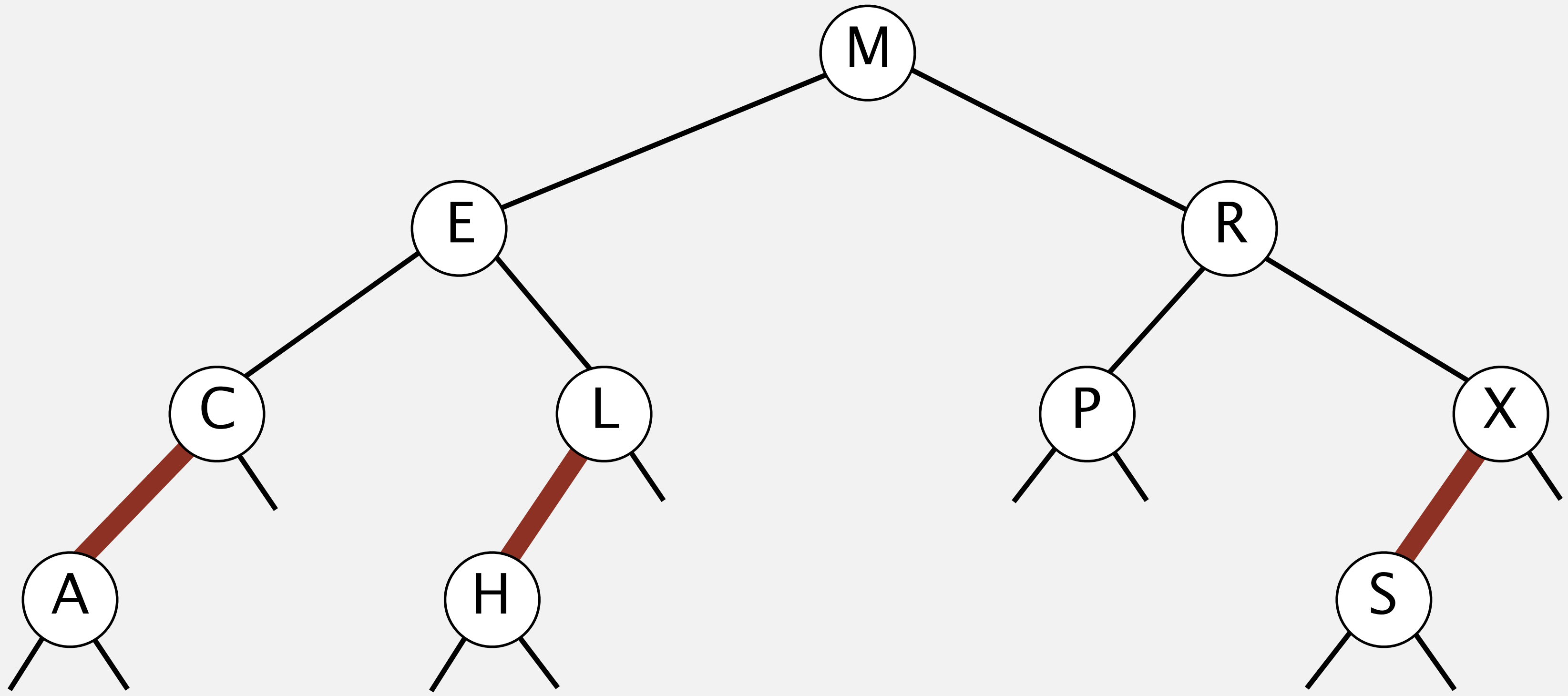
- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

← to restore color invariants



Red-black BST construction demo

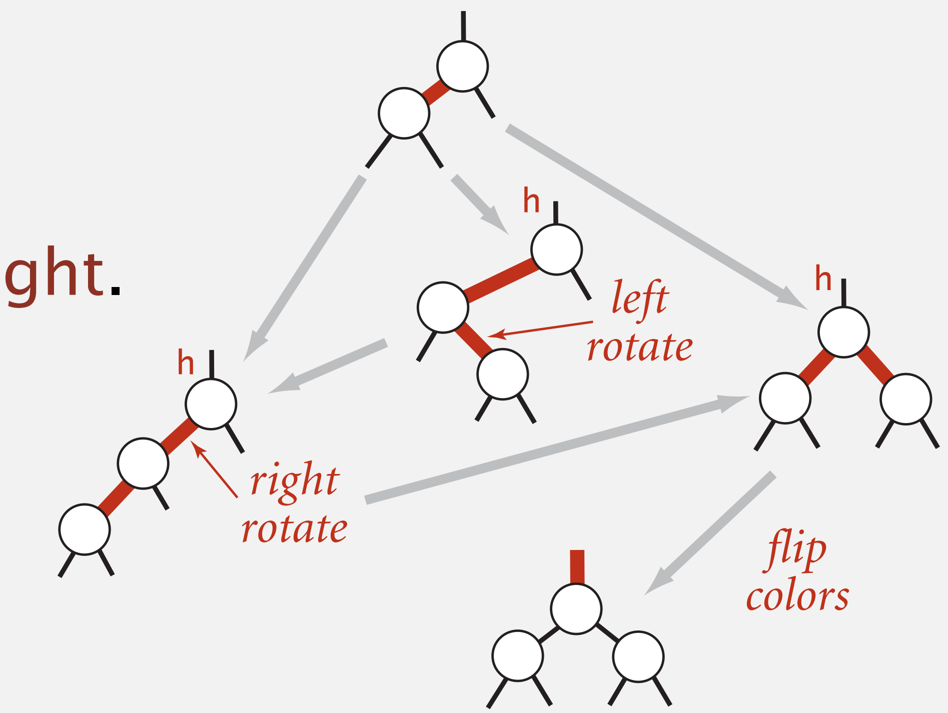
insert S E A R C H X M P L



Insertion into a LLRB tree: Java implementation

Can distill down to three cases!

- Right child red; left child black: rotate left.
- Left child red; left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))      h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))      flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

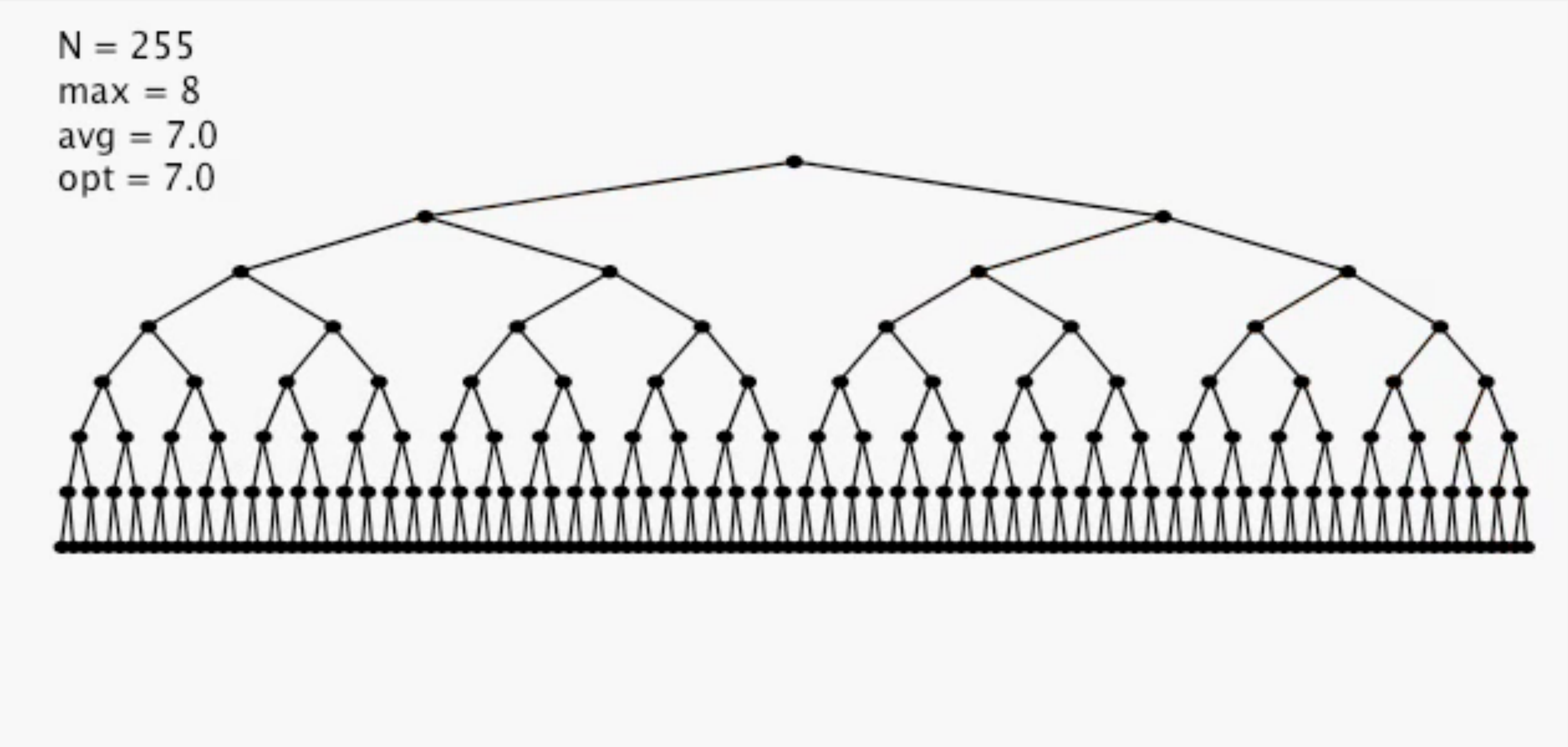
lean left

balance 4-node

split 4-node

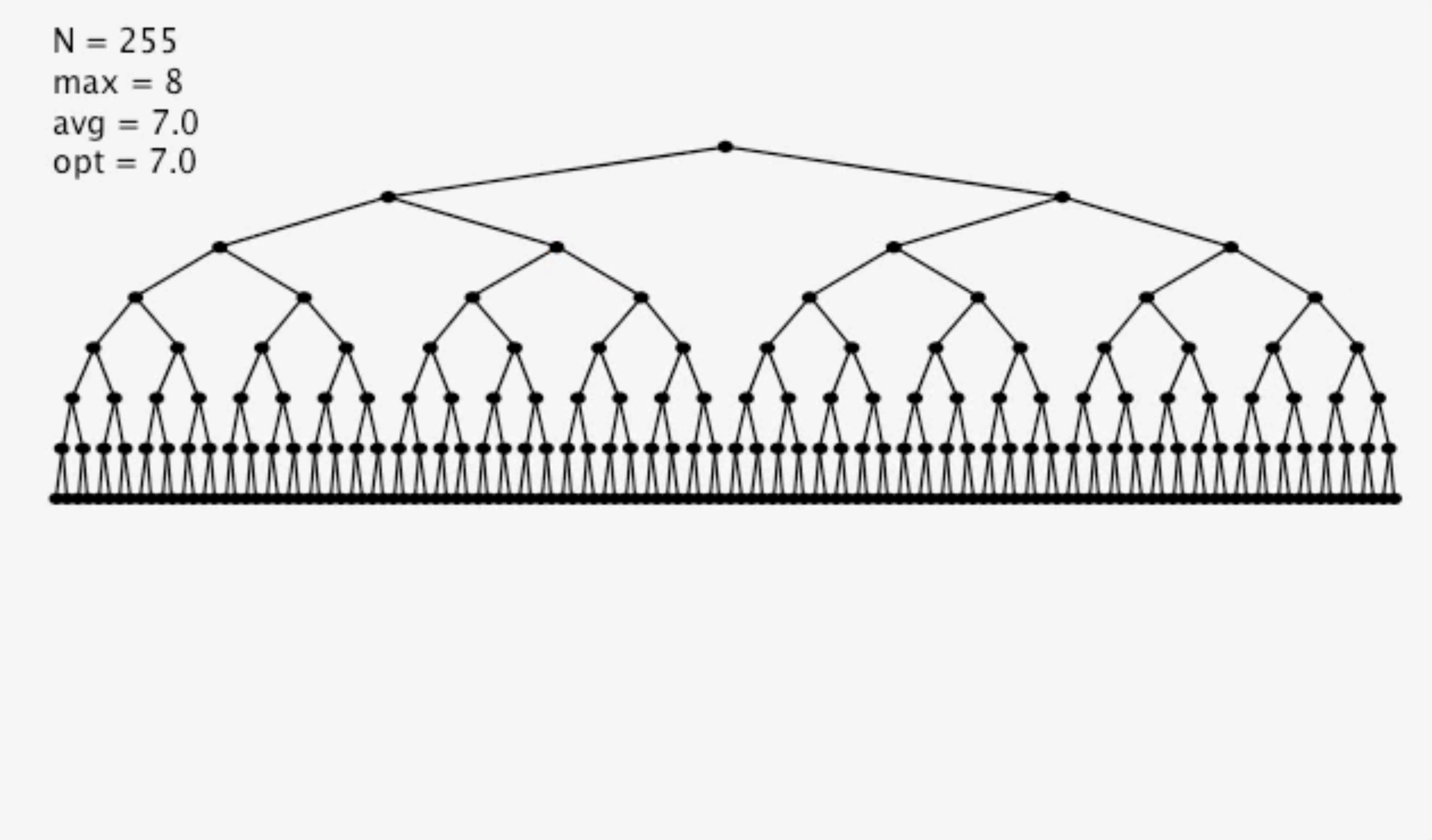
only a few extra lines of code provides near-perfect balance

Insertion into a LLRB tree: visualization



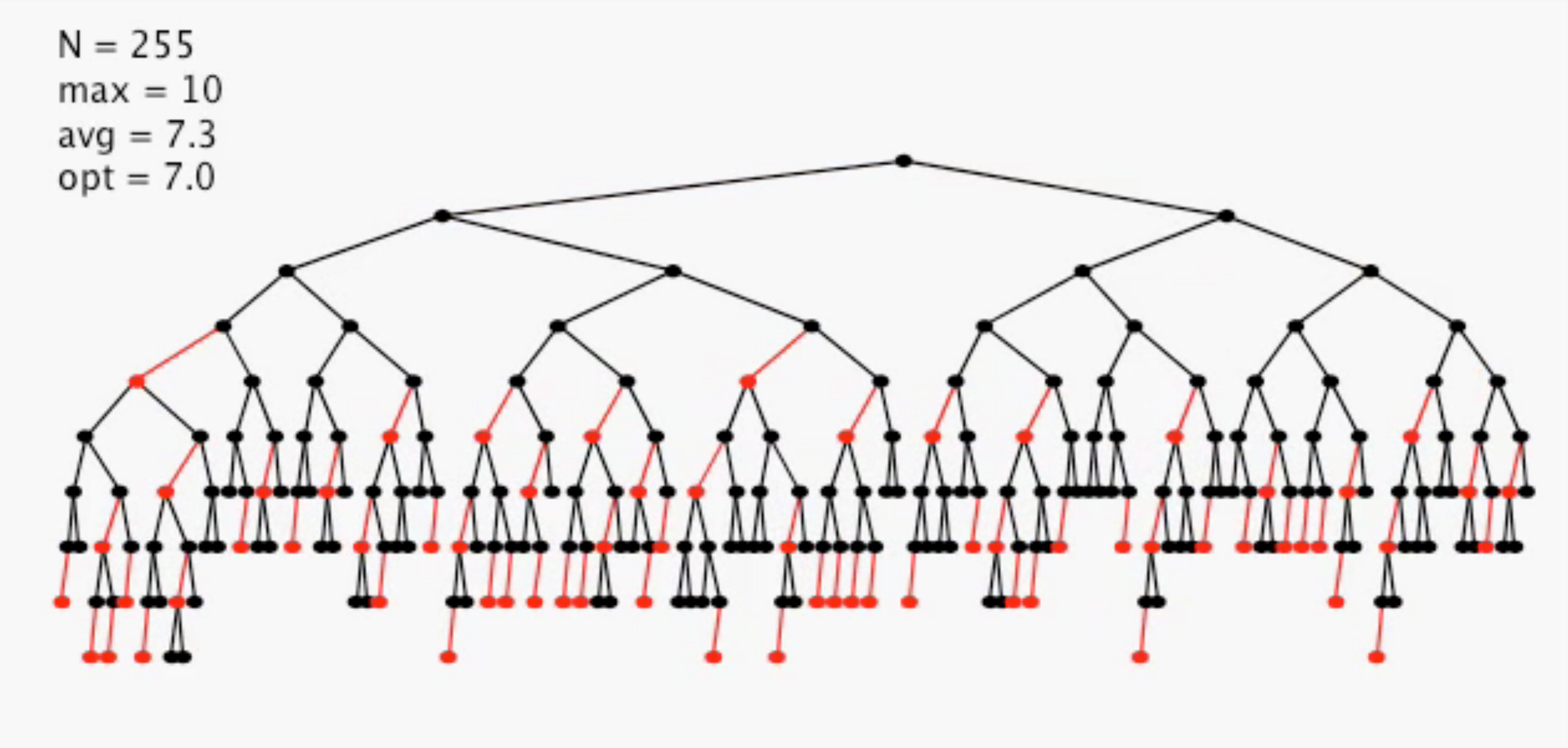
255 insertions in ascending order

Insertion into a LLRB tree: visualization



255 insertions in descending order

Insertion into a LLRB tree: visualization



255 random insertions

Balanced search trees: quiz 4

What is the maximum height of a LLRB tree with n keys?

A. $\sim \log_3 n$

B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

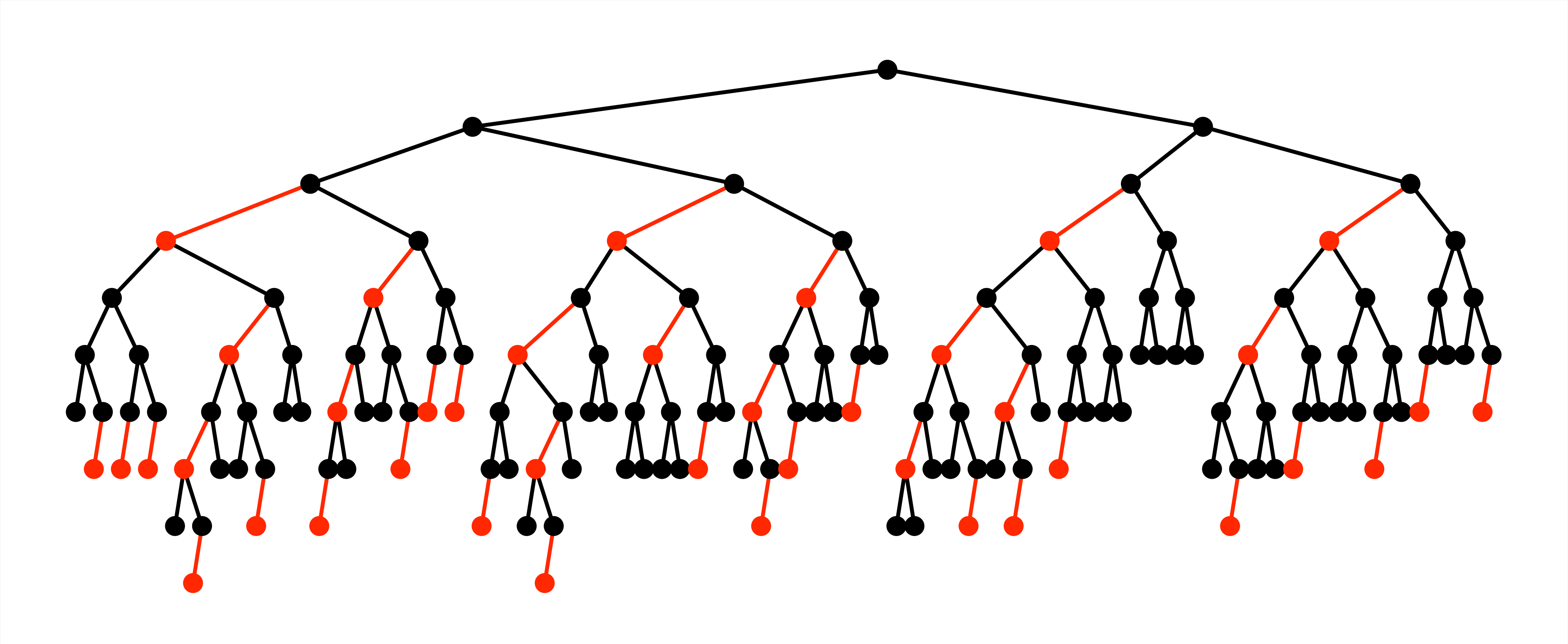
D. $\sim n$

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg n$ in the worst case.

Pf.

- Black height = height of corresponding 2–3 tree $\leq \lg n$.
- Never two red links in-a-row.



Empirical observation. Height of tree is $\sim 1.0 \lg n$ in typical applications.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

hidden constant c is small
(at most 2 lg n compares)

THREADED SET

Threaded set. Implement the following API:

public class ThreadedSet		
	`ThreadedSet()`	*create an empty threaded set*
`void`	`add(String s)`	*add the string to the set (if it is not already in the set)*
`boolean`	`contains(String s)`	*is the string s in the set?*
`String`	`previousKey(String s)`	*the string added to the set immediately before s (null if s is the first string added or s not in set)*

Performance requirement. $\log n$ time per operation (worst case).

War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
Xerox Palo Alto Research Center,
Palo Alto, California, and
Carnegie-Mellon University

and

Robert Sedgewick*
Program in Computer Science
Brown University
Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

War story: red–black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.

- Red–black BST.
- Exceeding height limit of 80 triggered error-recovery process.

should allow for $\leq 2^{40}$ keys

Extended telephone service outage.

- Main cause = height bound exceeded!
- Telephone company sues database provider.
- Legal testimony:

[redacted]

> *" If implemented properly, the height of a red–black BST with n keys is at most 2 lg n."* — *expert witness*

Balanced trees in the wild

Red–black trees are widely used as system symbol tables.

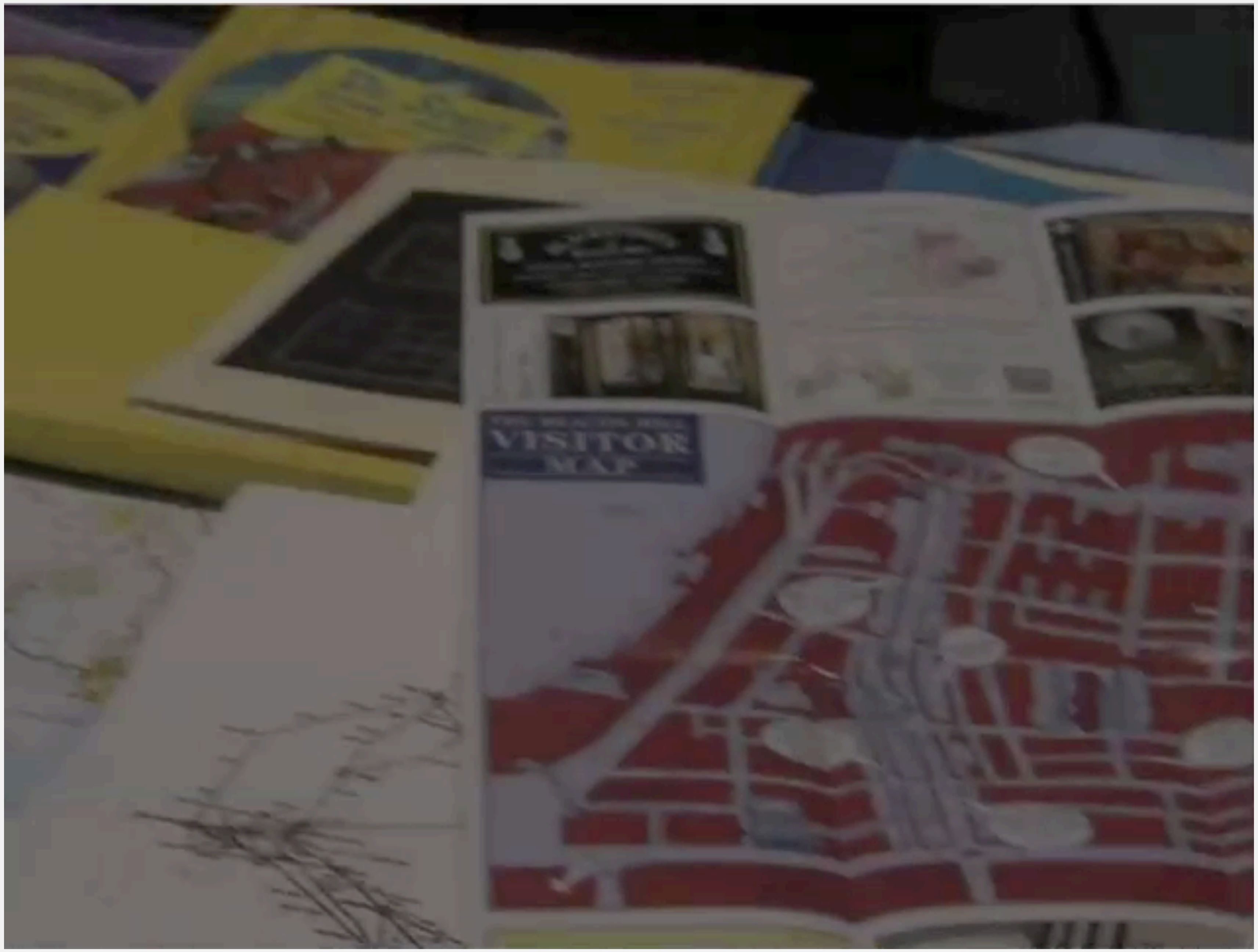
- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

Other balanced BSTs. AVL trees, splay trees, randomized BSTs, ...

B-trees (and cousins) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS, BTRFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

Red–black BSTs in the wild



Common sense. Sixth sense.
Together they're the
FBI's newest team.