Homework 5

Out: *Dec 5* Due: *Dec 13*

1. Prove von Neumann's min max theorem. You can assume LP duality.

2. (Braess's paradox; wellknown to transportation planners) Figure (a) depicts a simple network of roads (each is one-way for simplicity) from point s to t. The number on the edge is the time to traverse that road. When we say the travel time is x, we mean that the time scales *linearly* with the amount of traffic in it.

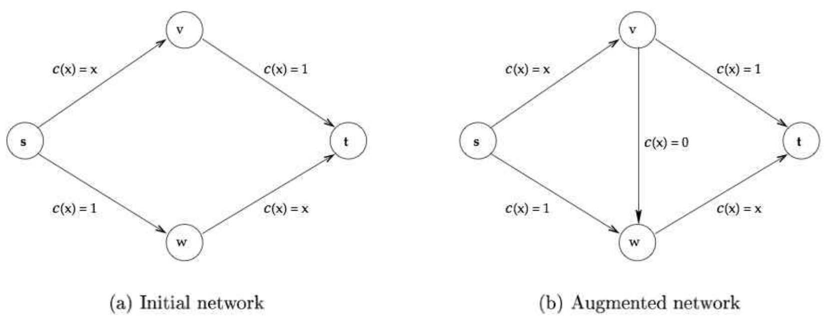

Figure 1: Braess's paradox

One unit of traffic (a large number of individual drivers) need to travel from s to t. (Actually assume is it just a tiny bit less than one unit.) Each driver's choice of route can be seen as a move in a multiplayer game. What is the Nash equilibrium and what is each driver's travel time to t in this equilibrium?

Figure (b) depicts the same network with a new superfast highway constructed from v to w. What is the new Nash equilibrium and the new travel time?

3. Show that approximating the number of simple cycles within a factor 100 in a directed graph is NP-hard. (Hint: Show that if there is a polynomial-time algorithm for this task, then we can solve the Hamiltonian cycle problem in directed graphs, which is NP-hard. Here the exact constant 100 is not important, and can even be replaced by, say, n.)

4. (Extra credit) (*Sudan's list decoding*) Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \in F^2$ where $F = GF(q)$ and $q \gg n$. We say that a polynomial $p(x)$ *describes* k of these pairs if $p(a_i) = b_i$ for k values of i. This question concerns an algorithm that recovers p even if $k < n/2$ (in other words, a majority of the values are wrong).

(a) Show that there exists a bivariate polynomial $Q(z, x)$ of degree at most $\lceil\sqrt{n}\rceil + 1$ in z and x such that $Q(b_i, a_i) = 0$ for each $i = 1, \ldots, n$. Show also that there is an efficient (poly(n) time) algorithm to construct such a Q.

(b) Show that if $R(z, x)$ is a bivariate polynomial and $g(x)$ a univariate polynomial then $z - g(x)$ divides $R(z, x)$ iff $R(g(x), x)$ is the 0 polynomial.

(c) Suppose $p(x)$ is a degree d polynomial that describes k of the points. Show that if d is an integer and $k > (d + 1)(\lceil\sqrt{n}\rceil + 1)$ then $z - p(x)$ divides the bivariate polynomial $Q(z, x)$ described in part (a). (Aside: Note that this places an upper bound on the number of such polynomials. Can you improve this upper bound by other methods?)

(There is a randomized polynomial time algorithm due to Berlekamp that factors a bivariate polynomial. Using this we can efficiently recover all the polynomials p of the type described in (c). This completes the description of Sudan's algorithm for *list decoding.*)

5. (Extra Credit) Show that the simple SDP for Balanced Separator (the one without triangle inequality constraint) is a very poor approximation to the optimum on the cycle graph. (Just n nodes connected in a simple undirected cycle.)