PRINCETON UNIVERSITY FALL '14 COS 521:ADVANCED ALGORITHMS

Homework 3

Out: *Oct 23* Due: *Nov 10*

1. Compute the mixing time (both upper and lower bounds) of a graph on $2n$ nodes that consists of two complete graphs on n nodes joined by a single edge. (Hint: Use elementary probability calculations; no eigenvalues.)

2. Let M be the Markov chain of a d-regular graph that is connected. Each node has self-loops with probability $1/2$. We saw in class that 1 is an eigenvalue with eigenvector $\vec{1}$. Show that every other eigenvalue has magnitude at most $1-1/10n^2$. (Hint: First show that a connected graph cannot have 2 eigenvalues that are 1.) What does this imply about the mixing time for a random walk on this graph from an arbitrary starting point?

3. (Game-playing equilibria) Recall the game of Rock, Paper, Scissors. Let's make it quantitative it by saying that the winning player gets \$ 1 from the other whereas a draw results in no exchange of money. Suppose we make two copies of the multiplicative weight update algorithm to play each other over many iterations. Both start using the uniformly random strategy (i.e., play each of Rock/paper/scissors with probability 1/3) and learn from experience using the MW rule. One imagines that repeated play causes them to converge to some kind of *equilibrium*. (a) Predict by just calculation/introspection what this equilibrium is. (Be honest; it's Ok to be wrong!). (b) Run this experiment on Matlab or any other programming environment and report what you discovered and briefly explain it. (We'll discuss the result in class.)

4. This question will study how mixing can be much slower on directed graphs. Describe an n-node directed graph (with max indegree and outdegree at most 5) that is fully connected but where the random walk takes $\exp(\Omega(n))$ time to mix (and the walk ultimately does mix). Argue carefully.

5. Describe an example (i.e., an appropriate set of n points in $\Re^n$) that shows that the Johnson-Lindenstrauss dimension reduction method —precisely the transformation described in Lecture—the does *not* preserve ℓ_1 distances within even factor 2. (Extra credit: Show that no *linear transformation* suffices, let alone J-L.)

6. (Dimension reduction for SVM's with margin) Suppose we are given two sets P, N of unit vectors in $\Re^n$ with the guarantee that there exists a hyperplane $a \cdot x = 0$ such that every point in P is on one side and every point in N is on the other. Furthermore, the ℓ_2 distance of each point in P and N to this hyperplane is at least ϵ. Then show using the Johnson Lindenstrauss lemma that a random linear mapping to $O(\log n/\epsilon^2)$ dimensions and such that the points are still separable by a hyperplane with margin $\epsilon/2$.

7. Suppose you are trying to convince your friend that there is no perfect randomness in his head. One way to do it would be to show that if you ask him to write down 100 random bits (say) then his last 20 are fairly predictable after you see the first 80.

   Describe the design of such a predictor using a Markovian model, carefully describing any assumptions. Implement the predictor in any suitable environment and submit the code with your answer. Report the results from a couple of experiments of the following form. Ask a couple of friends to input 100 bits quickly (or 200 if he is patient), and see how well the model predicts the last 20 (or 50) bits. The metric for the model's success in prediction is
   Number of correct guesses − Number of incorrect guesses.
   In order to do better than random guessing this number should be fairly positive.

8. (Extra credit) Calculate the eigenvectors and eigenvalues of the n-dimensional boolean hypercube, which is the graph with vertex set $\{-1, 1\}^n$ and x, y are connected by an edge iff they differ in exactly one of the n locations. (Hint: Use symmetry extensively.)