COS 226 – Data Structures and Algorithms
Fall 2014 – Flipped Lecture Section
Group worksheet
Week 8 – 11.04.14
Topics covered: digraphs, MST
Solution

Instructions: This worksheet covers directed graphs (digraphs) and Minimum spanning trees (MST). Answer questions as a group (3-4 students)

1. Minimum spanning tree. (fin-s08)

Suppose you know the MST of a weighted graph G. Now, a new edge v-w of weight c is inserted into G to form a weighted graph G'. Design an O(V) time algorithm to determine if the MST in G is also an MST in G'
You may assume all edge weights are distinct. Your answer will be graded for correctness, clarity, and conciseness.

(a) State the algorithm.
(b) Explain briefly why it takes O(V) time.

Solution

Find the unique path between v and w in the MST. This takes $O(V)$ time using BFS or DFS because there are only $V-1$ edges in the MST subgraph. We claim that the MST in G is the same as the MST in G' if and only if every edge on the path has length less than c.

- If any edge on the path has weight greater than c, we can decrease the weight of the MST by swapping the largest weight edge on the path with v-w. Hence weight of the MST for G' is strictly less than the weight of the MST for G.
- If the weight of v-w is larger than any edge on the path between v and w, then the cycle property asserts that v-w is not in the MST for G' (because it is the largest weight edge on the cycle consisting of the path from v to w plus the edge v-w). Thus, the MST for G is also the MST for G'.

2. Problem identification [fin-s14]

You are applying for a job in a software company. Determine if each of the following tasks is possible, impossible or unknown.

_____ Given an undirected graph, determine if there exists a path of length $V - 1$ with no repeated vertices in time proportional to EV in the worst case.

_____ Given a digraph, determine if there exists a directed path between every pair of vertices in time proportional to $E + V$ in the worst case.

_____ Given a digraph, design an algorithm to determine whether it is a *rooted DAG* (i.e., a DAG in which there is a path from every vertex to some root r) in time proportional to $E + V$ in the worst case.

C *This is the* HAMILTON-PATH *problem, which is NP-complete. So, unless P = NP, there does not exist a polynomial-time algorithm for* HAMILTON-PATH, *but this is unknown.*

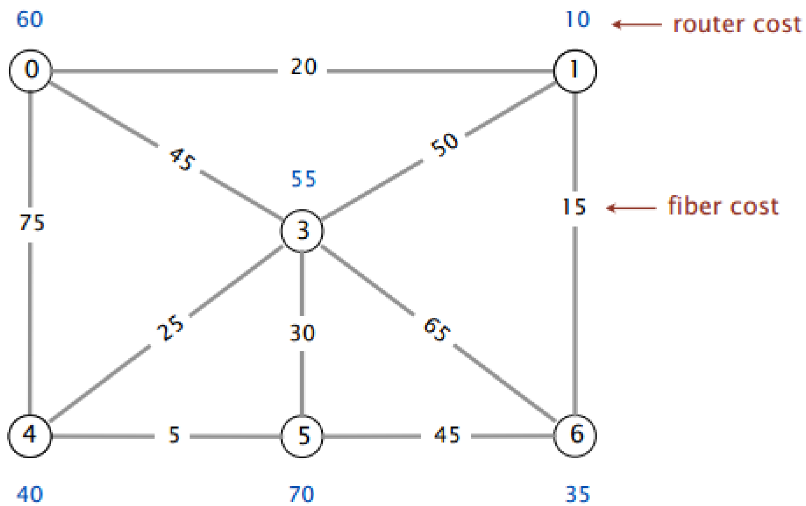
A *There is a directed path between every pair of vertices if and only if the digraph has a single strong component. You can determine this in linear time by either of the following two algorithms:*

- *Run Kosaraju-Sharir to compute the number of strong components.*
- *Pick an arbitrary vertex s; check whether every vertex is reachable from s (via DFS); and check whether s is reachable from every vertex (via DFS in the reverse digraph).*

A *You did this on the Wordnet assignment.*

3. Algorithm Design [fin-s14]

There are N dorm rooms, each of which needs a secure internet connection. It costs $w_i > 0$ dollars to install a secure router in dorm room i and it costs $c_{ij} > 0$ dollars to build a secure fiber connection between rooms i and j. A dorm room receives a secure internet connection if either there is a router installed there or there is some path of fiber connections between the dorm room and a dorm room with an installed router. The goal is to determine in which dorm rooms to install the secure routers and which pairs of dorm rooms to connect with fiber so as to minimize the total cost.

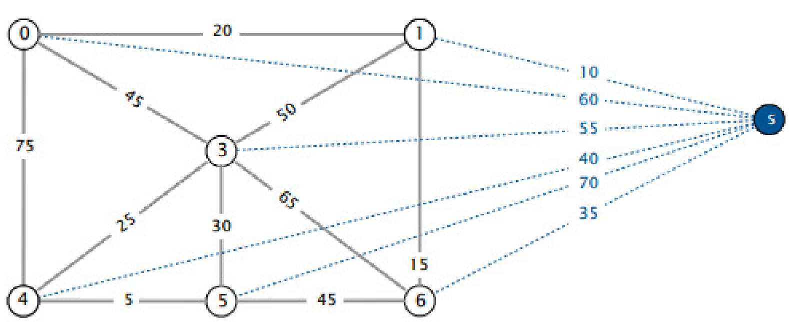


Formulate the problem as a minimum spanning tree problem. To demonstrate your formulation, modify the figure above to show the MST problem that you would solve to find the minimum cost set of routers and fiber connections

Solution

Create an edge-weighted graph with $N+1$ vertices (one for each dorm room plus an artificial source vertex s).

- Include an edge between i and j with weight c_{ij} (to represent potential fiber connections).
- Include an edge between s and i with cost w_i (to represent potential routers).



Now, if the MST contains any edge of the form s-i, then we install a router in dorm room i; if the MST contains any edge i-j not of this form, then we build a fiber connection between dorm rooms i and j.