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3.3 BALANCED SEARCH TREES

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

Last updated on Oct 19, 2015, 4:15 PM

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
goal	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`

Challenge. Guarantee performance.

optimized for teaching and coding;
introduced to the world in this course!

This lecture. 2–3 trees, left-leaning red–black BSTs, B-trees.

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3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

2-3 tree

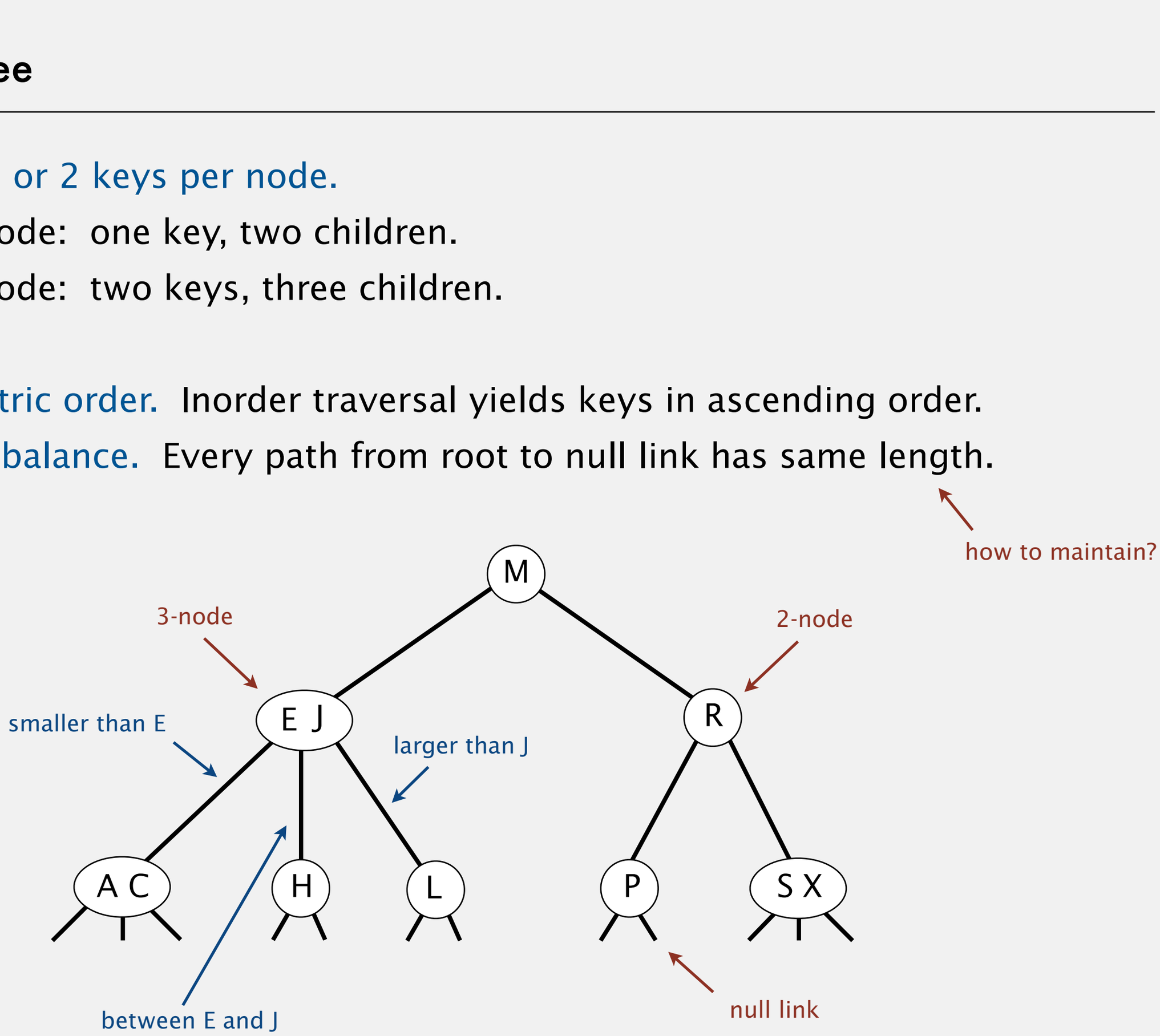
Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.

how to maintain?

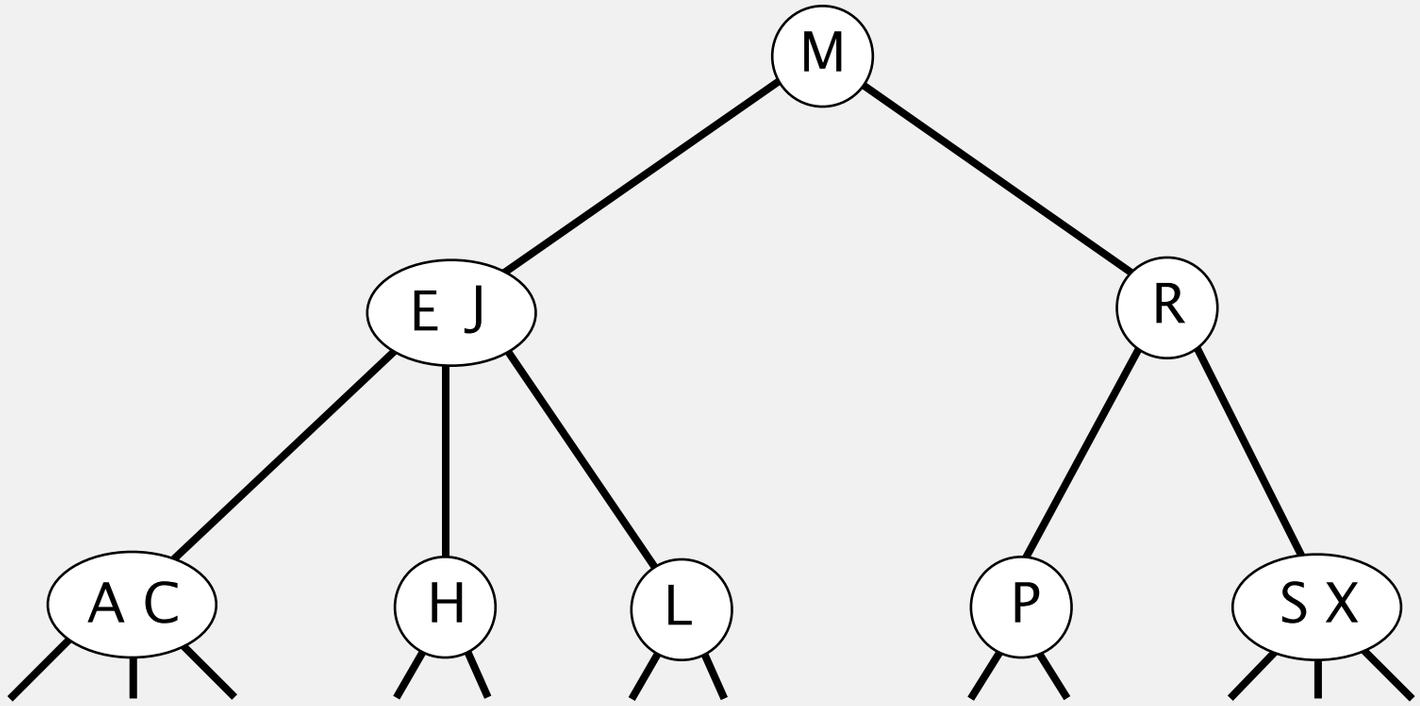


2-3 tree demo

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

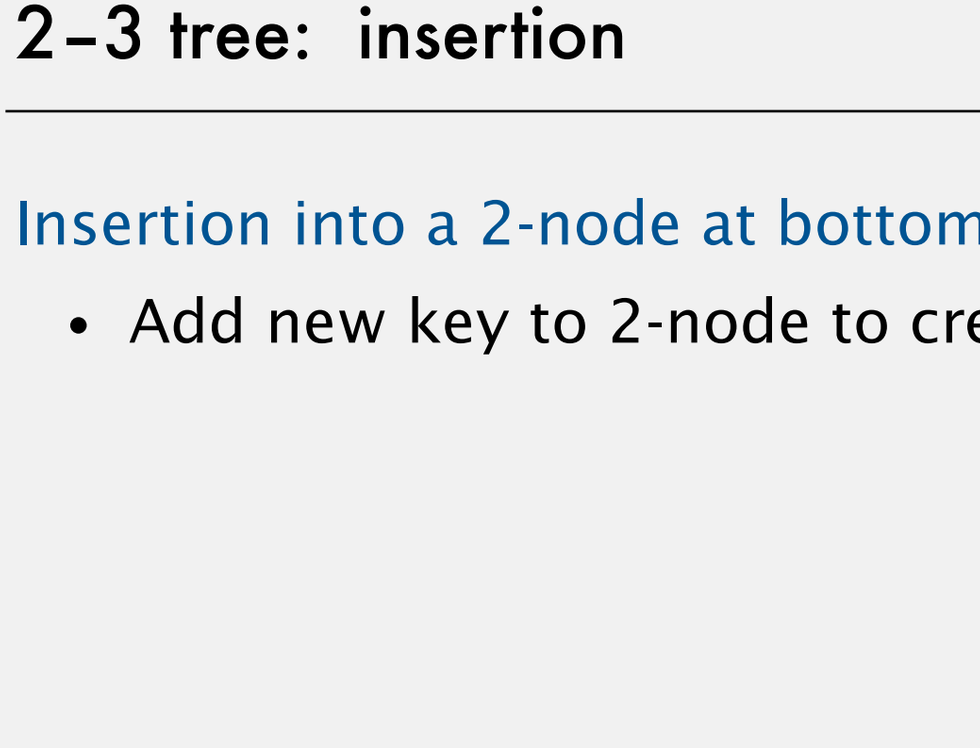


2-3 tree: insertion

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G

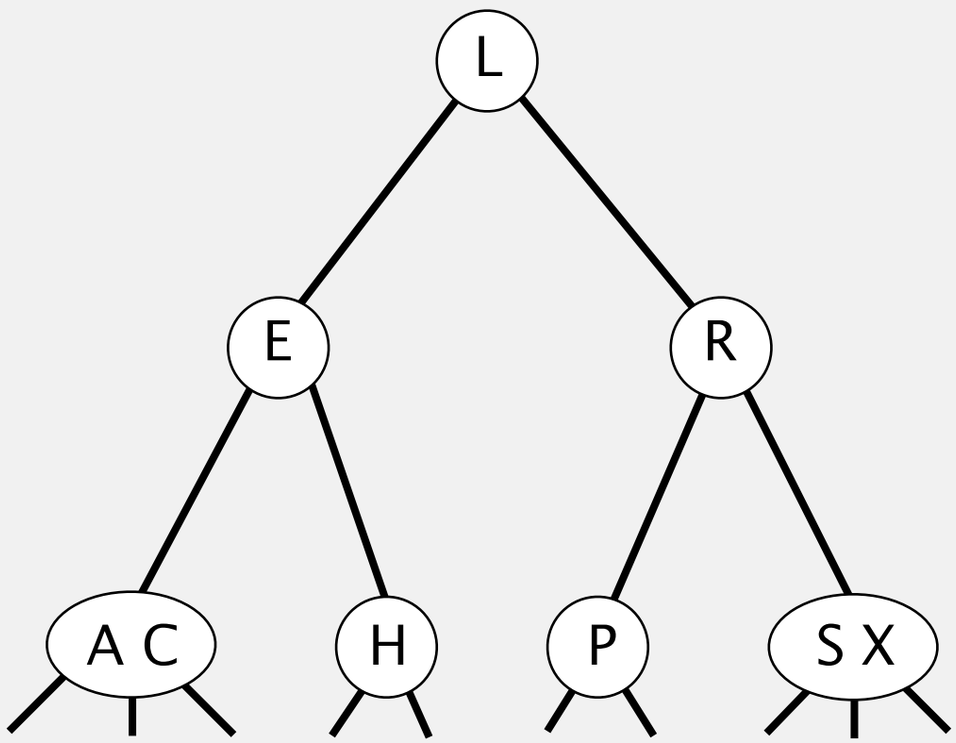


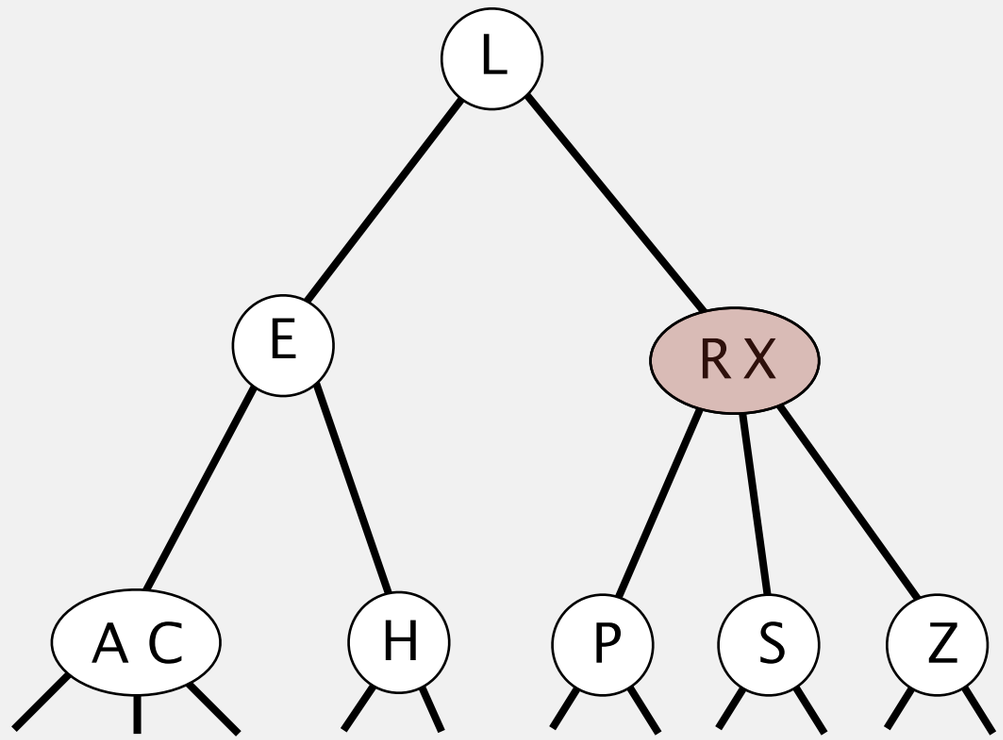
2-3 tree: insertion

Insertion into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert Z





2-3 tree: global properties

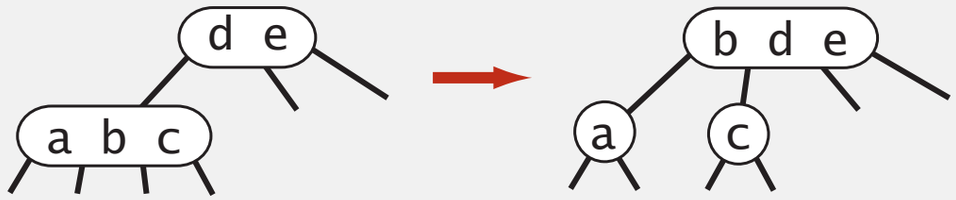
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.

root

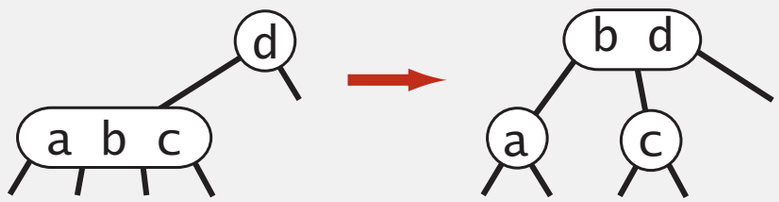
parent is a 3-node

left

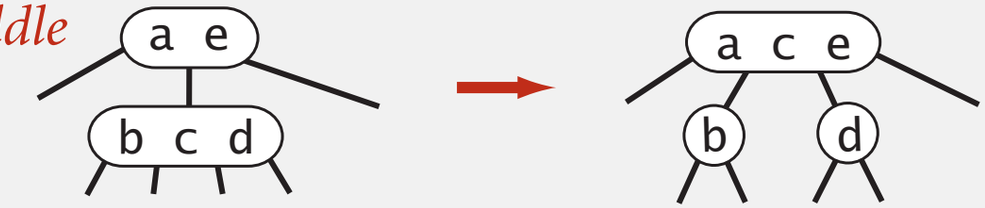


parent is a 2-node

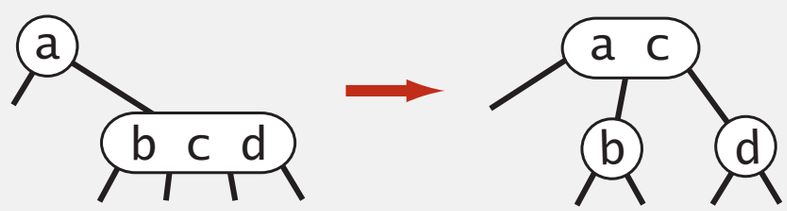
left



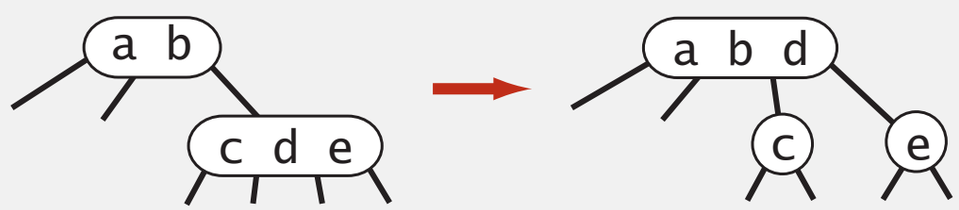
middle



right

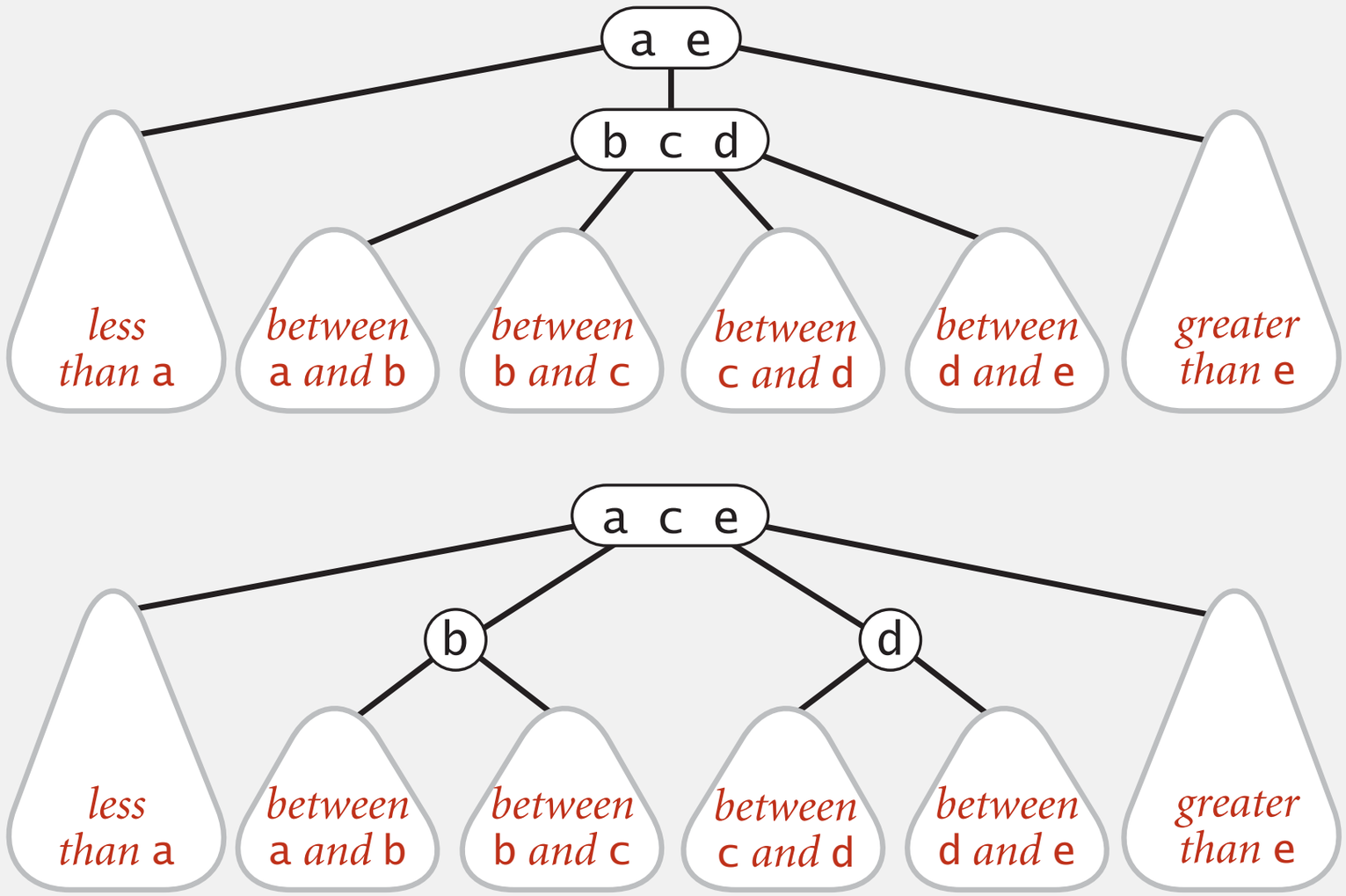


right



2-3 tree: performance

Splitting a 4-node is a local transformation: constant number of operations.



Balanced search trees: quiz 1

What is the range of heights of a 2–3 tree with N keys (best / worst case)?

A. $\sim \log_4 N$ / $\sim \log_3 N$

B. $\sim \log_3 N$ / $\sim \log_2 N$

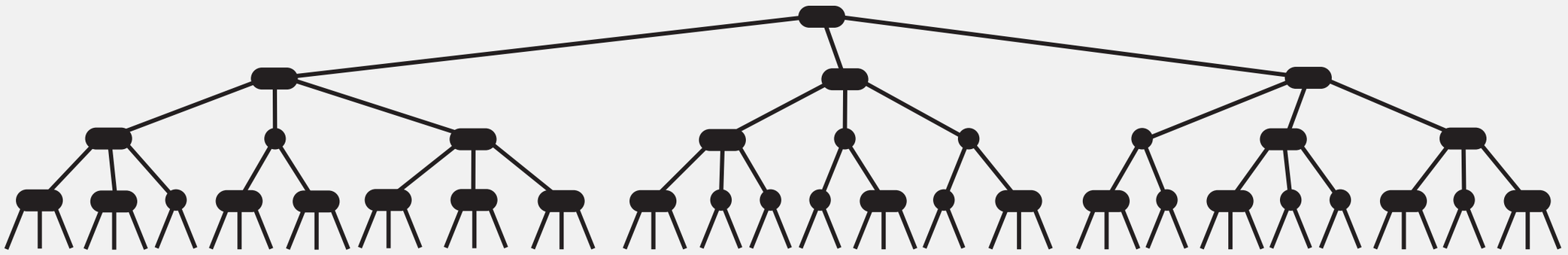
C. $\sim \log_3 N$ / $\sim 2 \log_2 N$

D. $\sim \log_3 N$ / $\sim N$

E. *I don't know.*

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



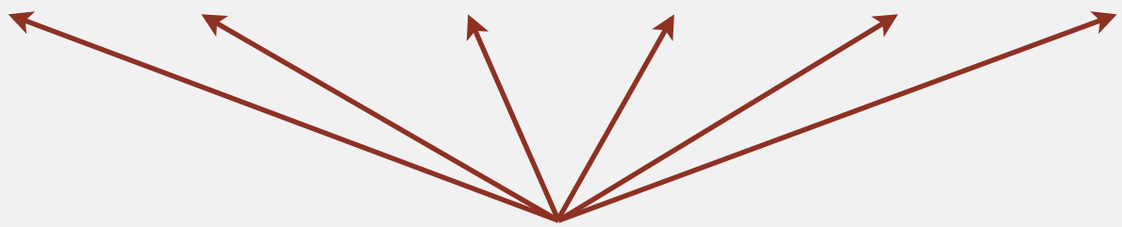
Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`



but hidden constant c is large
(depends upon implementation)

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

fantasy code

```
public void put(Key key, Value val)
{
   Node x = root;
   while (x.getTheCorrectChild(key) != null)
   {
      x = x.getTheCorrectChildKey();
      if (x.is4Node()) x.split();
   }
   if      (x.is2Node()) x.make3Node(key, val);
   else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it, but there's a better way.

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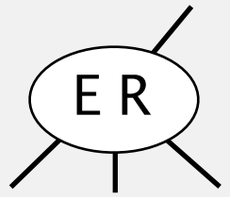
3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

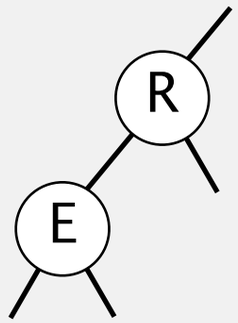
How to implement 2-3 trees with binary trees?

Challenge. How to represent a 3 node?



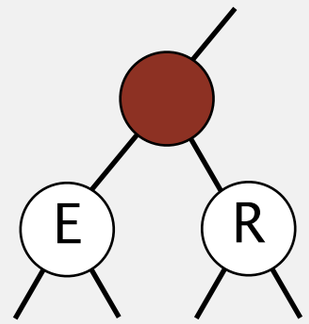
Approach 1. Regular BST.

- No way to tell a 3-node from a 2-node.
- Cannot map from BST back to 2–3 tree.



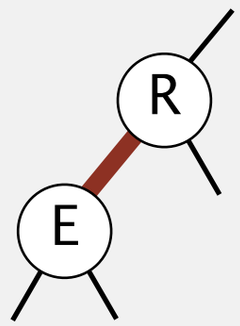
Approach 2. Regular BST with red "glue" nodes.

- Wastes space, wasted link.
- Code probably messy.



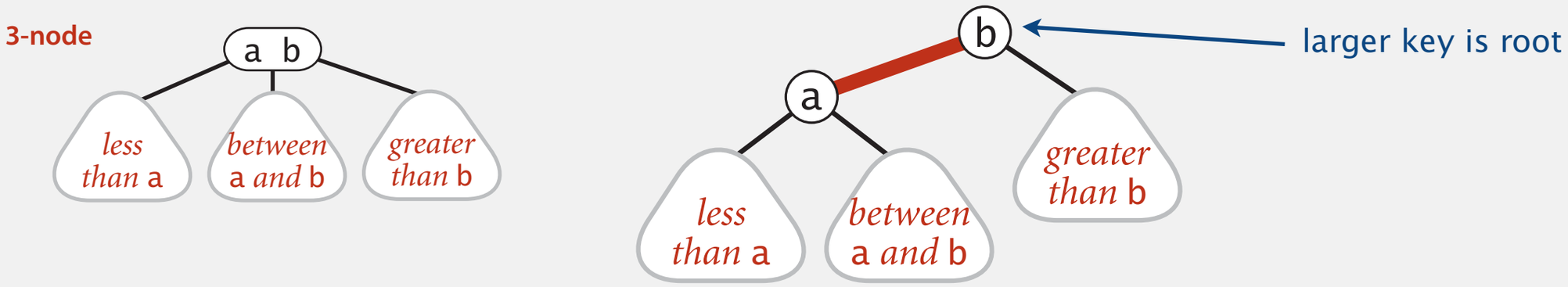
Approach 3. Regular BST with red "glue" links.

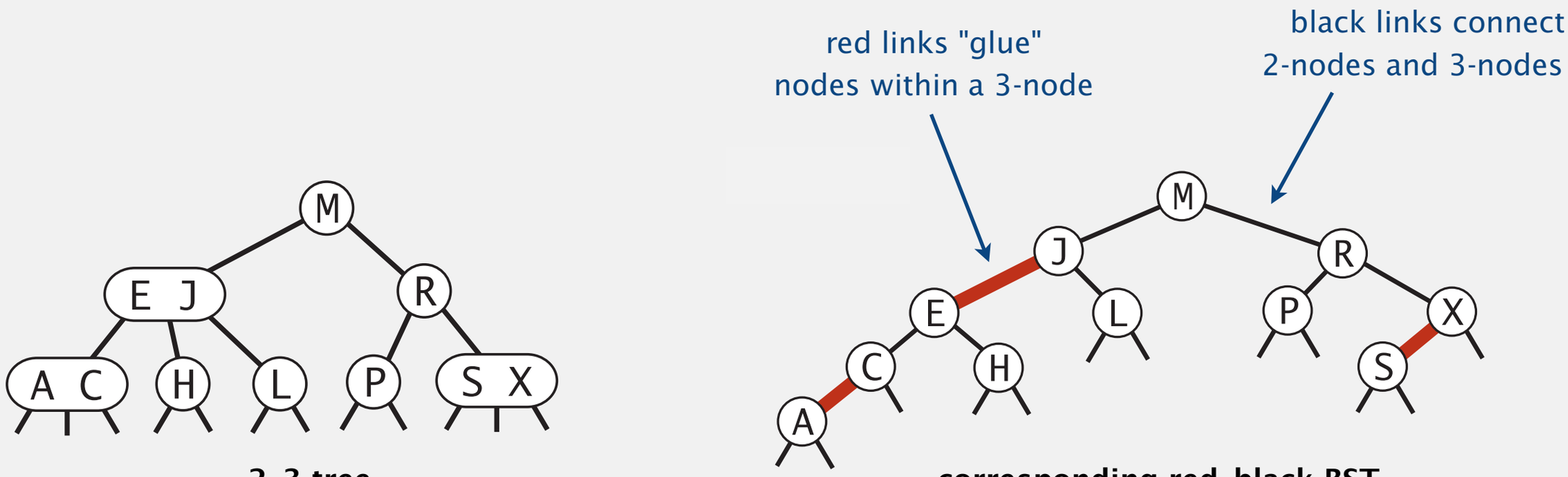
- Widely used in practice.
- Arbitrary restriction: red links lean left.



Left-leaning red–black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.



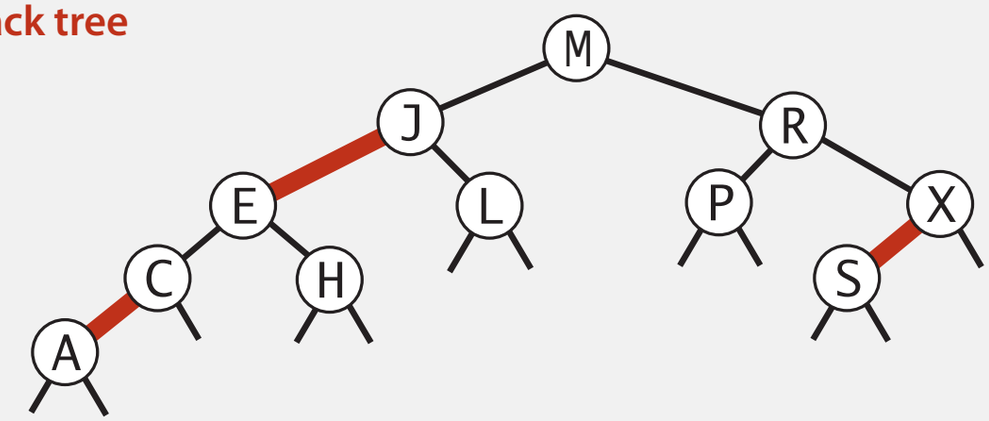


2–3 tree **corresponding red–black BST**

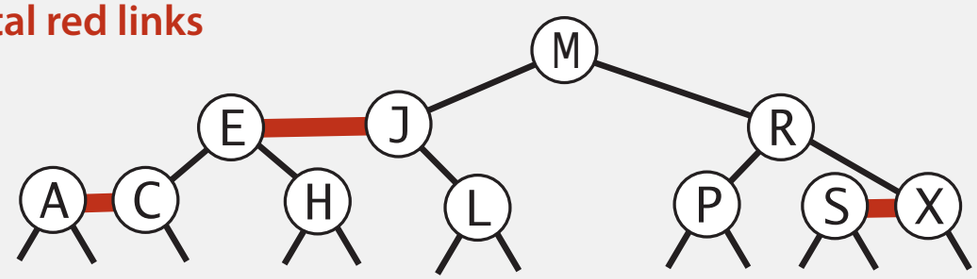
Left-leaning red–black BSTs: 1–1 correspondence with 2–3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

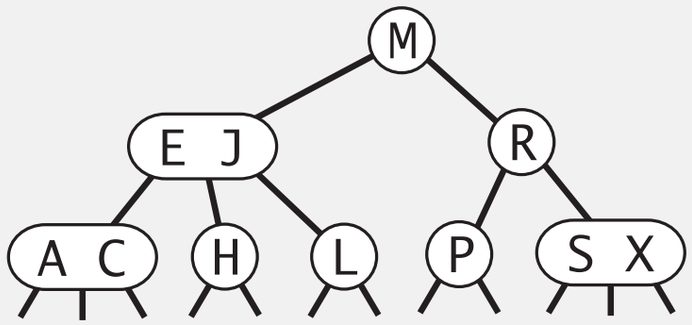
red–black tree



horizontal red links



2-3 tree

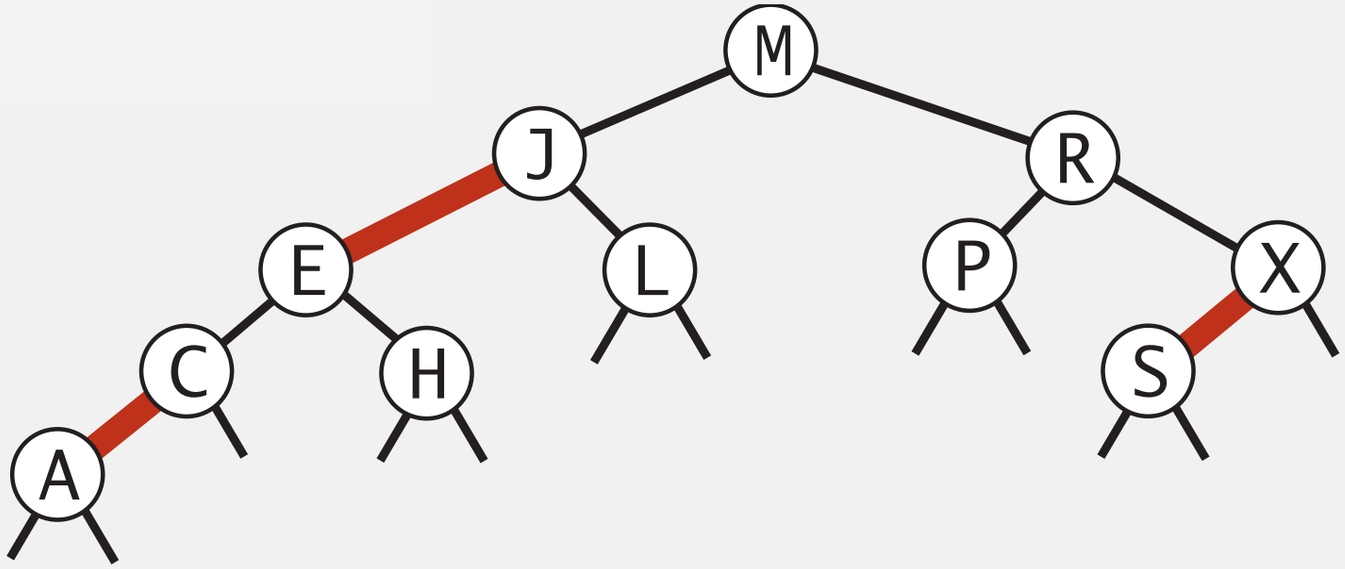


An equivalent definition

A BST such that:

- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"

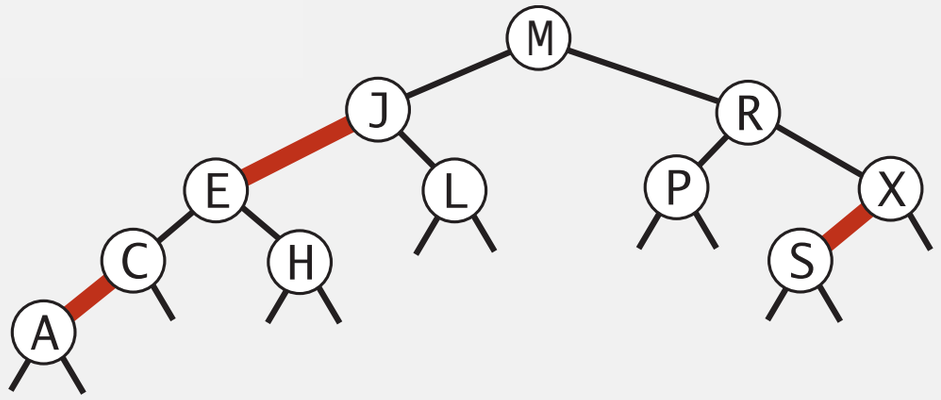


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Most other ops (e.g., floor, iteration, selection) are also identical.

Red-black BST representation

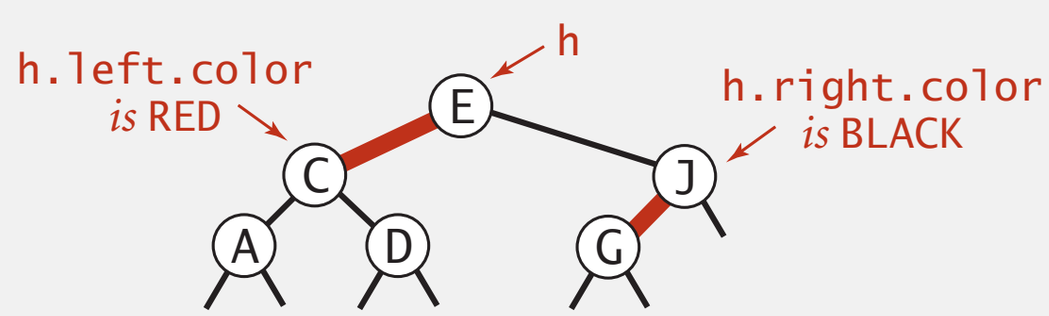
Each node is pointed to by precisely one link (from its parent) ⇒
can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}
```

```
private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



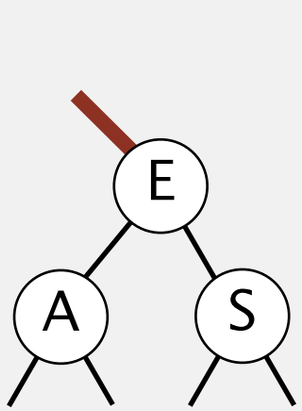
Insertion into a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

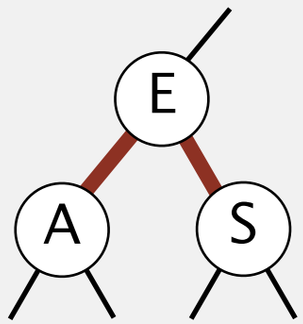
During internal operations, maintain:

- Symmetric order.
- Perfect black balance.

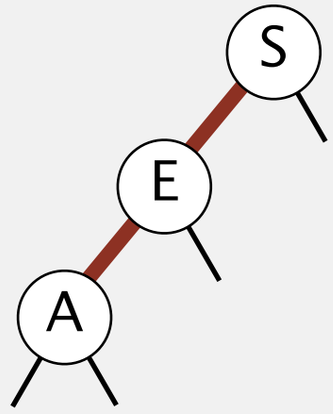
  [but not necessarily color invariants]



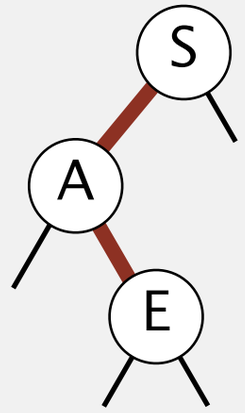
right-leaning red link



two red children (a temporary 4-node)



left-left red (a temporary 4-node)

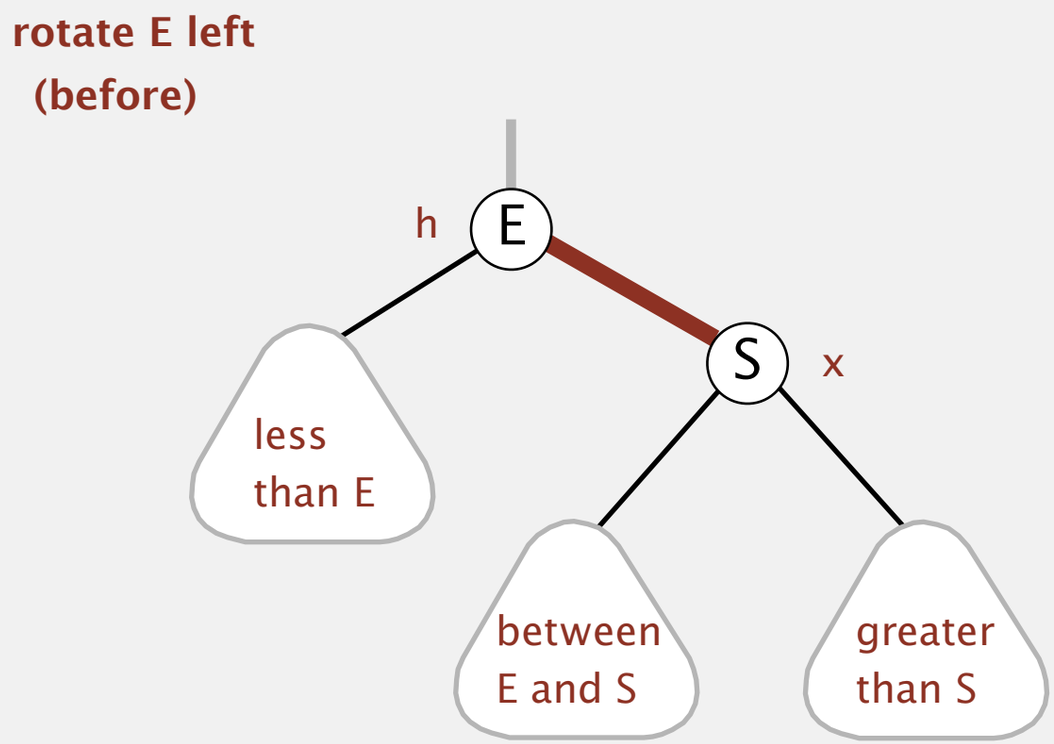


left-right red (a temporary 4-node)

How? Apply elementary red-black BST operations: rotation and color flip.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

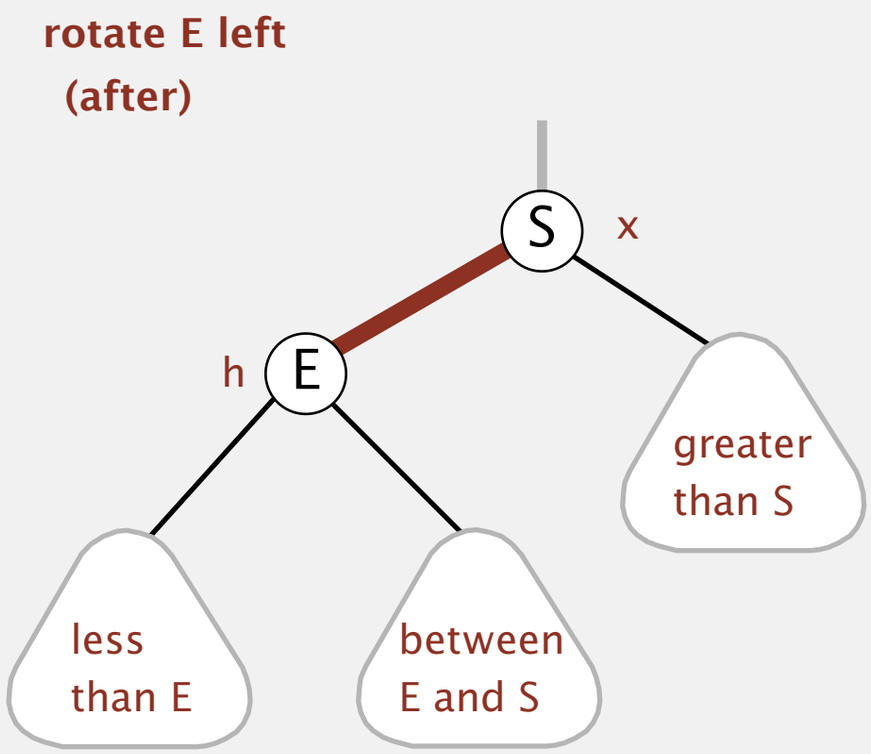



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

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   h.right = x.left;
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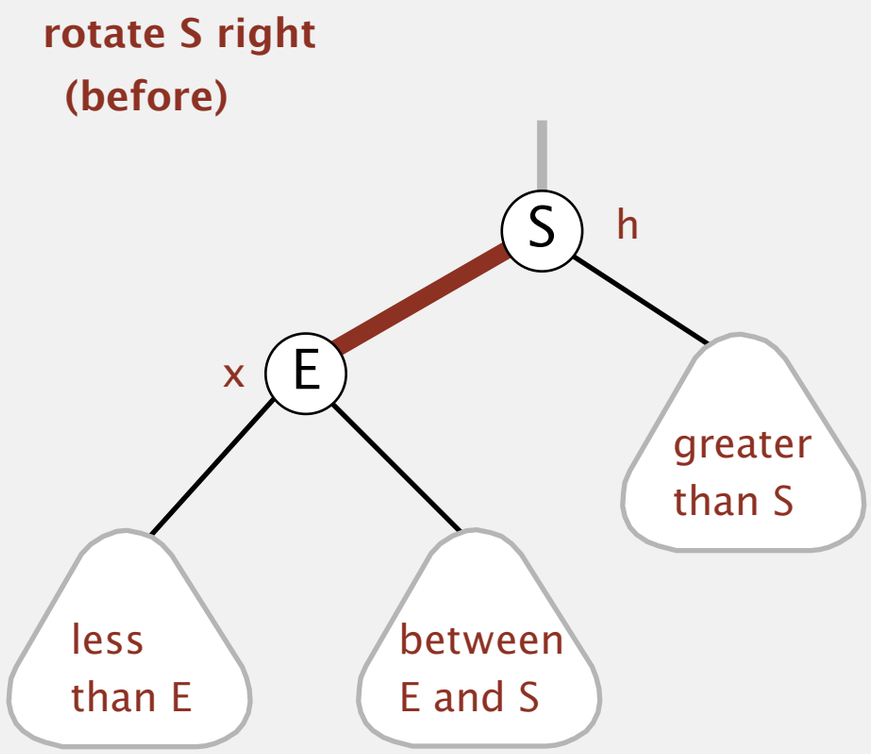
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

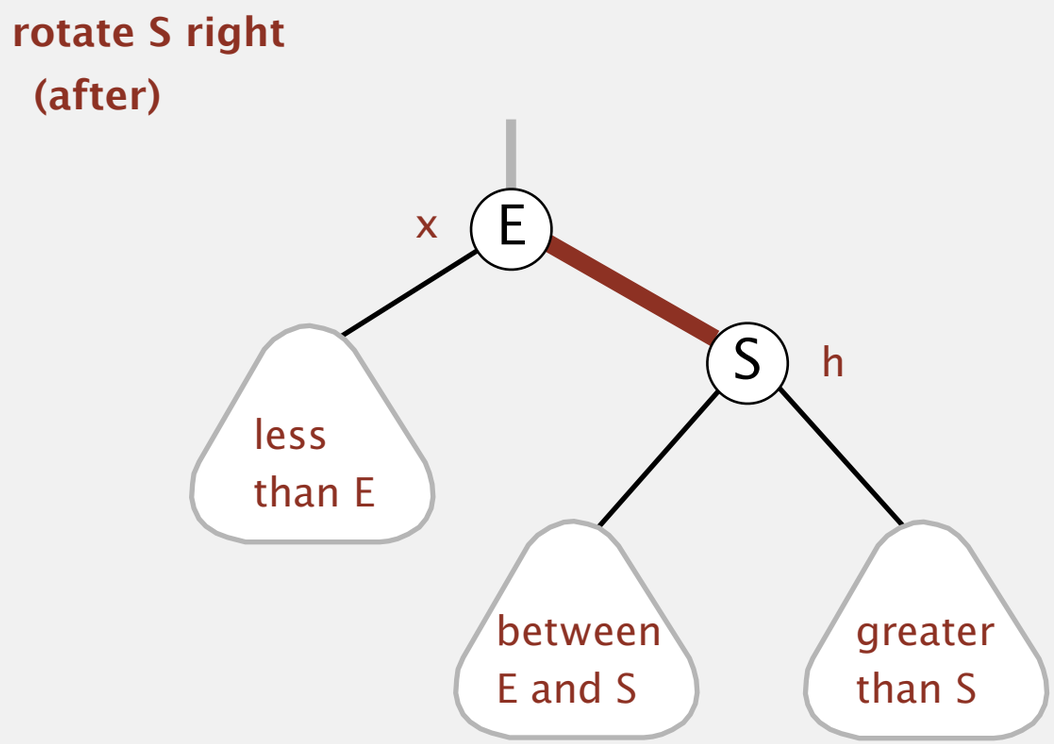



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

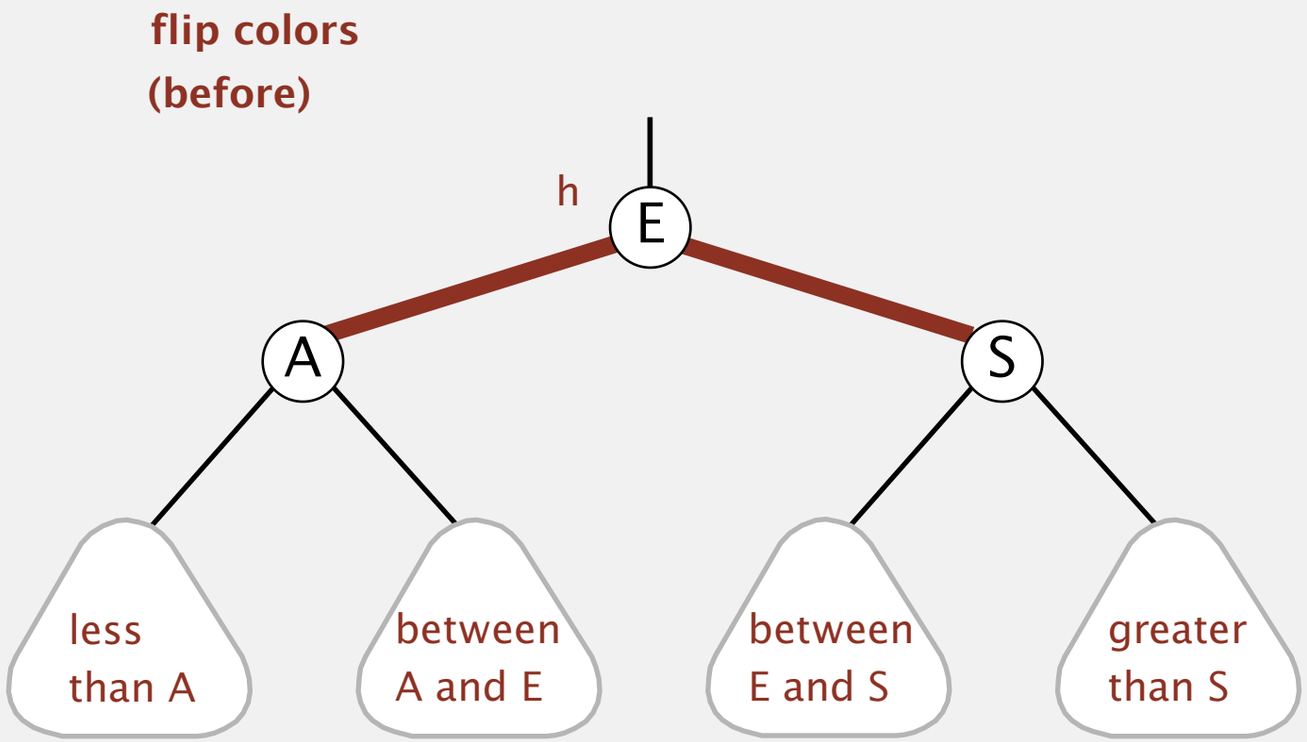



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

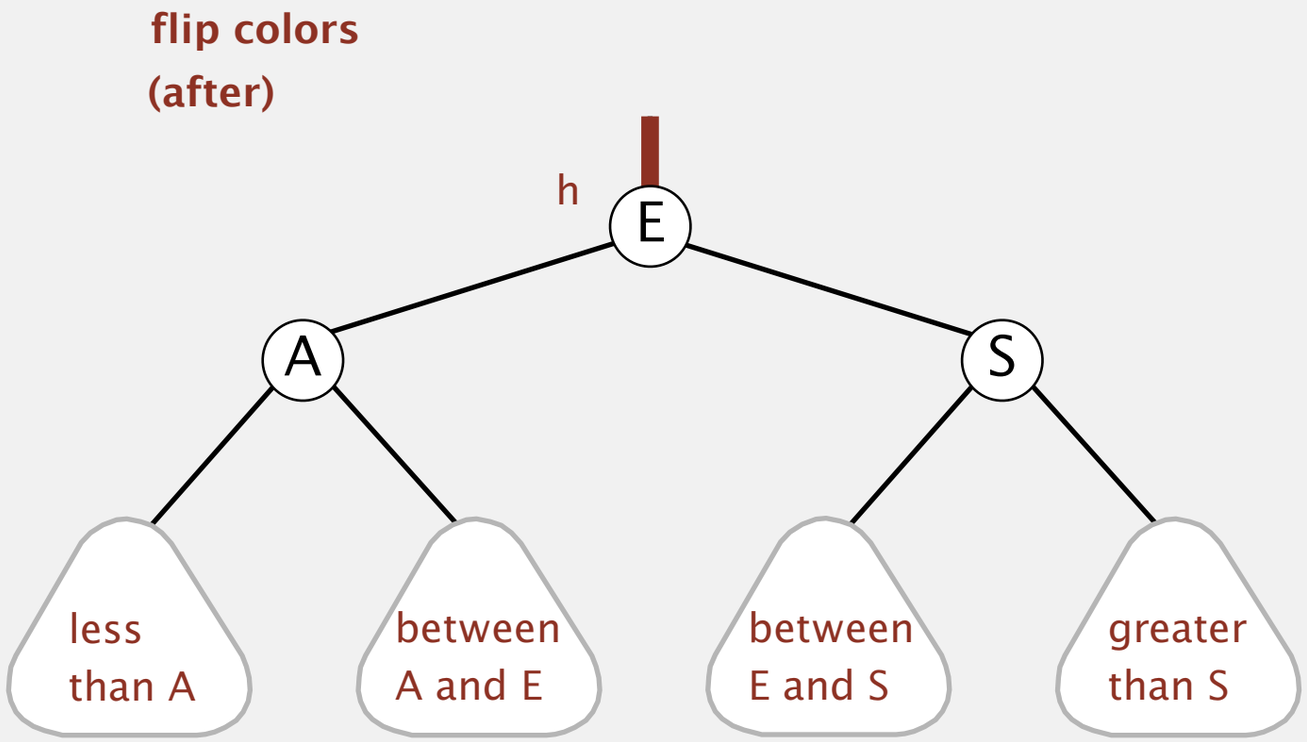



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.




```
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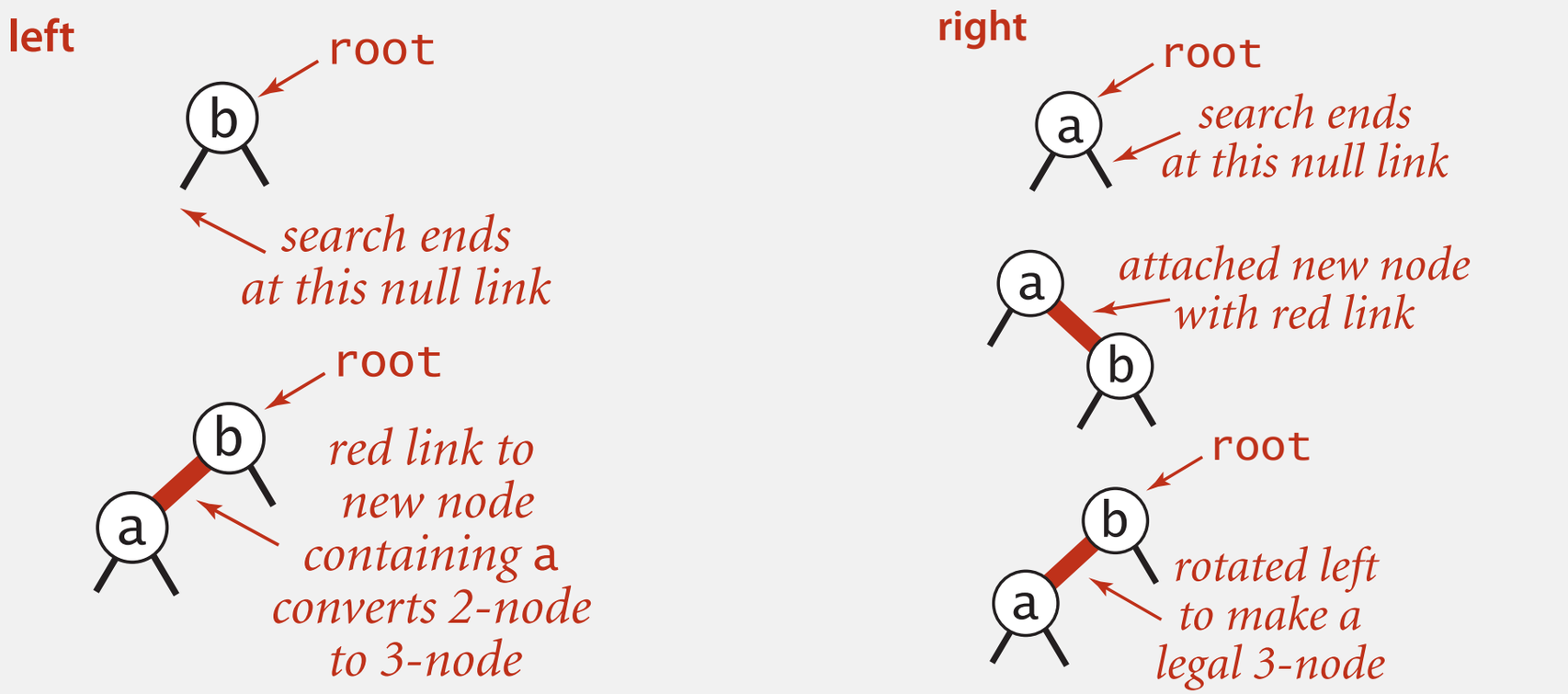
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Insertion into a LLRB tree

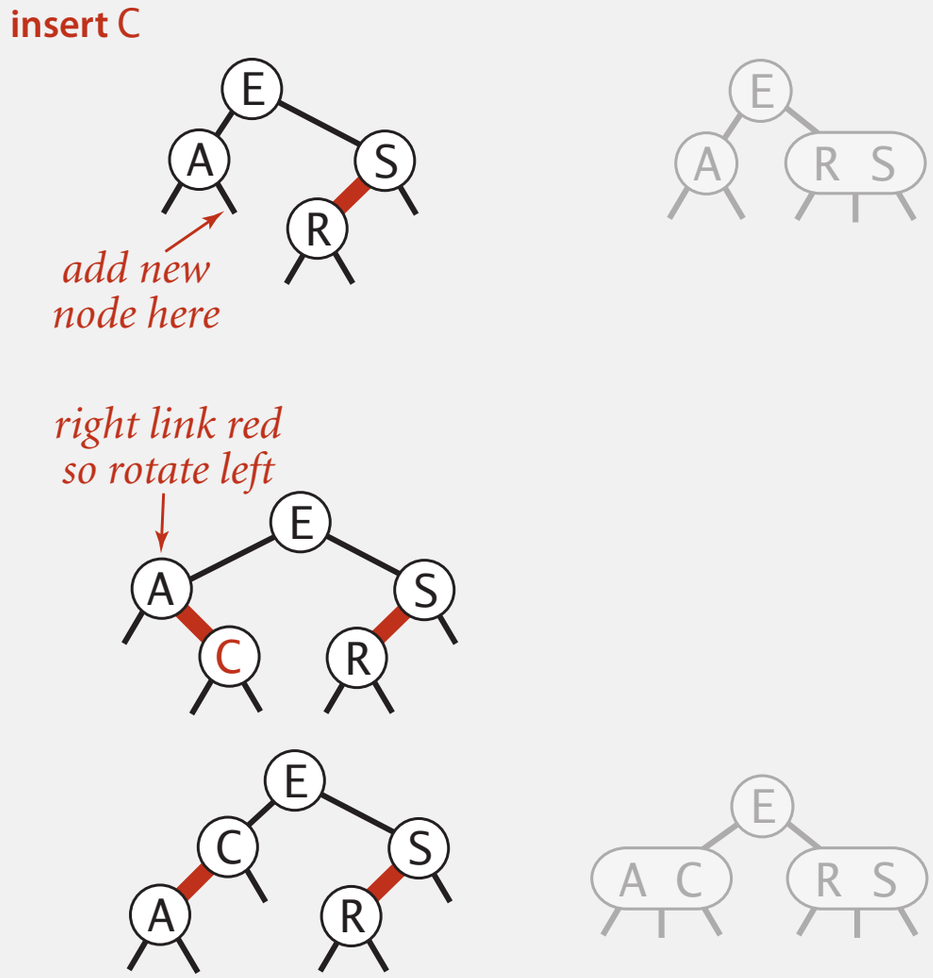
Warmup 1. Insert into a tree with exactly 1 node.



Insertion into a LLRB tree

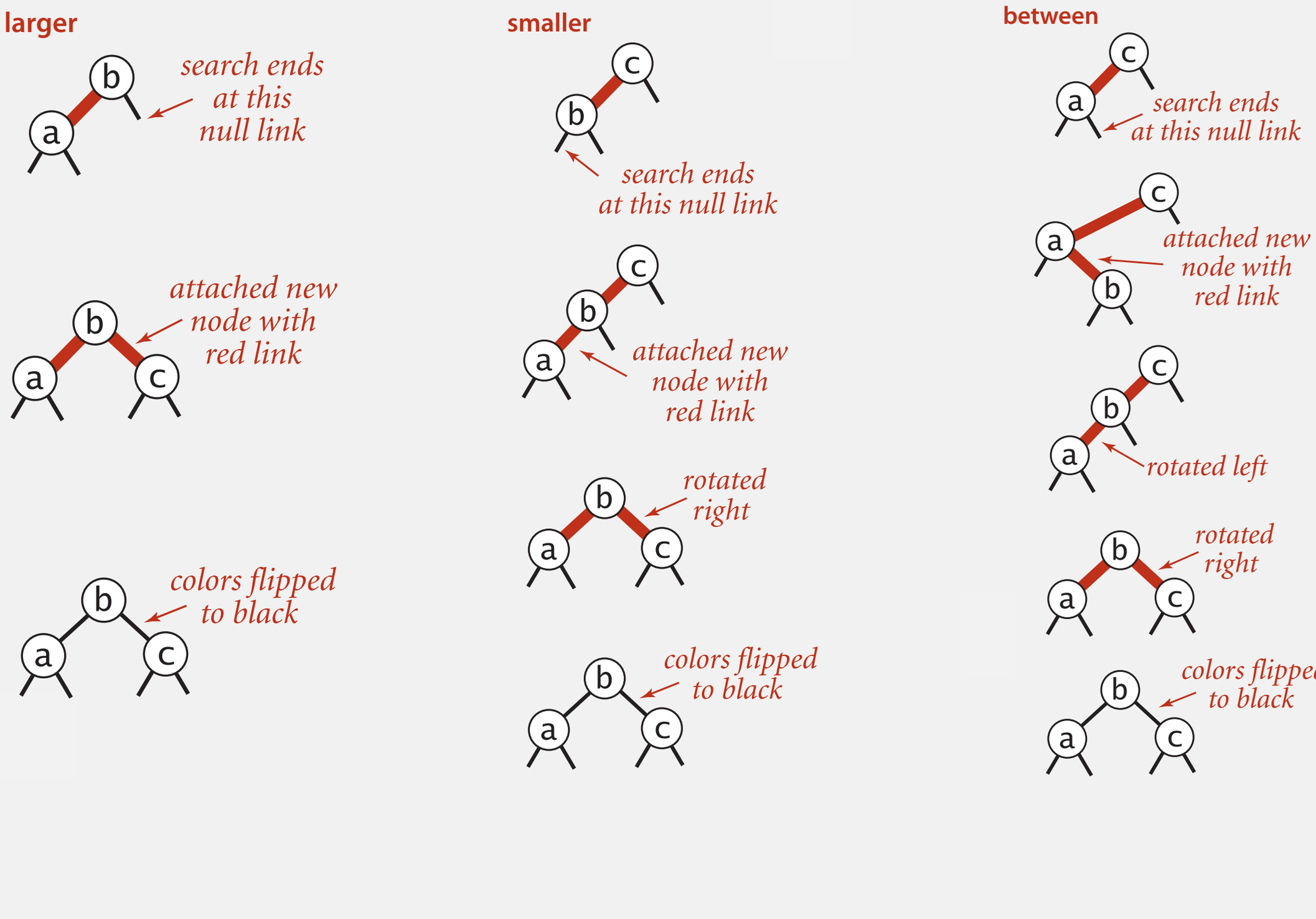
Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- If new red link is a right link, rotate left. ← to fix color invariants



Insertion into a LLRB tree

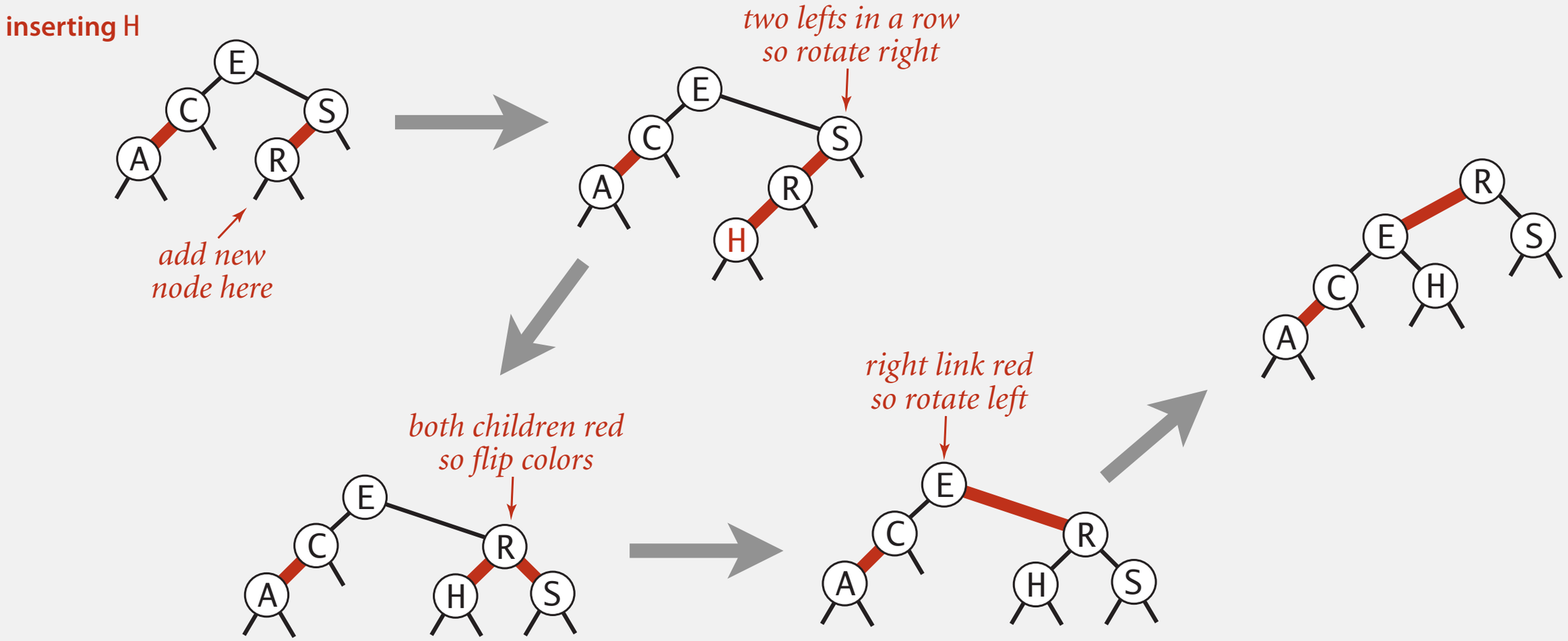
Warmup 2. Insert into a tree with exactly 2 nodes.



Insertion into a LLRB tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed). ← to fix color invariants

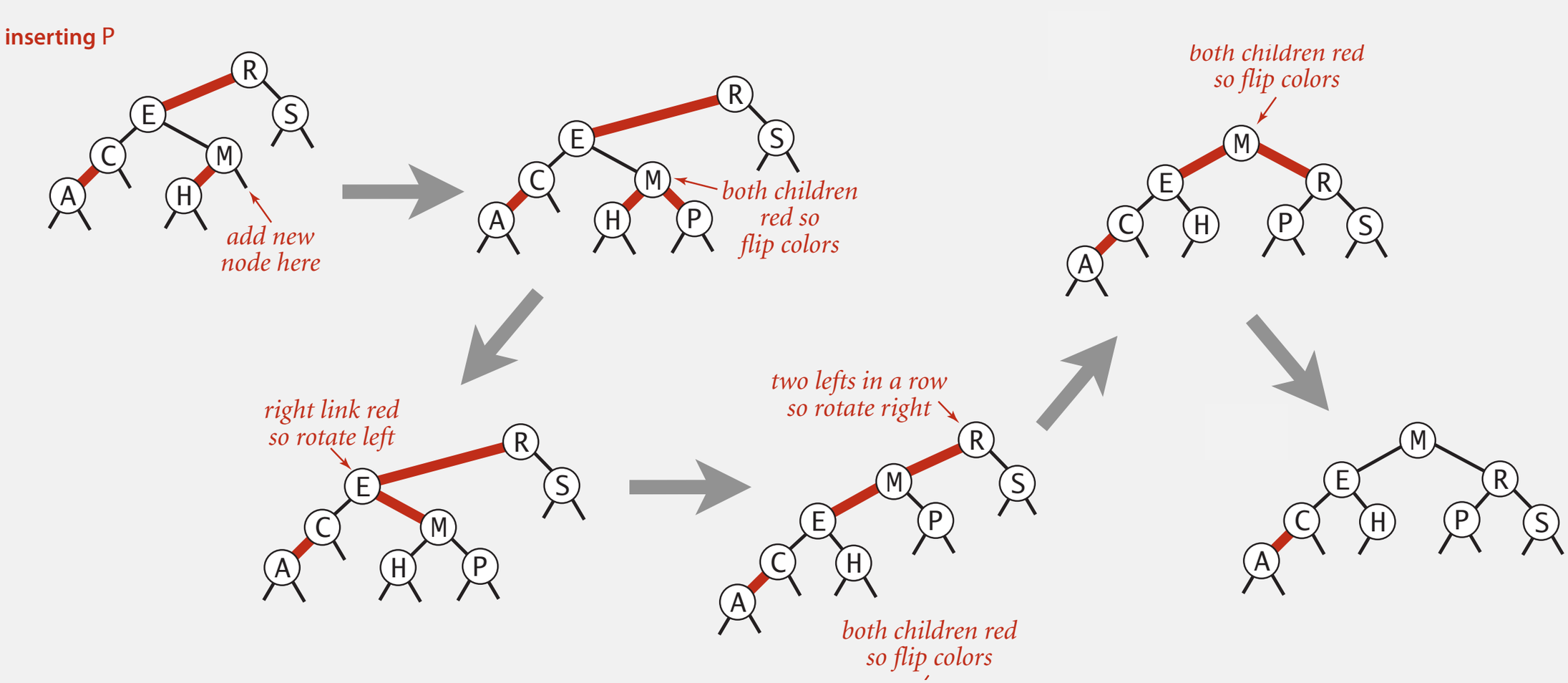


Insertion into a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

← to fix color invariants

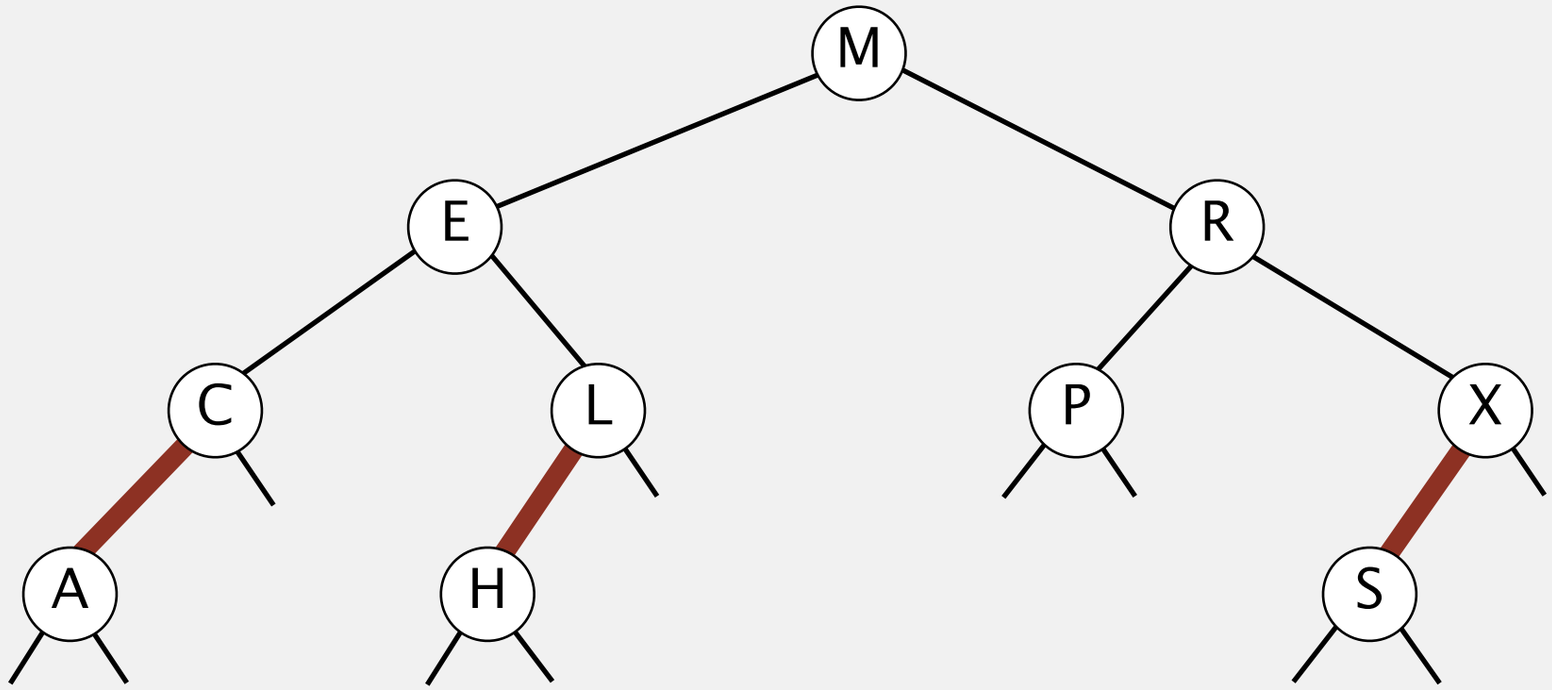


Red–black BST construction demo

insert S

Red-black BST construction demo

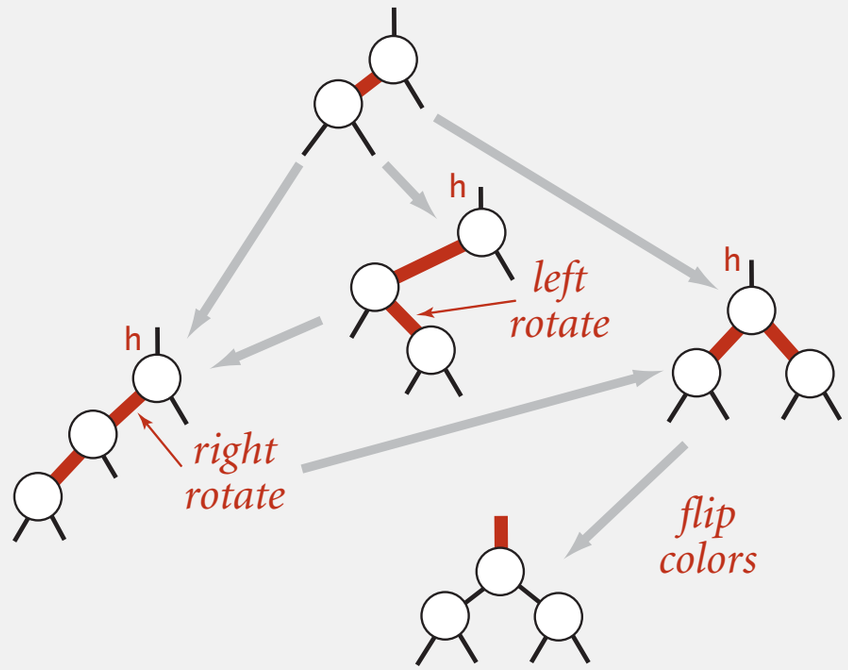
red-black BST



Insertion into a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

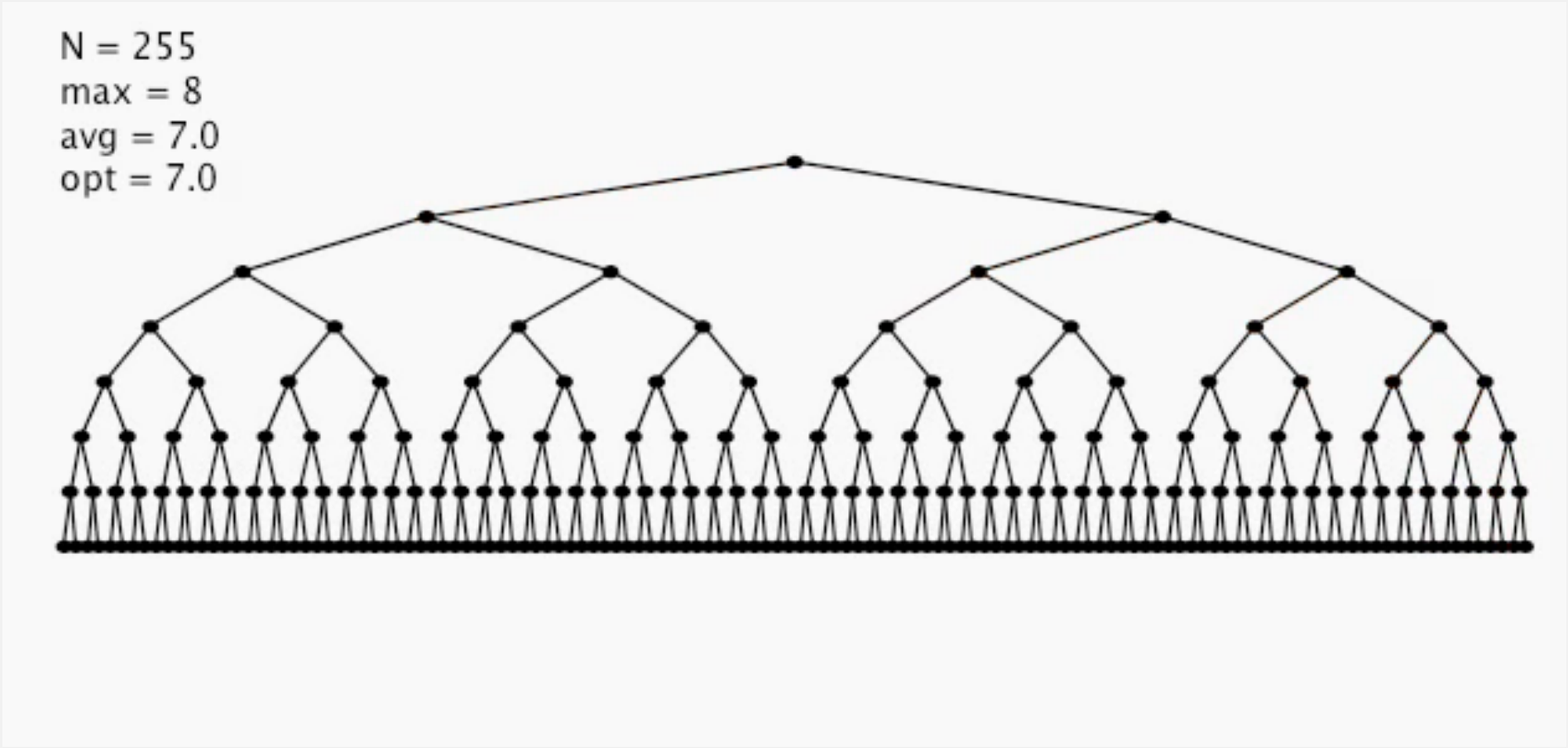
lean left

balance 4-node

split 4-node

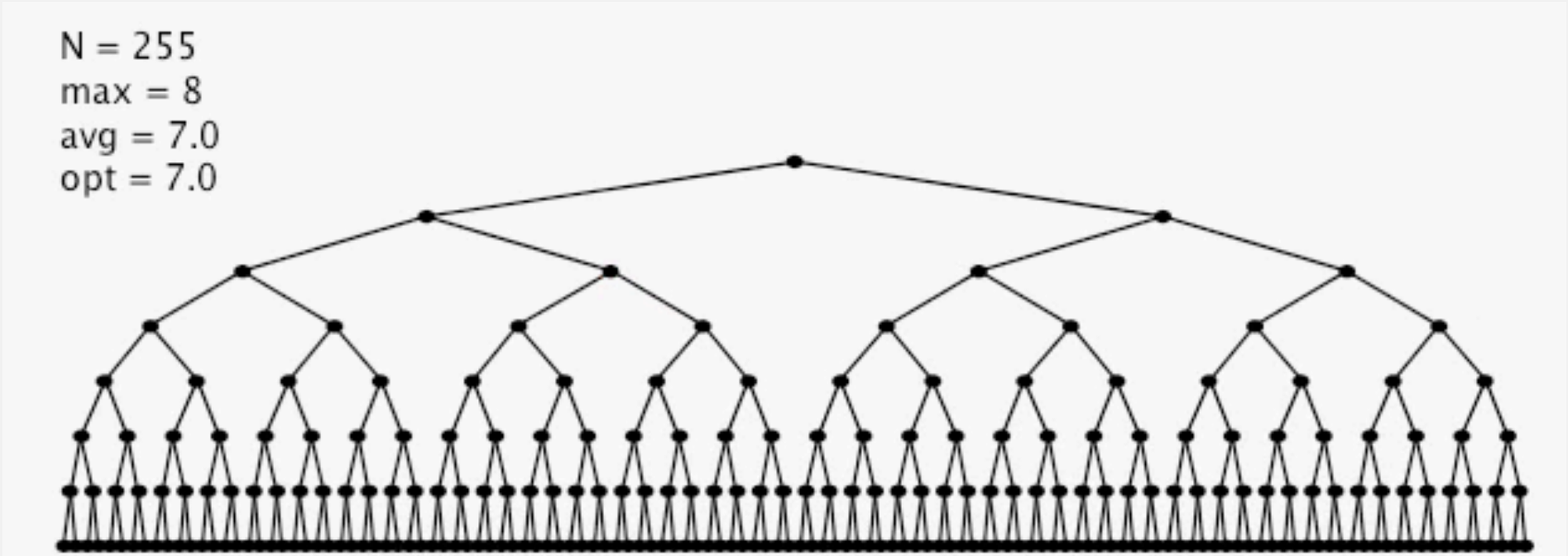
only a few extra lines of code provides near-perfect balance

Insertion into a LLRB tree: visualization



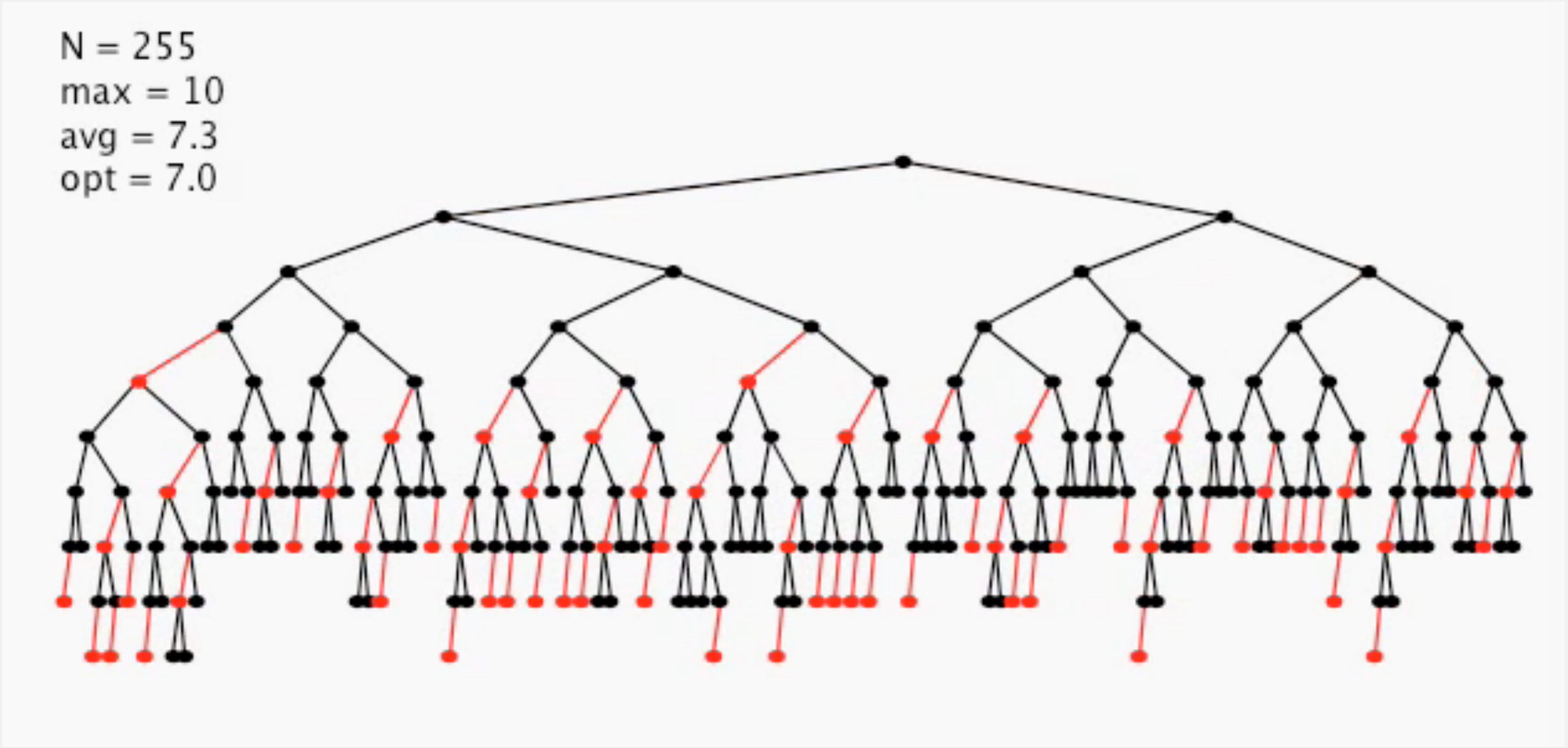
255 insertions in ascending order

Insertion into a LLRB tree: visualization



255 insertions in descending order

Insertion into a LLRB tree: visualization



255 random insertions

Balanced search trees: quiz 2

What is the height of a LLRB tree with N keys in the worst case?

A. $\sim \log_3 N$

B. $\sim \log_2 N$

C. $\sim 2 \log_2 N$

D. $\sim N$

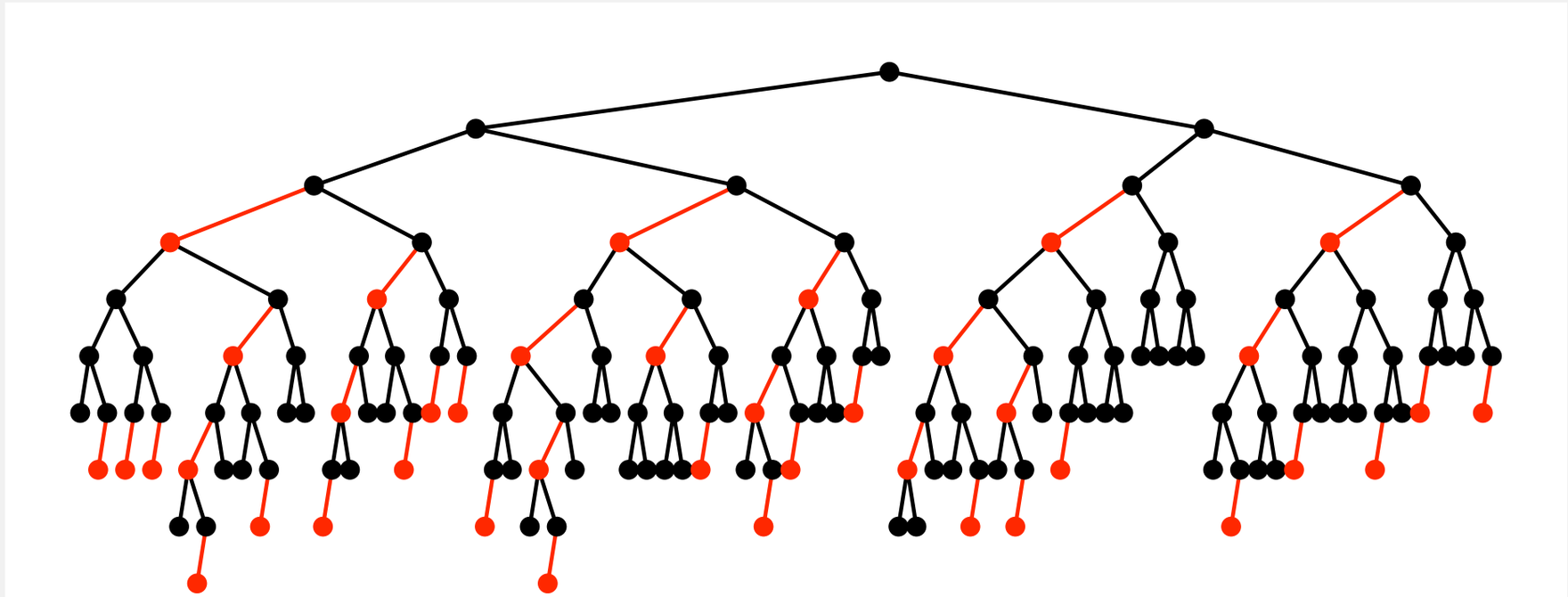
E. *I don't know.*

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

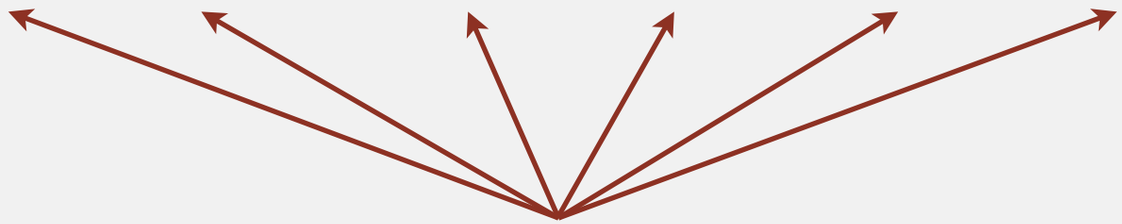
- Black height = height of corresponding 2–3 tree $\leq \lg N$.
- Never two red links in a row.



Property. Height of tree is ~ $1.0 \lg N$ in typical applications.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	N	$\log N$	N	N	✔	`compareTo()`
BST	N	N	N	$\log N$	$\log N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`
red-black BST	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`



hidden constant c is small
(at most 2 lg N compares)

War story: why red-black?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- InterPress.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

XEROX PARC

Xerox Alto

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
Xerox Palo Alto Research Center,
Palo Alto, California, and
Carnegie-Mellon University

and

Robert Sedgewick*
Program in Computer Science
Brown University
Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this

the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

War story: red–black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.

- Red–Black BST.
- Exceeding height limit of 80 triggered error-recovery process.

show allow for for up to 2^{40} keys

Extended telephone service outage.

did not rebalance BST during delete

- Main cause = height bound exceeded!
- Telephone company sues database provider.
- Legal testimony:

> *" If implemented properly, the height of a red–black BST with N keys is at most 2 lg N. "* — *expert witness*

http://algs4.cs.princeton.edu

3.3 Balanced Search Trees

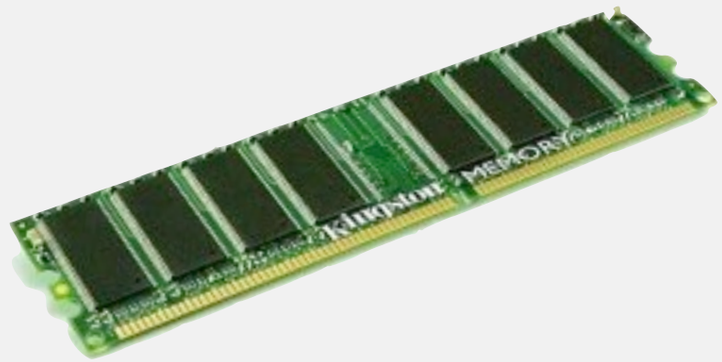
- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

File system model

Page. Contiguous block of data (e.g., a 4,096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).

slow



fast

Property. Time required for a probe is much larger than time to access data within a page.

Cost model. Number of probes.

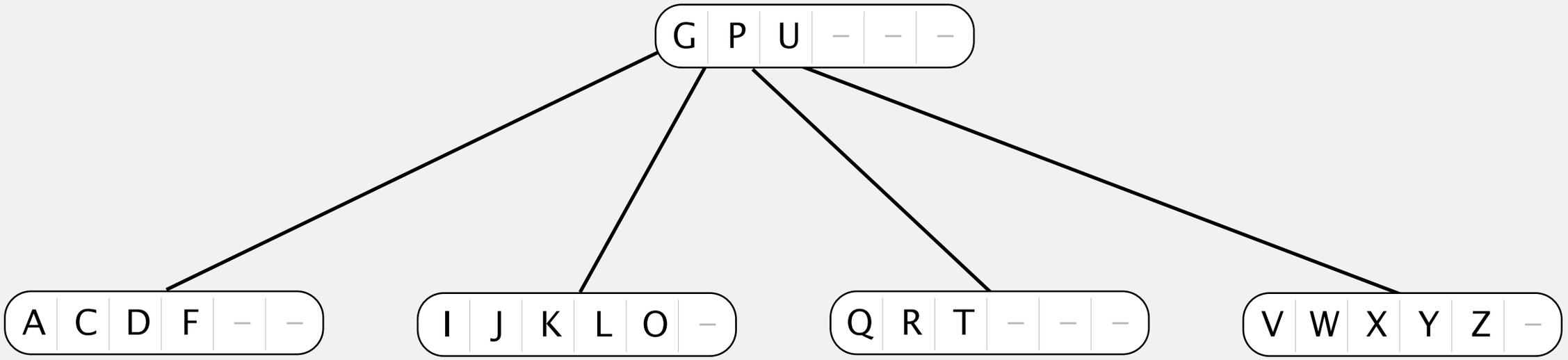
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2–3 trees by allowing up to M keys per node.

- At least $\lceil M / 2 \rceil$ keys in all nodes (except root).
- Every path from root to leaf has same number of links.

choose M as large as possible so that M keys fit in a page (M = 1,024 is typical)

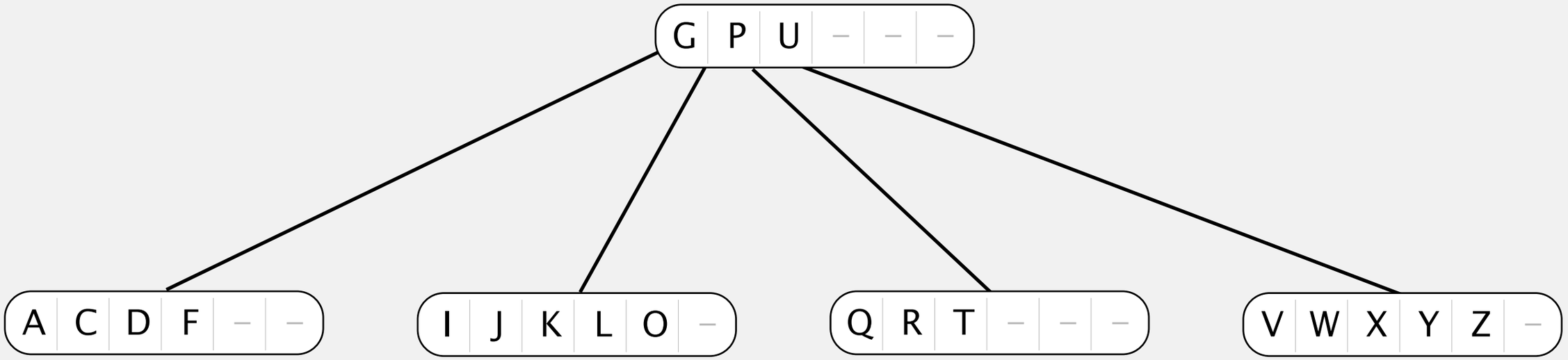


a B-tree (M = 6)

Search in a B-tree

- Start at root.
- Check if node contains key.
- Otherwise, find interval for search key and take corresponding link.

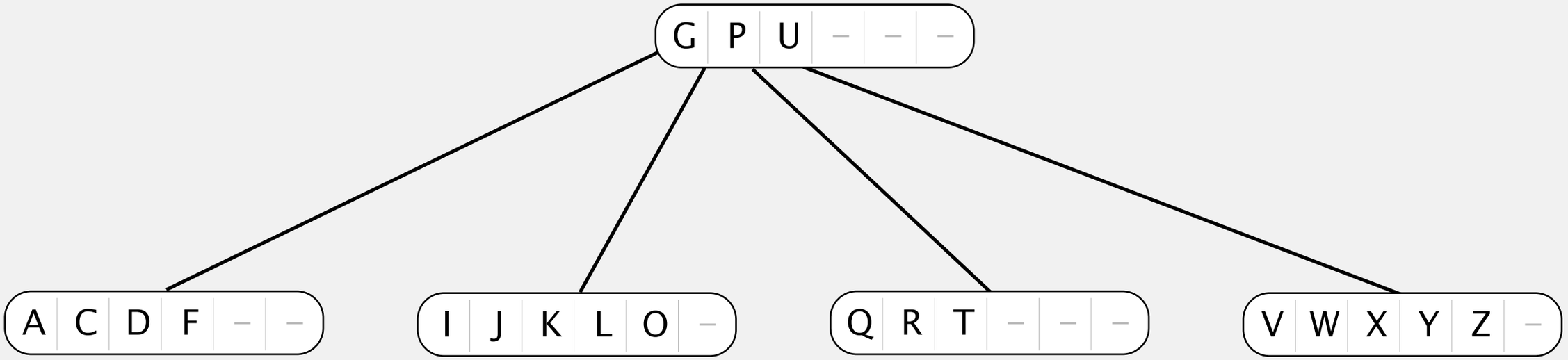
could use binary search
(but all ops are considered free)



a B-tree (M = 6)

Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with $M + 1$ keys on the way back up the B-tree (moving middle key to parent).



a B-tree (M = 6)

Balance in B-tree

Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\sim \log_M N$ and $\sim \log_{M/2} N$ probes.

Pf. All nodes (except possibly root) have between $\lfloor M / 2 \rfloor$ and M keys.

In practice. Number of probes is at most 4. ⟵ M = 1024; N = 62 billion
$\log_{M/2} N \leq 4$

Balanced search trees: quiz 3

What of the following does the B in B-tree not mean?

A. Bayer

B. Balanced

C. Binary

D. Boeing

E. *I don't know.*

> *" the more you think about what the B in B-trees could mean, the more you learn about B-trees and that is good. "*
>
> *– Rudolph Bayer*

Balanced trees in the wild

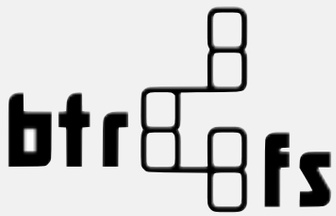
Red–Black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

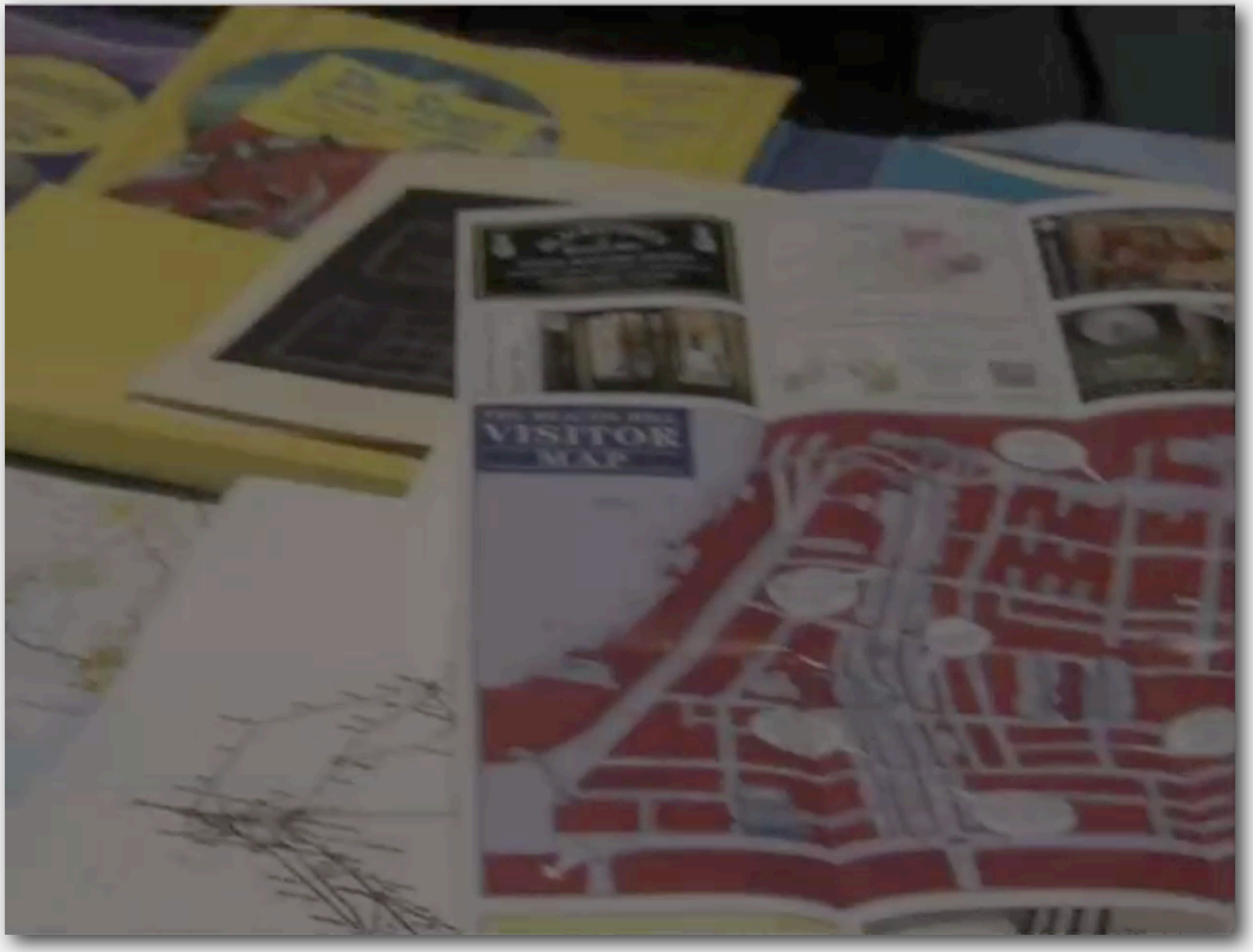
B-tree cousins. B+ tree, B*tree, B# tree, ...

B-trees (and cousins) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS, BTRFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.



Red–black BSTs in the wild



Common sense. Sixth sense.
Together they're the
FBI's newest team.

Red-black BSTs in the wild

ACT FOUR

FADE IN:

48 INT. FBI HQ - NIGHT 48

Antonio is at THE COMPUTER as Jess explains herself to Nicole and Pollock. The CONFERENCE TABLE is covered with OPEN REFERENCE BOOKS, TOURIST GUIDES, MAPS and REAMS OF PRINTOUTS.

JESS
It was the red door again.

POLLOCK
I thought the red door was the storage container.

JESS
But it wasn't red anymore. It was black.

ANTONIO
So red turning to black means... what?

POLLOCK
Budget deficits? Red ink, black ink?

NICOLE
Yes. I'm sure that's what it is. But maybe we should come up with a couple other options, just in case.

Antonio refers to his COMPUTER SCREEN, which is filled with mathematical equations.

ANTONIO
It could be an algorithm from a binary search tree. A red-black tree tracks every simple path from a node to a descendant leaf with the same number of black nodes.

JESS
Does that help you with girls?