http://algs4.cs.princeton.edu

1.4 ANALYSIS OF ALGORITHMS

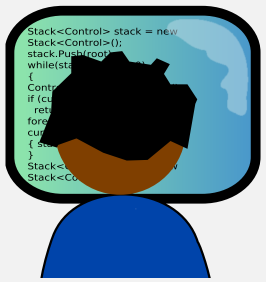
- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *memory*

http://algs4.cs.princeton.edu

1.4 Analysis of Algorithms

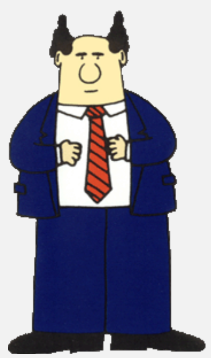
- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *memory*

Cast of characters

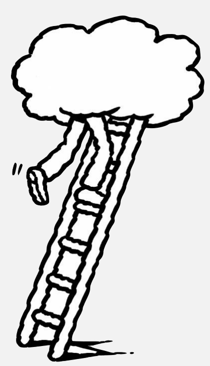


Programmer needs to develop a working solution.

Student (you) might play any or all of these roles someday.



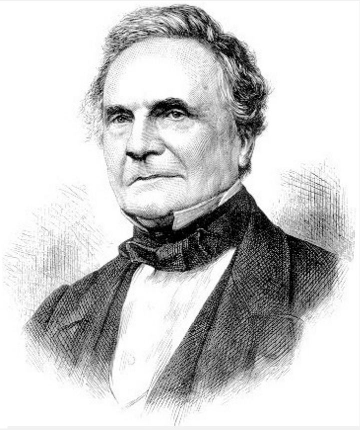
Client wants to solve problem efficiently.

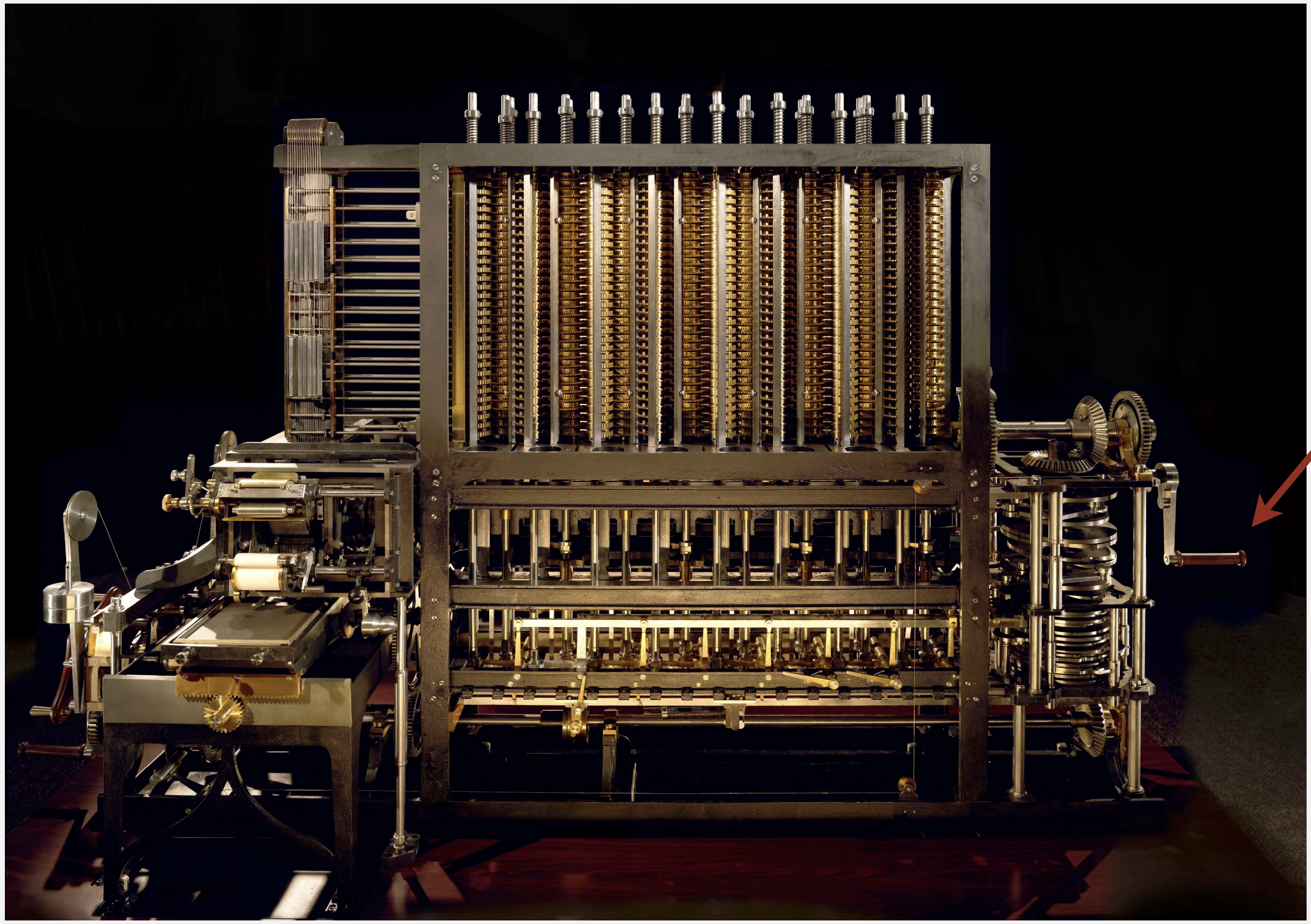


Theoretician seeks to understand.

Running time

" *As soon as an Analytical Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will then arise—By what course of calculation can these results be arrived at by the machine in the shortest time?* " — *Charles Babbage (1864)*





how many times do you have to turn the crank?

Reasons to analyze algorithms

Predict performance.

Compare algorithms.

this course
(COS 226)

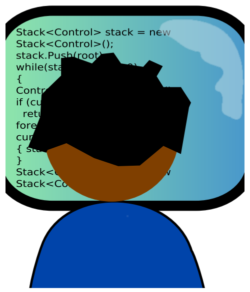
Provide guarantees.

theory of algorithms
(COS 423)

Understand theoretical basis.

Primary practical reason: avoid performance bugs.

client gets poor performance because programmer did not understand performance characteristics

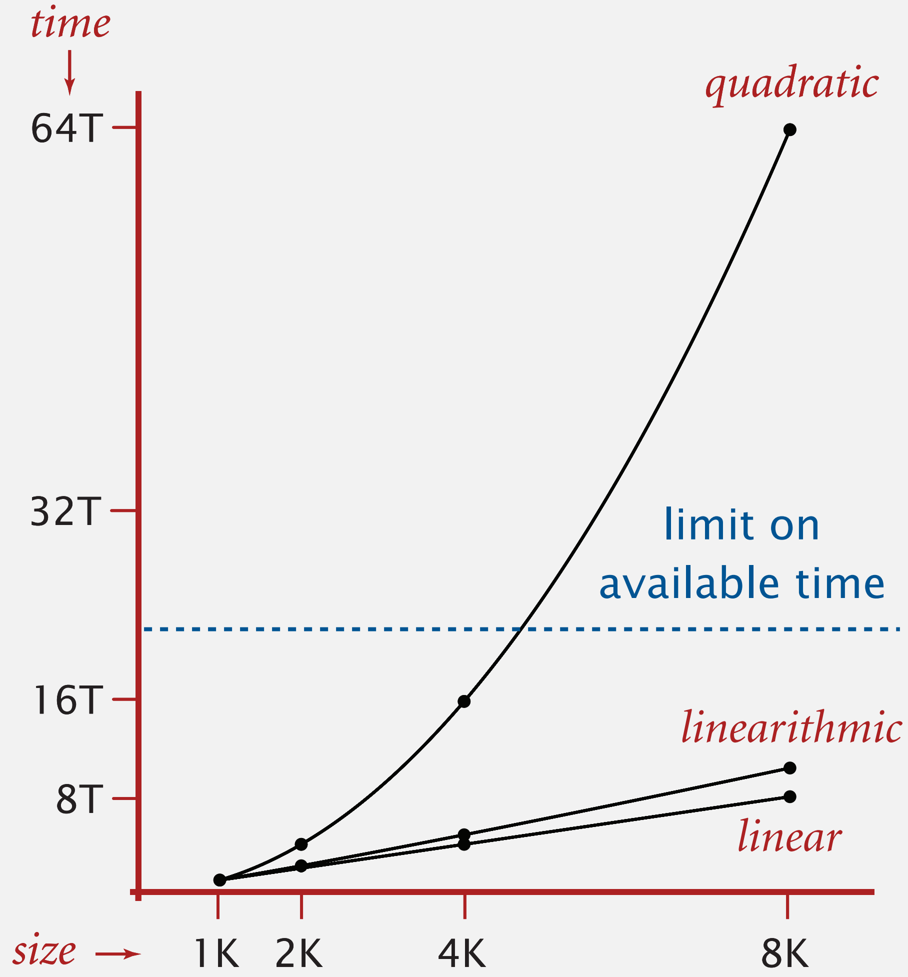


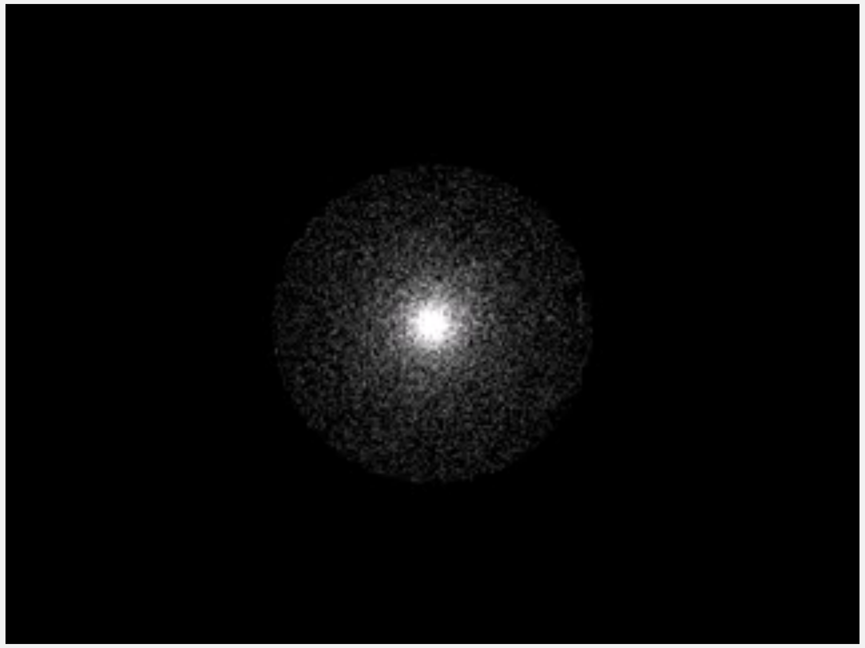
An algorithmic success story

N-body simulation.

- Simulate gravitational interactions among n bodies.
- Applications: cosmology, fluid dynamics, semiconductors, ...
- Brute force: n^2 steps.
- Barnes–Hut algorithm: $n \log n$ steps, enables new research.

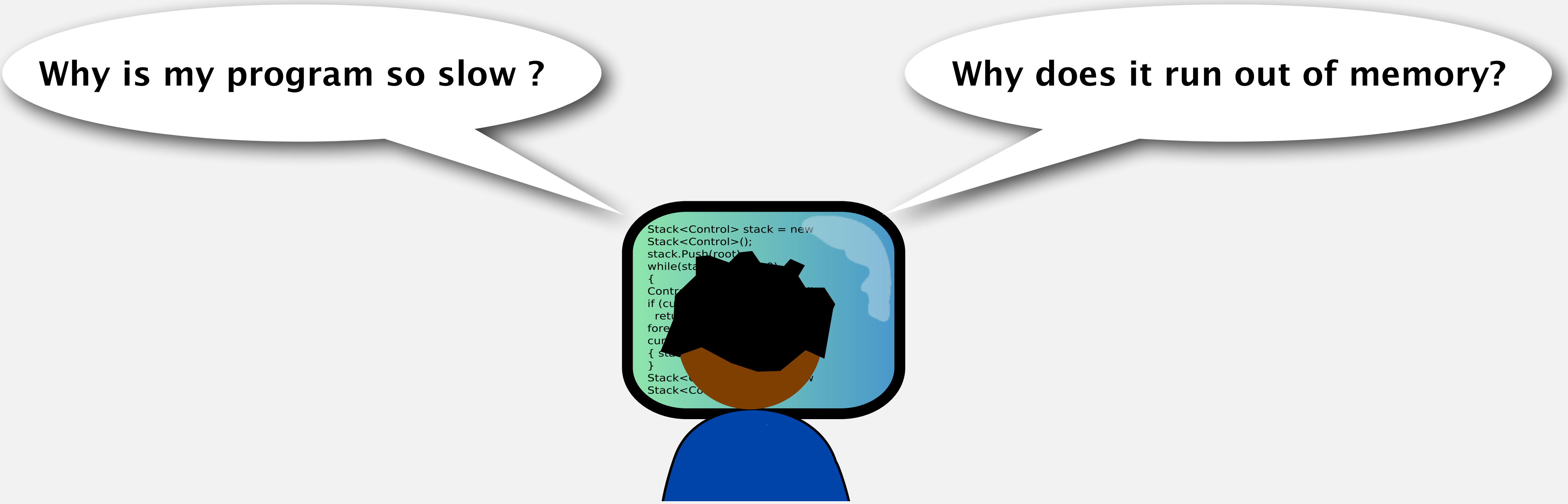
Andrew Appel
PU ’81





The challenge

Q. Will my program be able to solve a large practical input?



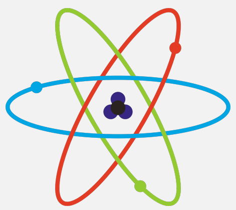
Insight. [Knuth 1970s] Use scientific method to understand performance.

Scientific method applied to the analysis of algorithms

A framework for predicting performance and comparing algorithms.

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

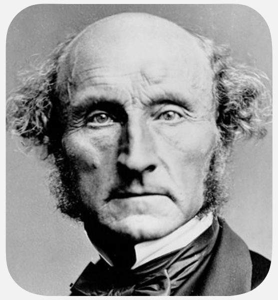


Principles.

- Experiments must be reproducible.
- Hypotheses must be falsifiable.

Francis Bacon

René Descartes



John Stuart Mills

Feature of the natural world. Computer itself.

http://algs4.cs.princeton.edu

1.4 ANALYSIS OF ALGORITHMS

- introduction
- *observations*
- mathematical models
- order-of-growth classifications
- memory

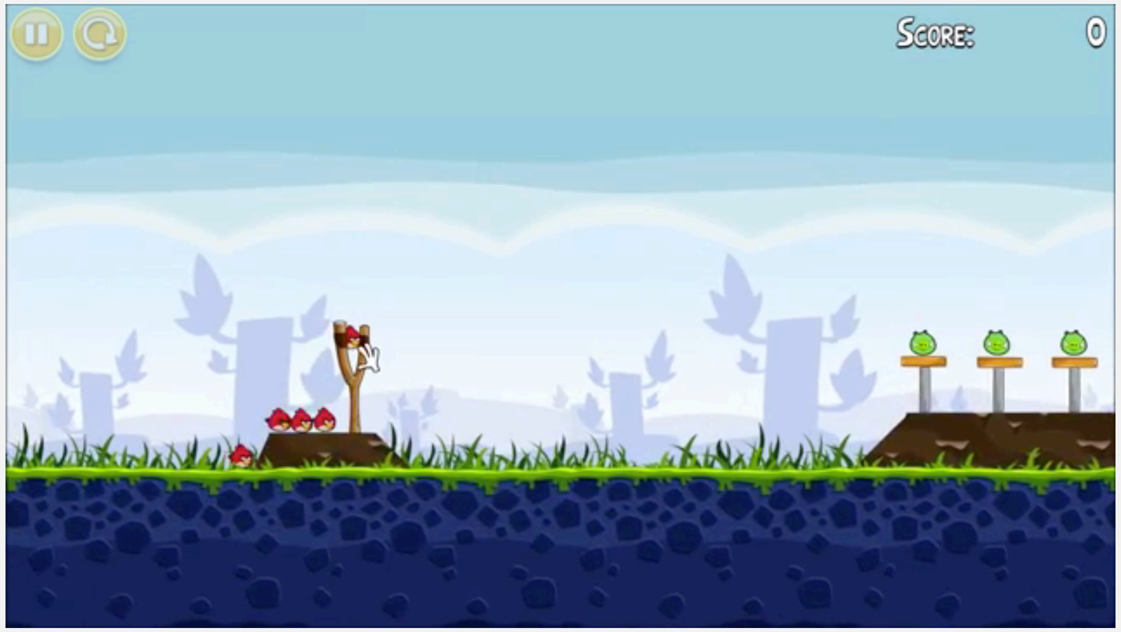
Example: 3-SUM

3-SUM. Given n distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0



Context. Related to problems in computational geometry.

3-SUM: brute-force algorithm

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int n = a.length;
      int count = 0;
      for (int i = 0; i < n; i++)
         for (int j = i+1; j < n; j++)
            for (int k = j+1; k < n; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }

   public static void main(String[] args)
   {
      In in = new In(args[0]);
      int[] a = in.readAllInts();
      StdOut.println(count(a));
   }
}
```

check each triple

for simplicity, ignore integer overflow

Measuring the running time

Q. How to time a program?

A. Manual.

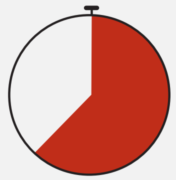
```
% java ThreeSum 1Kints.txt
```

tick tick tick

```
70
% java ThreeSum 2Kints.txt
```

tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

```
528
% java ThreeSum 4Kints.txt
```



tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

```
4039
```

Measuring the running time

Q. How to time a program?

A. Automatic.

public class Stopwatch	(part of `algs4.jar`)
public Stopwatch()	*create a new stopwatch*
public double elapsedTime()	*time since creation (in seconds)*

client code

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   int[] a = in.readAllInts();
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
   StdOut.println("elapsed time = " + time);
}
```

Empirical analysis

Run the program for various input sizes and measure running time.

Empirical analysis

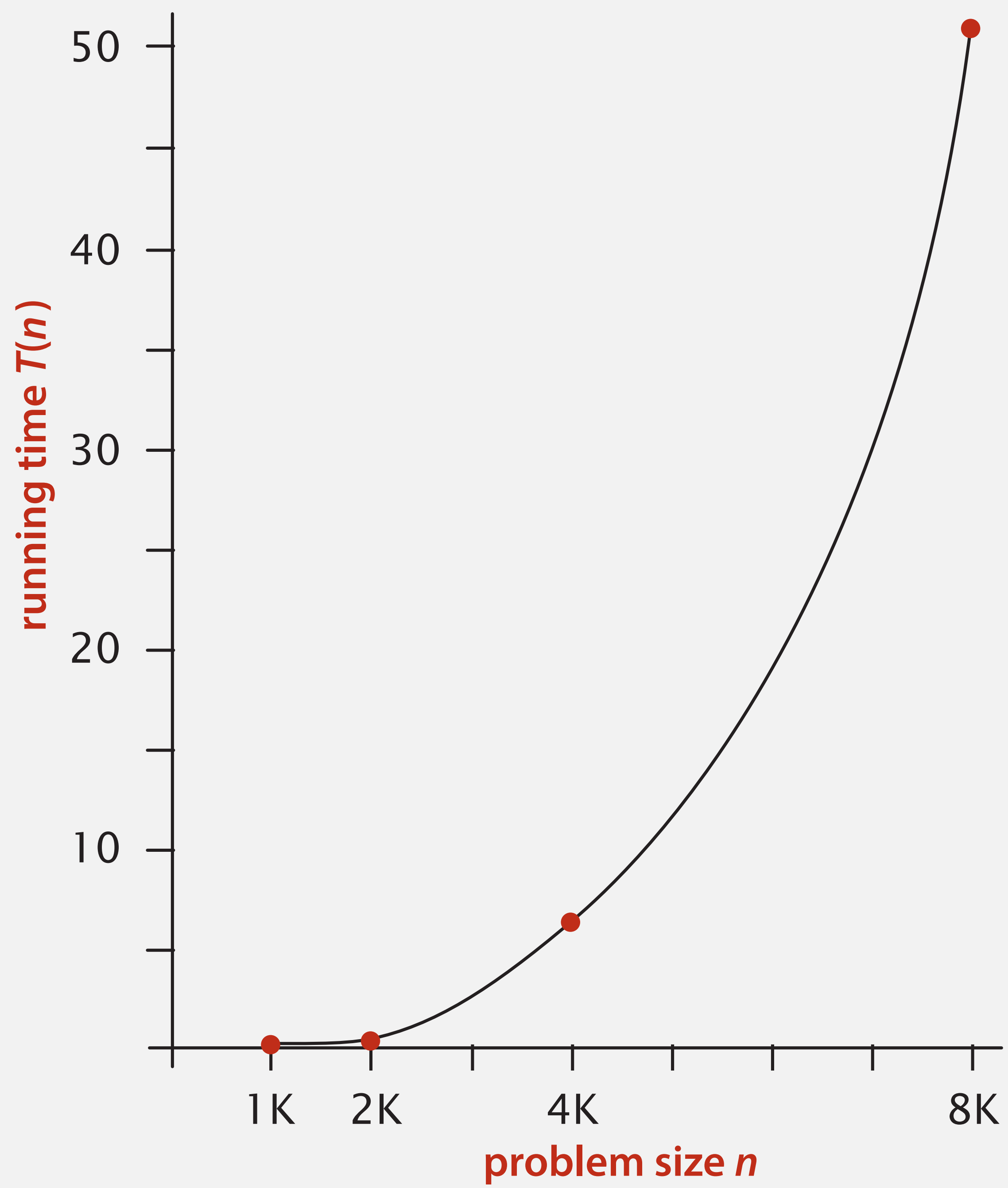
Run the program for various input sizes and measure running time.

n	time (seconds) †
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

† on a 2.8GHz Intel PU-226 with 64GB DDR E3 memory and 32MB L3 cache; running Oracle Java 1.7.0_45-b18 on Springdale Linux v. 6.5

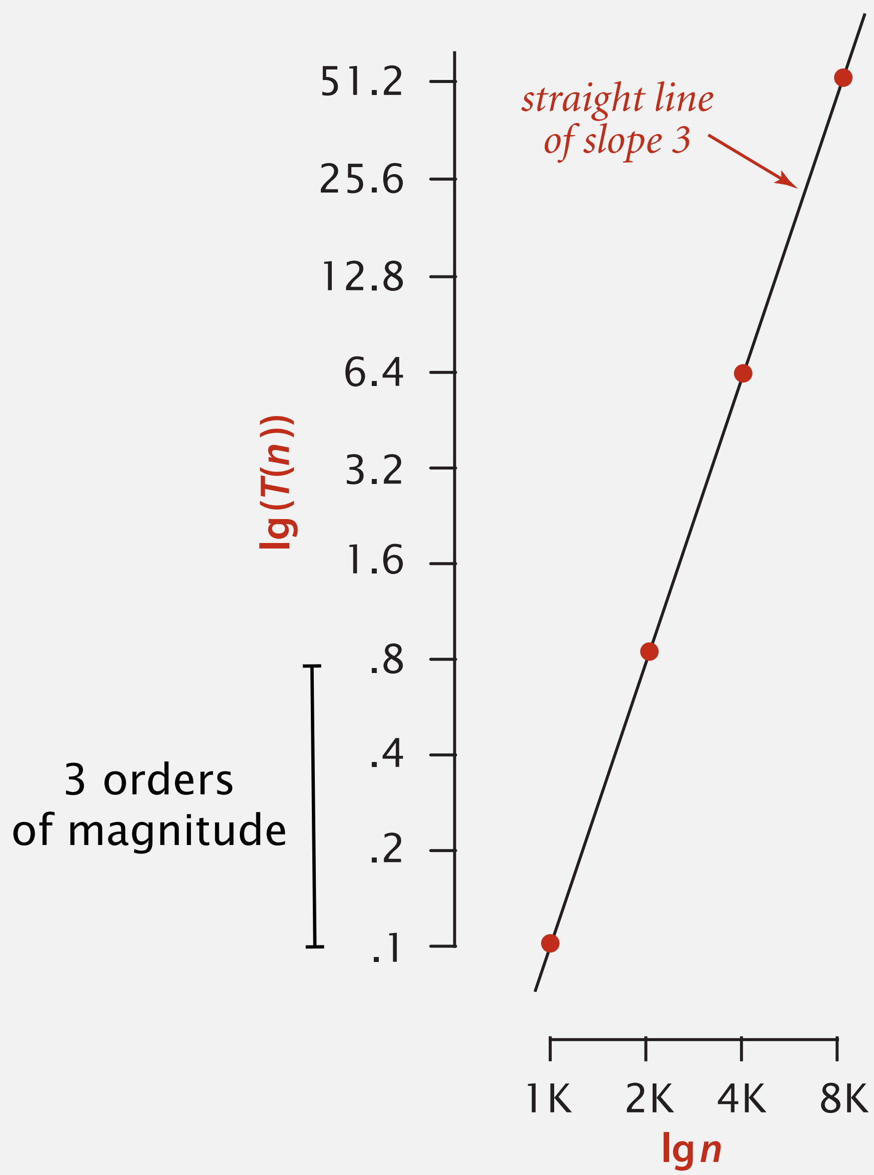
Data analysis

Standard plot. Plot running time $T(n)$ vs. input size n.



Data analysis

Log-log plot. Plot running time $T(n)$ vs. input size n using log-log scale.



$$\log_2(T(n)) = b \log_2 n + c$$
$$b = 2.999$$
$$c = -33.2103$$

$$T(n) = a\, n^b, \text{ where } a = 2^c$$

power law

slope

Regression. Fit straight line through data points: $a\, n^b$.

Hypothesis. The running time is about $1.006 \times 10^{-10} \times n^{2.999}$ seconds.

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times n^{2.999}$ seconds.

"order of growth" of running time is about n^3 [stay tuned]

Predictions.

- 51.0 seconds for $n = 8{,}000$.
- 408.1 seconds for $n = 16{,}000$.

Observations.

n	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Run program, doubling the size of the input.

n	time (seconds) †	ratio	lg ratio
250	0		–
500	0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8	3
8,000	51.1	8	3

$$\frac{T(n)}{T(n/2)} = \frac{an^b}{a(n/2)^b} = 2^b$$

$\log_2 (6.4 / 0.8) = 3.0$

seems to converge to a constant $b \approx 3$

Hypothesis. Running time is about $a\,n^{\,b}$ with $b = \log_2$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of n) and solve for a.

n	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1

$51.1 = a \times 8000^3$

$\Rightarrow \quad a = 0.998 \times 10^{-10}$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times n^3$ seconds.

almost identical hypothesis
to one obtained via regression

Analysis of algorithms quiz 1

Estimate the running time to solve a problem of size $n = 96{,}000$.

A. 39 *seconds*

B. 52 *seconds*

C. 117 *seconds*

D. 350 *seconds*

n	time (seconds)
1,000	0.02
2,000	0.05
4,000	0.20
8,000	0.81
16,000	3.25
32,000	13.01

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent b in power law $a\,n^{\,b}$

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

determines constant a in power law $a\,n^{\,b}$

14.04 LTS

Bad news. Sometimes difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

An aside

Algorithmic experiments are virtually free by comparison with other sciences.

Chemistry
(1 experiment)

Biology
(1 experiment)

Physics
(1 experiment)

Computer Science
(1 million experiments)

Bottom line. No excuse for not running experiments to understand costs.

http://algs4.cs.princeton.edu

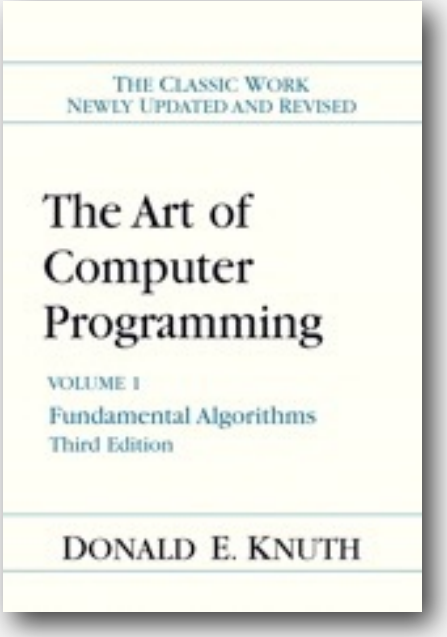
1.4 Analysis of Algorithms

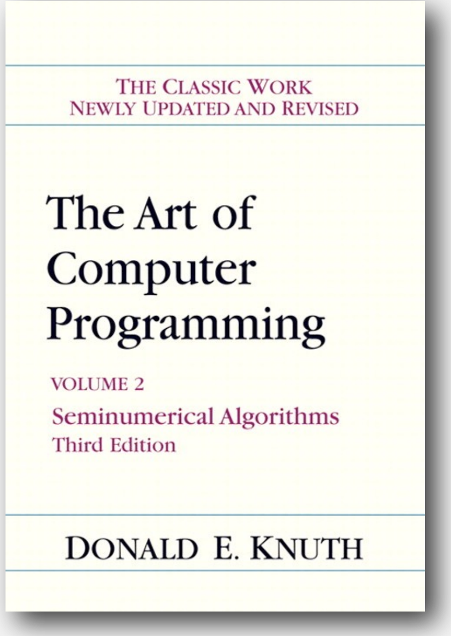
- introduction
- observations
- **mathematical models**
- order-of-growth classifications
- memory

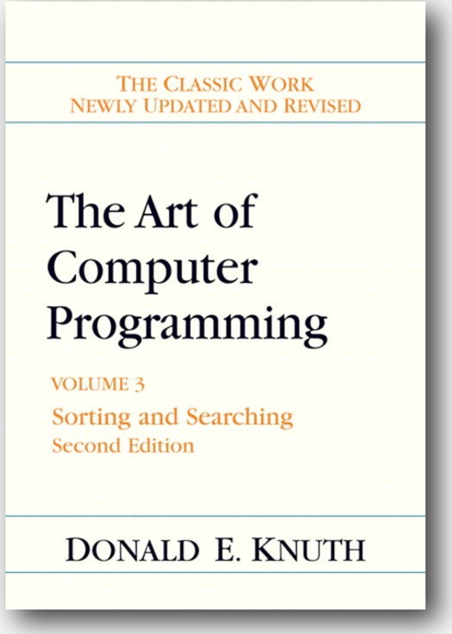
Mathematical models for running time

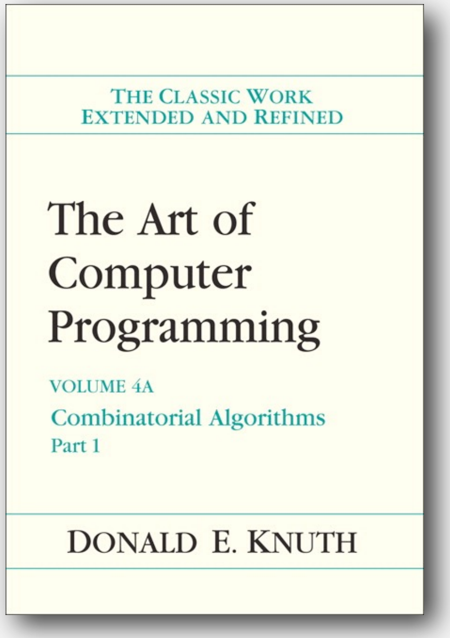
Total running time: sum of cost × frequency for all operations.

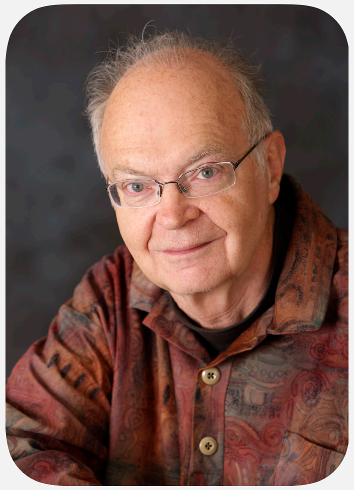
- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.











Donald Knuth
1974 Turing Award

Example: 1-SUM

Q. How many operations as a function of input size n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   if (a[i] == 0)
      count++;
```

n array accesses

operation	cost (ns) †	frequency
variable declaration	2/5	2
assignment statement	1/5	2
less than compare	1/5	$n + 1$
equal to compare	1/10	n
array access	1/10	n
increment	1/10	n to $2\,n$

† representative estimates (with some poetic license)

Analysis of algorithms quiz 2

How many array accesses as a function of n?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

A. $\frac{1}{2}\, n\, (n - 1)$

B. $n\, (n - 1)$

C. $n\, (n + 1)$

D. $2\, n^2$

Example: 2-SUM

Q. How many operations as a function of input size n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (n-1) = \frac{1}{2} n(n-1) = \binom{n}{2}$$

operation	cost (ns)	frequency
variable declaration	2/5	$n + 2$
assignment statement	1/5	$n + 2$
less than compare	1/5	$\frac{1}{2}(n+1)(n+2)$
equal to compare	1/10	$\frac{1}{2}n(n-1)$
array access	1/10	$\boxed{n(n-1)}$
increment	1/10	$\frac{1}{2}n(n+1)$ to n^2

$1/4\ n^2 + 13/20\ n + 13/10$ ns

to

$3/10\ n^2 + 3/5\ n + 13/10$ ns

(tedious to count exactly)

Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (n-1) \;=\; \frac{1}{2}\,n(n-1) \;=\; \binom{n}{2}$$

operation	cost (ns)	frequency
variable declaration	2/5	$n + 2$
assignment statement	1/5	$n + 2$
less than compare	1/5	$\frac{1}{2}\,(n+1)\,(n+2)$
equal to compare	1/10	$\frac{1}{2}\,n\,(n-1)$
array access	**1/10**	$n\,(n-1)$
increment	1/10	$\frac{1}{2}\,n\,(n+1)$ to n^2

cost model = array accesses

(we assume compiler/JVM do not optimize any array accesses away!)

Simplification 2: tilde notation

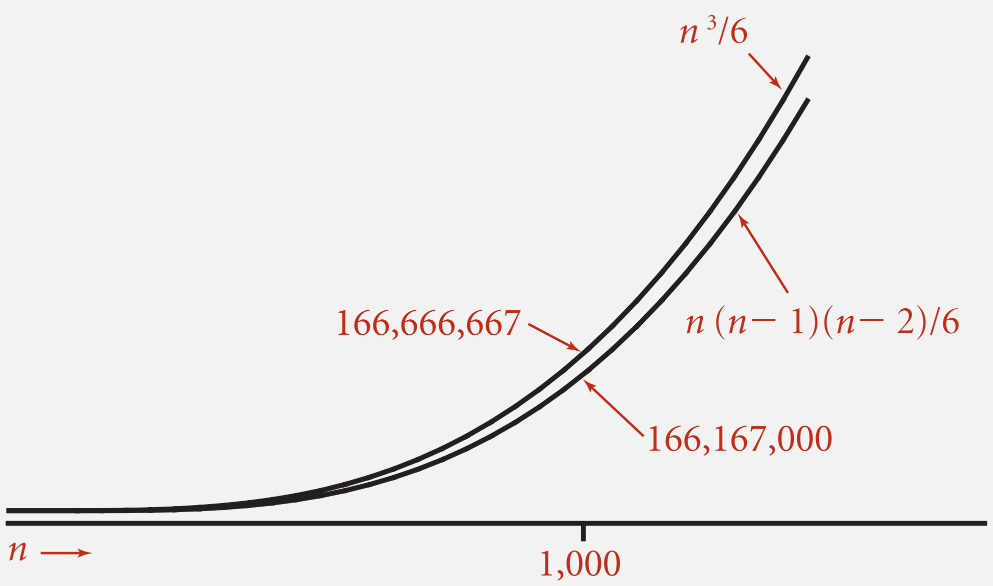
- Estimate running time (or memory) as a function of input size n.
- Ignore lower order terms.

Ex 1. $\frac{1}{6} n^3 + 20 n + 16 \quad \sim \frac{1}{6} n^3$

Ex 2. $\frac{1}{6} n^3 + 100 n^{4/3} + 56 \quad \sim \frac{1}{6} n^3$

Ex 3. $\frac{1}{6} n^3 \underbrace{- \frac{1}{2} n^2 + \frac{1}{3} n}_{\text{discard lower-order terms}} \quad \sim \frac{1}{6} n^3$

(e.g., n = 1000: 166.67 million vs. 166.17 million)



Leading-term approximation

Rationale.

- When n is large, terms are negligible.
- When n is small, we don't care.

Technical definition. $f(n) \sim g(n)$ means $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size n.
- Ignore lower order terms.

operation	frequency	tilde notation
variable declaration	$n + 2$	$\sim n$
assignment statement	$n + 2$	$\sim n$
less than compare	$\frac{1}{2}(n + 1)(n + 2)$	$\sim \frac{1}{2} n^2$
equal to compare	$\frac{1}{2} n (n - 1)$	$\sim \frac{1}{2} n^2$
array access	$n (n - 1)$	$\sim n^2$
increment	$\frac{1}{2} n (n + 1)$ to n^2	$\sim \frac{1}{2} n^2$ to $\sim n^2$

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size n?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (n-1) = \frac{1}{2}n(n-1) = \binom{n}{2}$$

A. $\sim n^2$ array accesses.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      for (int k = j+1; k < n; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \sim \frac{1}{6}n^3$$

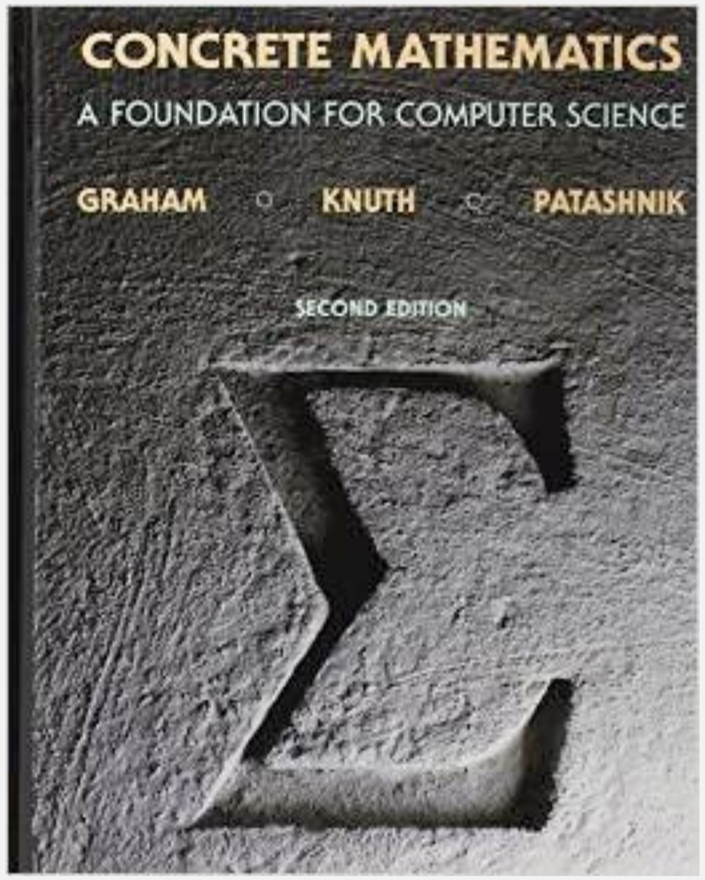
A. $\sim \frac{1}{2} n^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course (COS 340).



Estimating a discrete sum

Q. How to estimate a discrete sum?

A2. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \ldots + n.$

$$\sum_{i=1}^{n} i \sim \int_{x=1}^{n} x\,dx \sim \frac{1}{2}n^2$$

Ex 2. $1 + 1/2 + 1/3 + \ldots + 1/n.$

$$\sum_{i=1}^{n} \frac{1}{i} \sim \int_{x=1}^{n} \frac{1}{x}\,dx \sim \ln n$$

Ex 3. 3-sum triple loop.

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=j}^{n} 1 \sim \int_{x=1}^{n} \int_{y=x}^{n} \int_{z=y}^{n} dz\,dy\,dx \sim \frac{1}{6}n^3$$

Ex 4. $1 + ½ + ¼ + ⅛ + \ldots$

$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

integral trick doesn't always work!

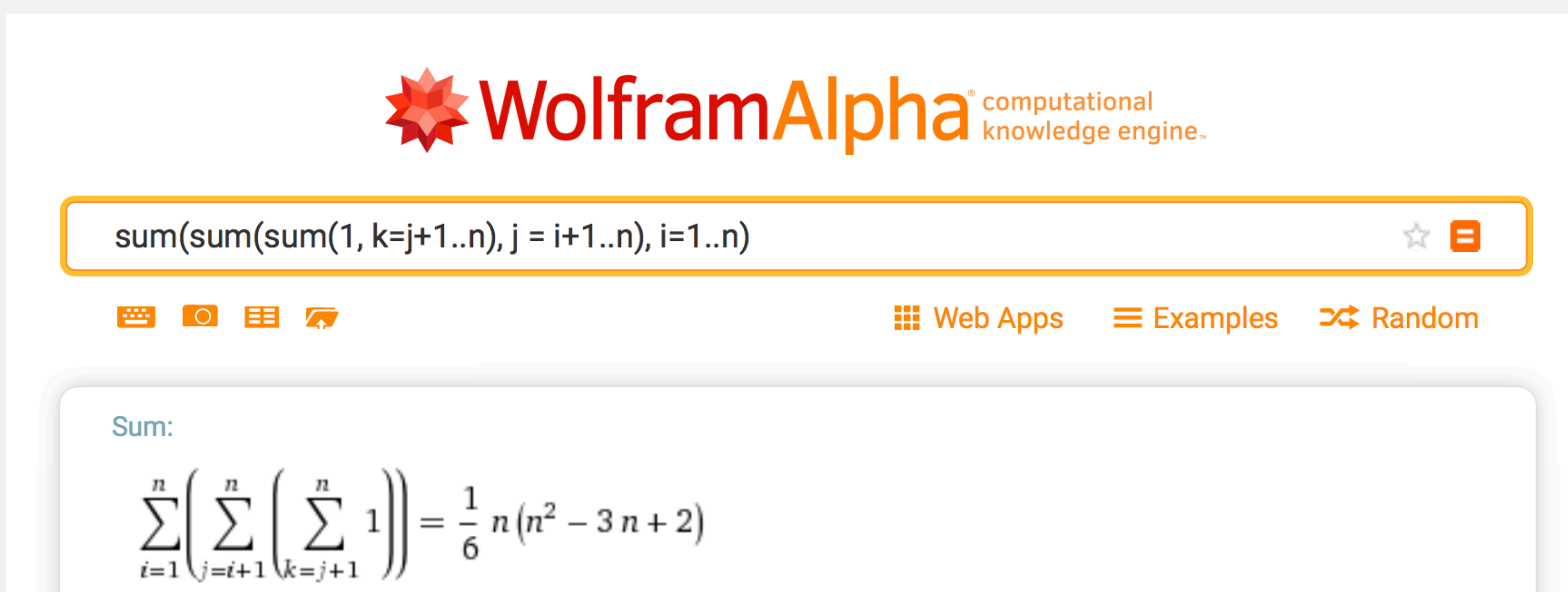
Estimating a discrete sum

Q. How to estimate a discrete sum?

A3. Use Maple or Wolfram Alpha.

```
[wayne:nobel.princeton.edu] > maple15
    |\^/|     Maple 15 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2011
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> factor(sum(sum(sum(1, k=j+1..n), j = i+1..n), i = 1..n));

                                n (n - 1) (n - 2)
                                -----------------
                                        6
```

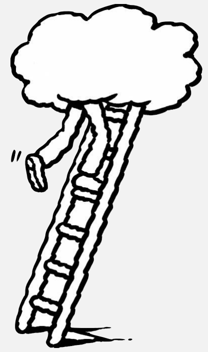


Mathematical models for running time

In theory, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.



costs (depend on machine, compiler)

$$T_n = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

$A =$ array access
$B =$ integer add
$C =$ integer compare
$D =$ increment
$E =$ variable assignment

frequencies
(depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(n) \sim c\, n^3$.

Analysis of algorithms quiz 3

How many array accesses as a function of n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      for (int k = 1; k < n; k = k*2)
         if (a[i] + a[j] >= a[k])
            count++;
```

A. $\sim n^2 \lg n$

B. $\sim 3/2\ n^2 \lg n$

C. $\sim 1/2\ n^3$

D. $\sim 3/2\ n^3$

http://algs4.cs.princeton.edu

1.4 Analysis of Algorithms

- introduction
- observations
- mathematical models
- order-of-growth classifications
- memory

Common order-of-growth classifications

Definition. If $f(n) \sim c\, g(n)$ for some constant $c > 0$, then the order of growth of $f(n)$ is $g(n)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the running time of this code is n^3.

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      for (int k = j+1; k < n; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. Mathematical analysis of running times.

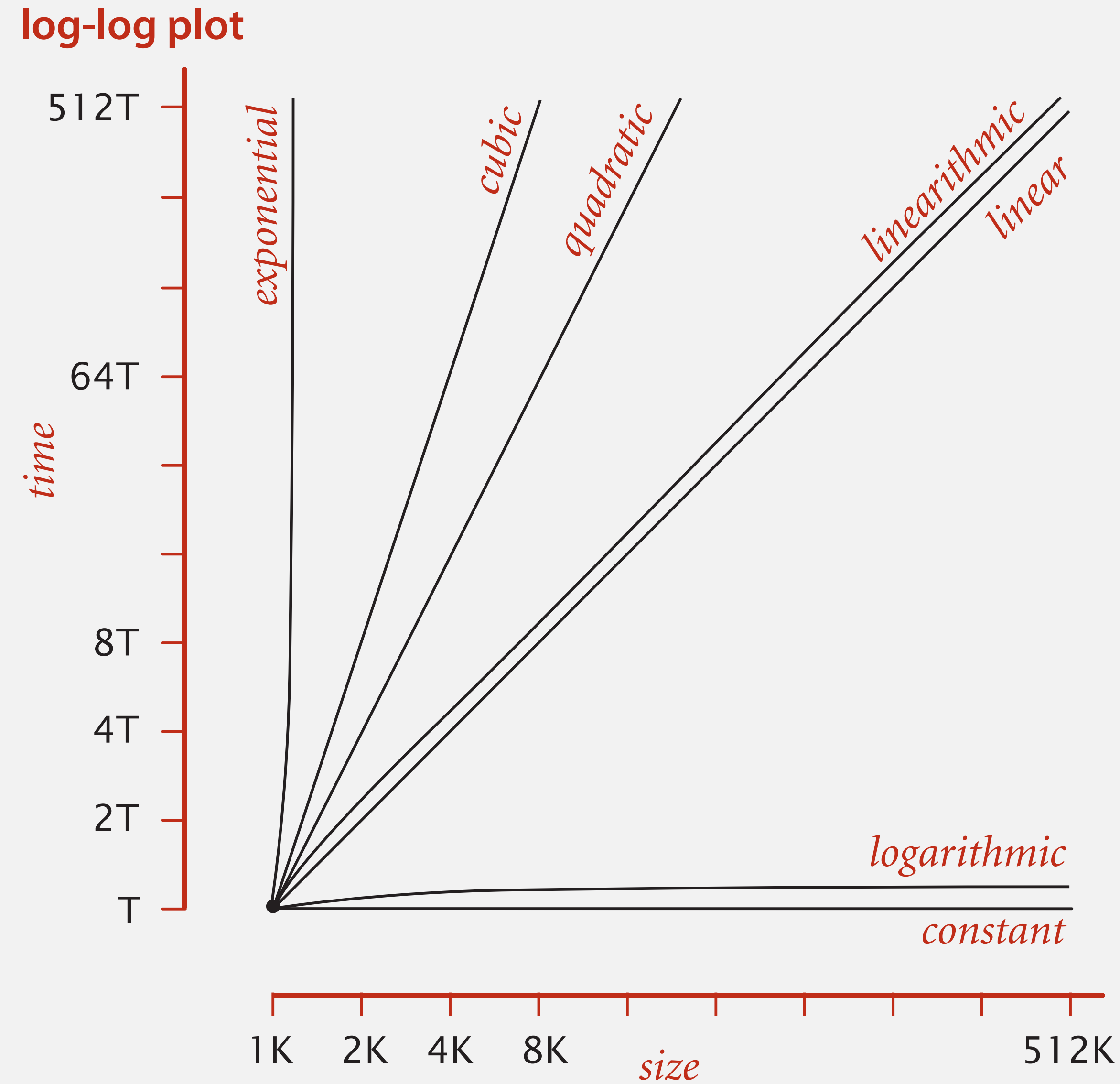
where leading coefficient depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$1, \;\log n, \; n, \; n \log n, \; n^2, \; n^3$, and 2^n

suffices to describe the order of growth of most common algorithms.



Typical orders of growth

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2n)$ / $T(n)$
1	**constant**	`a = b + c;`	statement	add two numbers	1
$\log n$	**logarithmic**	`while (n > 1) { n = n/2; ... }`	divide in half	binary search	~ 1
n	**linear**	`for (int i = 0; i < n; i++) { ... }`	single loop	find the maximum	2
$n \log n$	**linearithmic**	*see mergesort lecture*	divide and conquer	mergesort	~ 2
n^2	**quadratic**	`for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) { ... }`	double loop	check all pairs	4
n^3	**cubic**	`for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) for (int k = 0; k < n; k++) { ... }`	triple loop	check all triples	8
2^n	**exponential**	*see combinatorial search lecture*	exhaustive search	check all subsets	2^n

Binary search

Goal. Given a sorted array and a key, find index of the key in the array?

Binary search. Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.

6	13	14	25	33	43	51	53	64	72	84	93	95	96	97
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Binary search: implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

Extra, Extra - Read All About It: Nearly All Binary Searches and Mergesorts are Broken

Friday, June 02, 2006

Posted by Joshua Bloch, Software Engineer

I remember vividly Jon Bentley's first Algorithms lecture at CMU, where he asked all of us incoming Ph.D. students to write a binary search, and then dissected one of our implementations in front of the class. Of course it was broken, as were most of our implementations. This made a real impression on me, as did the treatment of this material in his wonderful *Programming Pearls* (Addison-Wesley, 1986; Second Edition, 2000). The key lesson was to carefully consider the invariants in your programs.

http://googleresearch.blogspot.com/2006/06/extra-extra-read-all-about-it-nearly.html

Binary search: Java implementation

Invariant. If `key` appears in array `a[]`, then `a[lo]` ≤ `key` ≤ `a[hi]`.

```
public static int binarySearch(int[] a, int key)
{
   int lo = 0, hi = a.length - 1;
   while (lo <= hi)
   {
      int mid = lo + (hi - lo) / 2;
      if      (key < a[mid]) hi = mid - 1;
      else if (key > a[mid]) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

why not mid = (lo + hi) / 2 ?

one "3-way compare"

Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \log_2 n$ key compares to search in a sorted array of size n.

Def. $T(n)$ = max # key compares to search a sorted subarray of length ≤ n.

Binary search recurrence. $T(n) \le T(n/2) + 1$ for $n > 1$, with $T(1) = 1$.

left or right half (floored division)

possible to implement with one 2-way compare (instead of 3-way)

Pf sketch. [assume n is a power of 2]

$$
\begin{aligned}
T(n) &\le T(n/2) + 1 && \text{[given]}\\
&\le T(n/4) + 1 + 1 && \text{[apply recurrence to first term]}\\
&\le T(n/8) + 1 + 1 + 1 && \text{[apply recurrence to first term]}\\
&\;\vdots\\
&\le T(n/n) + \underbrace{1 + 1 + \ldots + 1}_{\log_2 n} && \text{[stop applying, } T(1) = 1 \text{]}\\
&= 1 + \log_2 n
\end{aligned}
$$

http://algs4.cs.princeton.edu

1.4 ANALYSIS OF ALGORITHMS

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *memory*

Basics

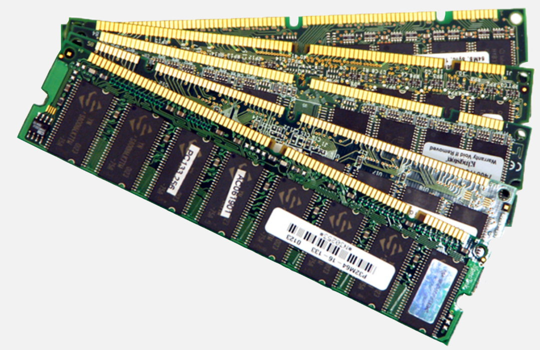
NIST most computer scientists

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million or 2^{20} bytes.

Gigabyte (GB). 1 billion or 2^{30} bytes.



64-bit machine. We assume a 64-bit machine with 8-byte pointers.

some JVMs “compress” ordinary object pointers to 4 bytes to avoid this cost

Typical memory usage for primitive types and arrays

type	bytes
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

primitive types

type	bytes
`char[]`	$2n + 24$
`int[]`	$4n + 24$
`double[]`	$8n + 24$

one-dimensional arrays

type	bytes
`char[][]`	$\sim 2\ m\ n$
`int[][]`	$\sim 4\ m\ n$
`double[][]`	$\sim 8\ m\ n$

two-dimensional arrays

Typical memory usage for objects in Java

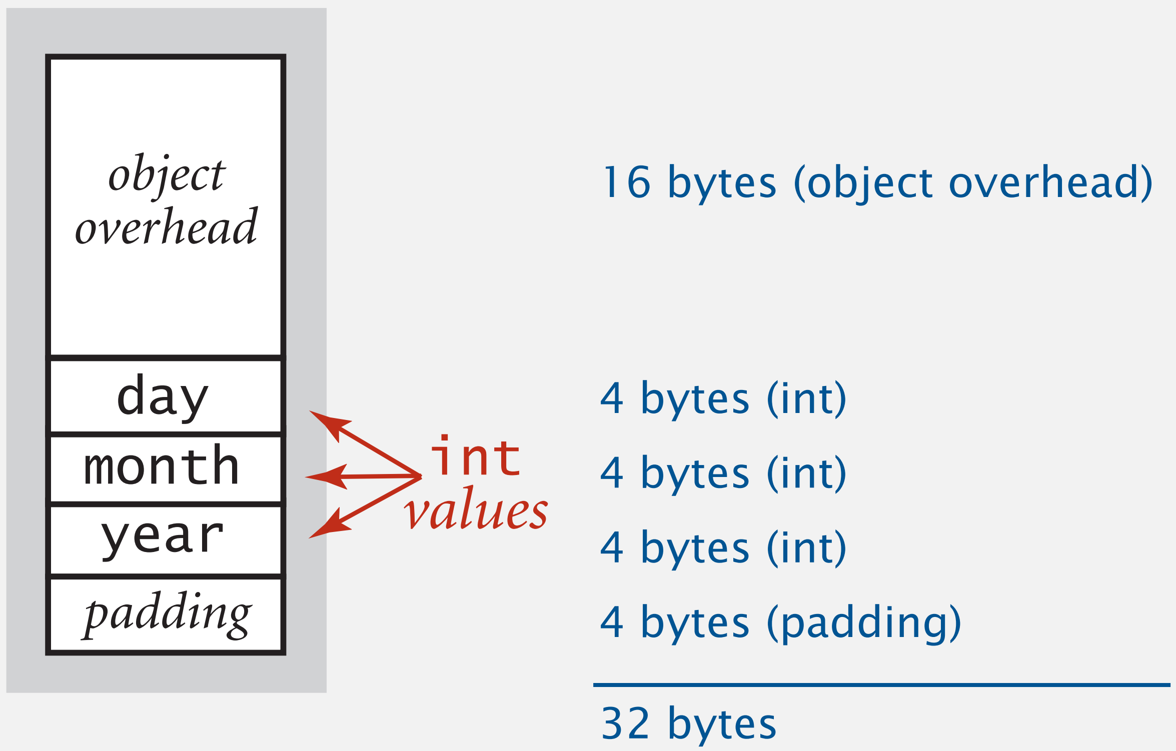
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A `Date` object uses 32 bytes of memory.

```
public class Date
{
   private int day;
   private int month;
   private int year;
...
}
```



Typical memory usage summary

Total memory usage for a data type value:

- Primitive type: 4 bytes for `int`, 8 bytes for `double`, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable.
- Padding: round up to multiple of 8 bytes.

+ 8 extra bytes per inner class object
(for reference to enclosing class)

Note. Depending on application, we may want to count memory for any referenced objects (recursively).

Analysis of algorithms quiz 4

How much memory does a `WeightedQuickUnionUF` use as a function of n ?

A. $\sim 4\,n$ *bytes*

B. $\sim 8\,n$ *bytes*

C. $\sim 4\,n^2$ *bytes*

D. $\sim 8\,n^2$ *bytes*

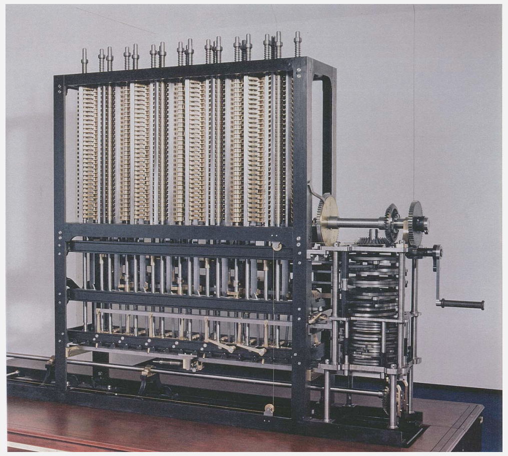
```
public class WeightedQuickUnionUF
{
   private int[] parent;
   private int[] size;
   private int count;

   public WeightedQuickUnionUF(int n)
   {
      parent = new int[n];
      size   = new int[n];

      count = 0;
      for (int i = 0; i < n; i++)
         parent[i] = i;
      for (int i = 0; i < n; i++)
         size[i] = 1;
   }
   ...
}
```

Turning the crank: summary

Empirical analysis.



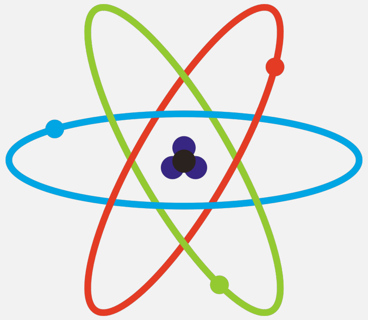
- Execute program to perform experiments.
- Assume power law.
- Formulate a hypothesis for running time.
- Model enables us to make predictions.

Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to explain behavior.

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \lceil n/2^{h+1} \rceil \, h \;\sim\; n$$

Scientific method.



- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.