COMPUTER SCIENCE
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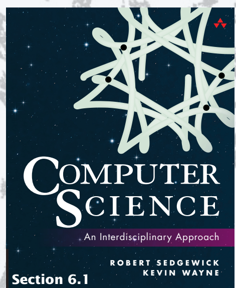

http://introcs.cs.princeton.edu

20. Combinational Circuits

Combinational circuits

Q. What is a combinational circuit?

A. A digital circuit (all signals are 0 or 1) with no feedback (no loops).

analog circuit: signals vary continuously

sequential circuit: loops allowed (stay tuned)

Q. Why combinational circuits?

A. Accurate, reliable, general purpose, fast, cheap.

Basic abstractions
- On and off.
- Wire: propagates on/off value.
- Switch: controls propagation of on/off values through wires.

Applications. Smartphone, tablet, game controller, antilock brakes, *microprocessor*, ...

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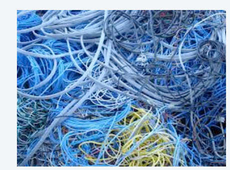
20. Combinational Circuits

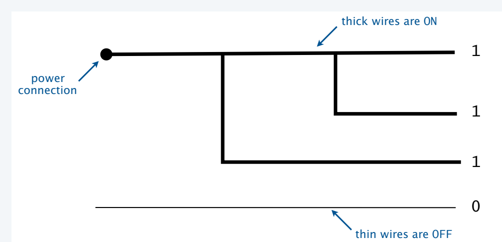
- **Building blocks**
- Boolean algebra
- Digital circuits
- Adder

CS.20.A.Circuits.Basics

Wires

Wires propagate on/off values
- ON (1): connected to power.
- OFF (0): not connected to power.
- Any wire connected to a wire that is ON is also ON.
- Drawing convention: "flow" from top, left to bottom, right.

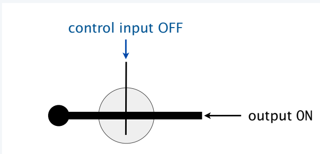


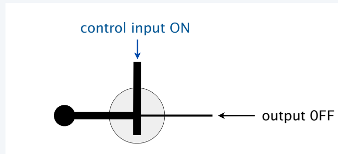


Controlled Switch

Switches control propagation of on/off values through wires.

- Simplest case involves two connections: control (input) and output.
- control OFF: output ON
- control ON: output OFF

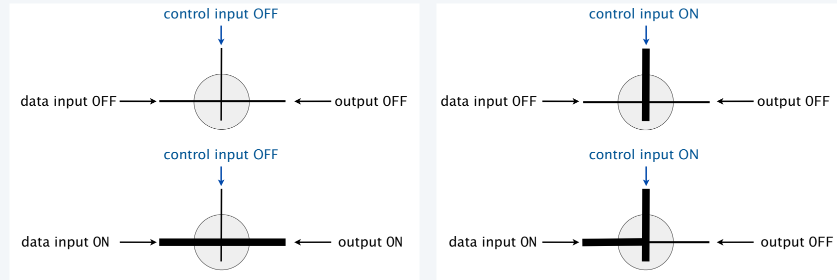




Controlled Switch

Switches control propagation of on/off values through wires.

- General case involves *three* connections: control input, *data input* and output.
- control OFF: output is connected to input
- control ON: output is disconnected from input



Idealized model of *pass transistors* found in real integrated circuits.

Controlled switch: example implementation

A *relay* is a physical device that controls a switch with a magnet

- 3 connections: input, output, control.
- Magnetic force pulls on a contact that cuts electrical flow.

First level of abstraction

Switches and wires model provides separation between physical world and logical world.

- We assume that switches operate as specified.
- That is the only assumption.
- Physical realization of switch is irrelevant to design.

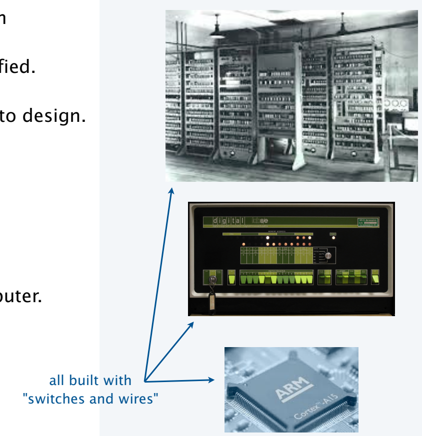
Physical realization dictates *performance*

- Size.
- Speed.
- Power.

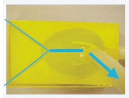
New technology immediately gives new computer.

Better switch? Better computer.

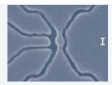
Basis of Moore's law.



Switches and wires: a first level of abstraction

technology	*"information"*	*switch*
pneumatic	air pressure	
fluid	water pressure	
relay	electric potential	

Amusing attempts that do not scale but prove the point

technology	*switch*
relay	
vacuum tube	
transistor	
"pass transistor" in integrated circuit	
atom-thick transistor	

Real-world examples that prove the point

Switches and wires: a first level of abstraction

VLSI = Very Large Scale Integration

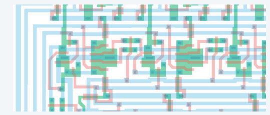
Technology
Deposit materials on substrate.

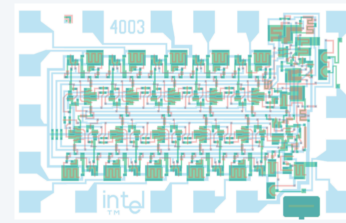
Key properties
Lines are wires.
Certain crossing lines are controlled switches.

Key challenge in physical world
Fabricating physical circuits with billions of wires and controlled switches

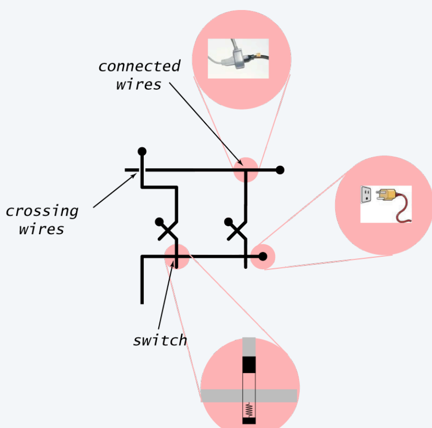
Key challenge in "abstract" world
Understanding behavior of circuits with billions of wires and controlled switches

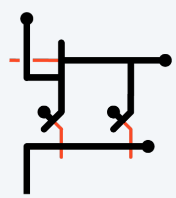
Bottom line. Circuit = Drawing (!)





Circuit anatomy





Need more levels of abstraction to understand circuit behavior

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20. Combinational Circuits

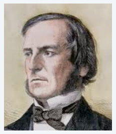
- Building blocks
- **Boolean algebra**
- Digital circuits
- Adder

CS.20.B.Circuits.Algebra

Boolean algebra

Developed by George Boole in 1840s to study logic problems
- Variables represent *true* or *false* (1 or 0 for short).
- Basic operations are AND, OR, and NOT (see table below).

Widely used in mathematics, logic and computer science.



George Boole
1815–1864

operation	*Java notation*	*logic notation*	*circuit design (this lecture)*
AND	x && y	$x \wedge y$	xy
OR	x || y	$x \vee y$	$x + y$
NOT	! x	$\neg x$	x'

← various notations in common use

DeMorgan's Laws

Example: (stay tuned for proof)

$$(xy)' = (x' + y')$$
$$(x + y)' = x'y'$$

Relevance to circuits. Basis for next level of abstraction.

Truth tables

A truth table is a systematic way to define a Boolean function
- One row for each possible set of argument values.
- Each row gives the function value for the specified argument values.
- N inputs: 2^N rows needed.

NOT

x	x'
0	1
1	0

AND

x	y	xy
0	0	0
0	1	0
1	0	0
1	1	1

OR

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

NOR

x	y	*NOR*
0	0	1
0	1	0
1	0	0
1	1	0

XOR

x	y	*XOR*
0	0	0
0	1	1
1	0	1
1	1	0

Truth table proofs

Truth tables are convenient for establishing identities in Boolean logic
- One row for each possibility.
- Identity established if columns match.

Proofs of DeMorgan's laws

x	y	xy	$(xy)'$	x	y	x'	y'	$x' + y'$
0	0	0	1	0	0	1	1	1
0	1	0	1	0	1	1	0	1
1	0	0	1	1	0	0	1	1
1	1	1	0	1	1	0	0	0

$(xy)' = (x' + y')$

x	y	$x + y$	$(x + y)'$ (NOR)	x	y	x'	y'	$x'y'$ (NOR)
0	0	0	1	0	0	1	1	1
0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	0	1	0
1	1	1	0	1	1	0	0	0

$(x + y)' = x'y'$

All Boolean functions of two variables

Q. How many Boolean functions of two variables?

A. 16 (all possibilities for the 4 bits in the truth table column).

Truth tables for all Boolean functions of 2 variables

x	y	*ZERO*	*AND*		x		y	*XOR*	*OR*	*NOR*	*EQ*	$\neg y$		$\neg x$		*NAND*	*ONE*
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Functions of three and more variables

Q. How many Boolean functions of *three* variables?

A. 256 (all possibilities for the 8 bits in the truth table column).

x	*y*	*z*	*AND*	*OR*	*NOR*	*MAJ*	*ODD*
0	0	0	0	0	1	0	0
0	0	1	0	1	0	0	1
0	1	0	0	1	0	0	1
0	1	1	0	1	0	1	0
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	0
1	1	0	0	1	0	1	0
1	1	1	1	1	0	1	1

Some Boolean functions of 3 variables

Examples

all extend to *N* variables

AND	logical AND	0 iff *any* inputs is 0 (1 iff all inputs 1)
OR	logical OR	1 iff *any* input is 1 (0 iff all inputs 0)
NOR	logical NOR	0 iff *any* input is 1 (1 iff all inputs 0)
MAJ	majority	1 iff more inputs are 1 than 0
ODD	odd parity	1 iff an odd number of inputs are 1

Q. How many Boolean functions of *N* variables?

A. 2^{2^N}

N	*number of Boolean functions with N variables*
2	$2^4 = 16$
3	$2^8 = 256$
4	$2^{16} = 65{,}536$
5	$2^{32} = 4{,}294{,}967{,}296$
6	$2^{64} = 18{,}446{,}744{,}073{,}709{,}551{,}616$

Universality of AND, OR and NOT

Every Boolean function can be represented as a sum of products
- Form an AND term for each 1 in Boolean function.
- OR all the terms together.

x	*y*	*z*	*MAJ*	*x'yz*	*xy'z*	*xyz'*	*xyz*	*x'yz + xy'z + xyz' + xyz = MAJ*
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Expressing MAJ as a sum of products

Def. A set of operations is *universal* if every Boolean function can be expressed using just those operations.

Fact. { AND, OR, NOT } is universal.

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20. Combinational Circuits

- Building blocks
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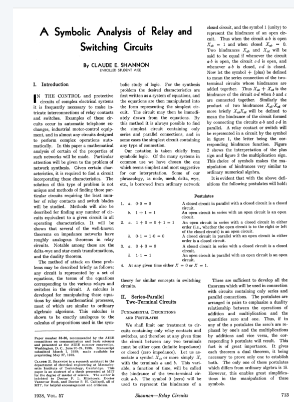
CS.20.C.Circuits.Digital

A basis for digital devices

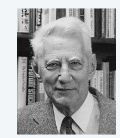
Claude Shannon connected circuit design with *boolean algebra* in 1937.

"Possibly the most important, and also the most famous, master's thesis of the [20th] century."

– Howard Gardner


A Symbolic Analysis of Relay and Switching Circuits

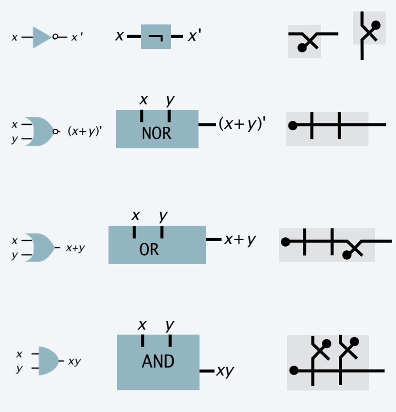
By CLAUDE E. SHANNON


Claude Shannon 1916–2001

Key idea. Can use boolean algebra to systematically analyze circuit behavior.

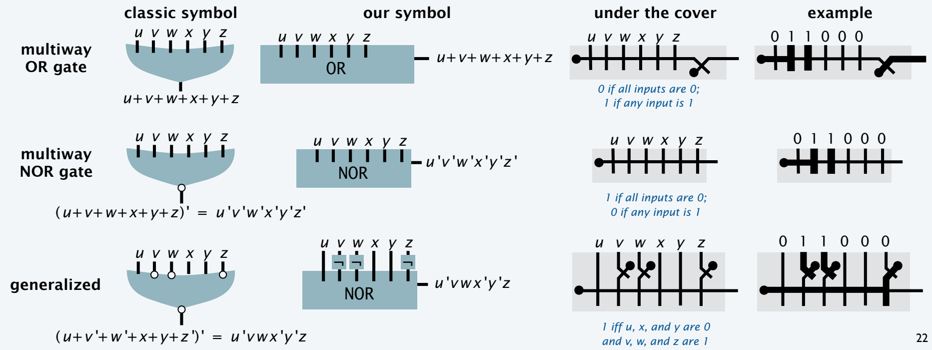
A second level of abstraction: logic gates

boolean function	notation	truth table	proof
NOT	x'	x, x': 0→1, 1→0	1 iff x is 0
NOR	(x + y)'	x y NOR: 0 0 1; 0 1 0; 1 0 0; 1 1 0	1 iff x and y are both 0
OR	x + y	x y OR: 0 0 0; 0 1 1; 1 0 1; 1 1 1	x+y = ((x + y)')'
AND	xy	x y AND: 0 0 0; 0 1 0; 1 0 0; 1 1 1	xy = (x' + y')'



Gates with arbitrarily many inputs

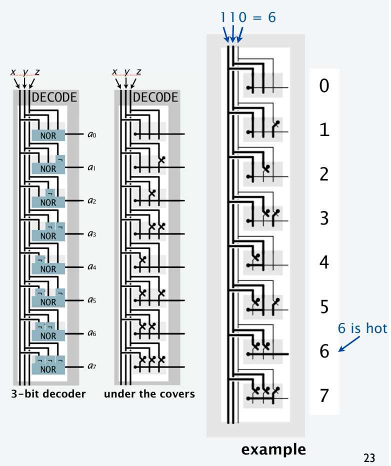
Multiway gates.
- OR: 1 if any input is 1; 0 if all inputs are 0.
- NOR: 0 if any input is 1; 1 if all inputs are 0.
- Generalized: Negate some inputs.



Generalized NOR gate application: Decoder

A *decoder* uses a binary address to switch on a single output line
- *n* address inputs, 2^n outputs.
- Uses all 2^n different generalized *NOR* gates.
- Addressed output line is 1; all others are 0.

x	y	z	a_0 x'y'z' (x+y+z)'	a_1 x'y'z (x+y+z')'	a_2 x'yz' (x+y'+z)'	a_3 x'yz (x+y'+z')'	a_4 xy'z' (x'+y+z)'	a_5 xy'z (x'+y+z')'	a_6 xyz' (x'+y'+z)'	a_7 xyz (x'+y'+z')'
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1



Next. Circuits for *any* boolean function.

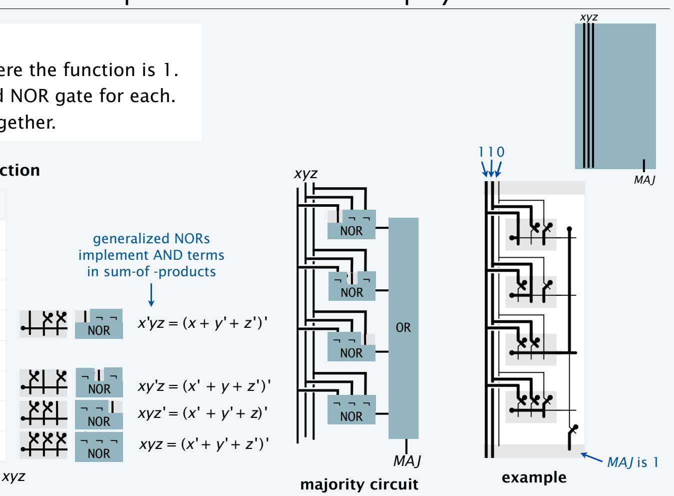
Creating a digital circuit that computes a boolean function: majority

Use the truth table
- Identify rows where the function is 1.
- Use a generalized NOR gate for each.
- OR the results together.

Example 1: Majority function

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

MAJ = x'yz + xy'z + xyz' + xyz

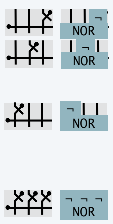


Creating a digital circuit that computes a boolean function: odd parity

Use the truth table
- Identify rows where the function is 1.
- Use a generalized NOR gate for each.
- OR the results together.

Example 2: Odd parity function

x	y	z	ODD
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1



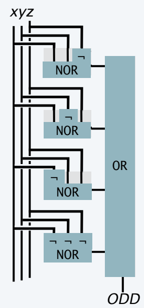
$x'y'z = (x + y + z')'$

$x'yz' = (x' + y' + z)'$

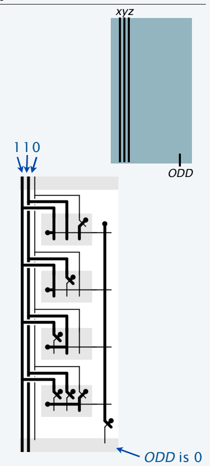
$xy'z' = (x' + y + z)'$

$xyz = (x' + y' + z')'$

$ODD = x'y'z + x'yz' + xy'z' + xyz$



odd parity circuit



example

Combinational circuit design: Summary

Problem: Design a circuit that computes a given boolean function.

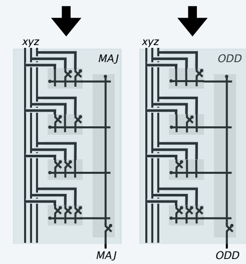
Ingredients
- OR gates.
- NOT gates.
- NOR gates.
- Wire.

Method
- Step 1: Represent input and output with Boolean variables.
- Step 2: Construct truth table to define the function.
- Step 3: Identify rows where the function is 1.
- Step 4: Use a generalized NOR for each and OR the results.

Bottom line (profound idea): Yields a circuit for ANY function.
Caveat (stay tuned): Circuit might be huge.

x	y	z	MAJ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

x	y	z	ODD
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

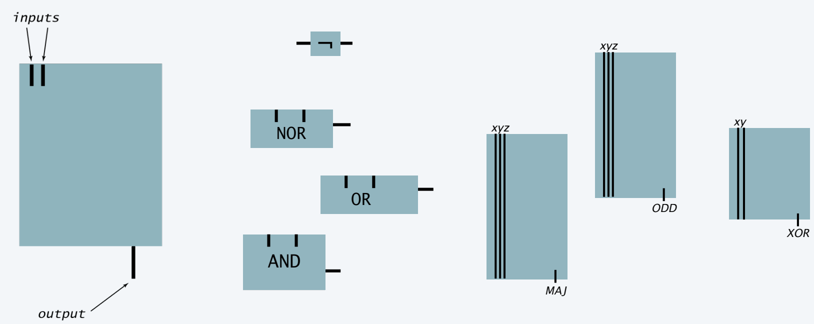


Self-assessment on combinational circuit design

Q. Design a circuit to implement XOR(x, y).

Encapsulation

Encapsulation in hardware design mirrors familiar principles in software design
- Building a circuit from wires and switches is the *implementation*.
- Define a circuit by its inputs and outputs is the *API*.
- We control complexity by *encapsulating* circuits as we do with *ADTs*.



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- **Adder**

CS.20.D.Circuits.Adder

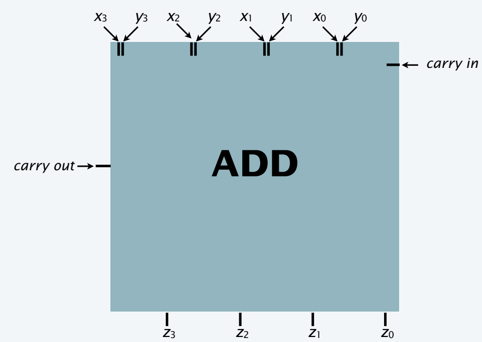
Let's make an adder circuit

Goal. $x + y = z$ for 4-bit binary integers. ← same ideas scale to 64-bit adder in your computer

- 4-bit adder: 9 inputs, 5 outputs.
- Each output is a boolean function of the inputs.

1	0	0	1	
	2	4	7	7
+	9	5	1	9
1	1	9	9	6

	1	1	0	0
	0	0	1	0
+	0	1	1	1
	1	0	0	1



carry out → c_4	c_3	c_2	c_1	c_0 ← carry in
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Let's make an adder circuit

Goal: $x + y = z$ for 4-bit integers.

Strawman solution: Build truth tables for each output bit.

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

4-bit adder truth table

c_0	x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	c_4	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	0	1	1
						. . .							
1	1	1	1	1	1	1	1	0	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1

$2^{8+1} = 512$ rows!

Q. Why is this a bad idea?

A. 128-bit adder: 2^{256+1} rows >> # electrons in universe!

Let's make an adder circuit

Goal: $x + y = z$ for 4-bit integers.

Do one bit at a time.

- Build truth table for carry bit.
- Build truth table for sum bit.

A surprise!

- Carry bit is MAJ.
- Sum bit is ODD.

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

carry bit

x_i	y_i	c_i	c_{i+1}	*MAJ*
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

sum bit

x_i	y_i	c_i	z_i	*ODD*
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

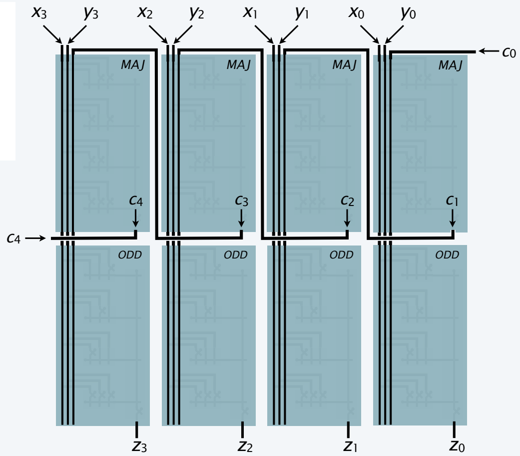
Let's make an adder circuit

Goal: $x + y = z$ for 4-bit integers.

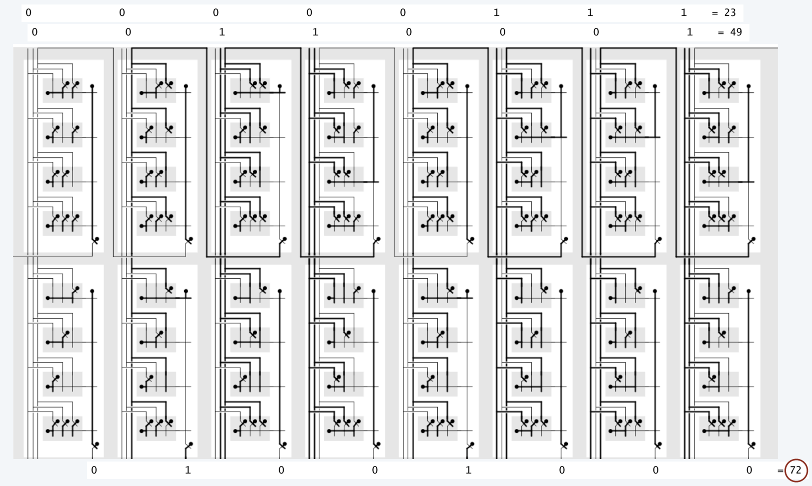
Do one bit at a time.

- Use MAJ and ODD circuits.
- Chain together 1-bit adders to "ripple" carries.

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0



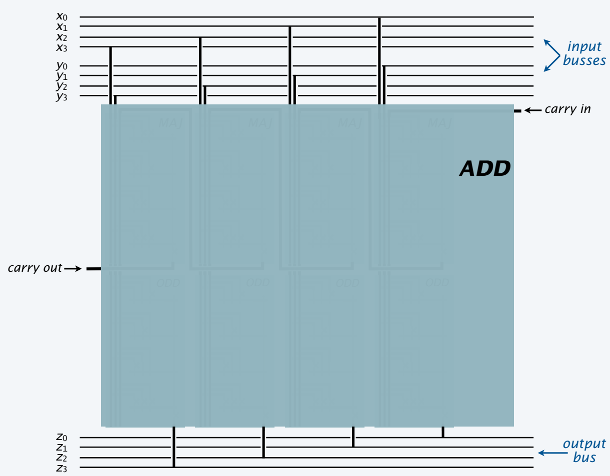
Adder example (8-bit)



Adder interface (4-bit)

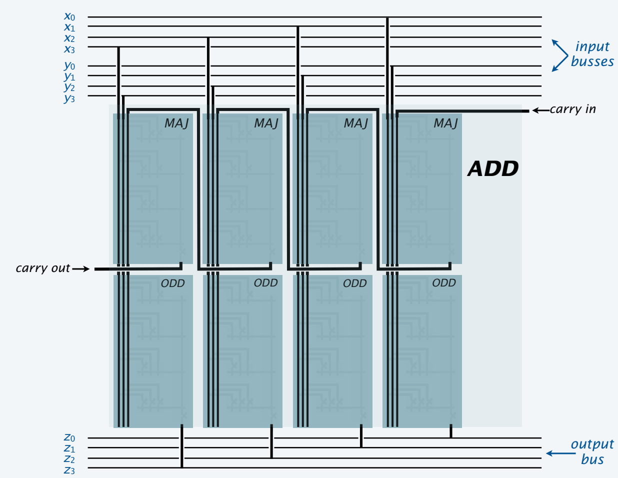
A bus is a group of wires that connect components (carrying data values).

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0



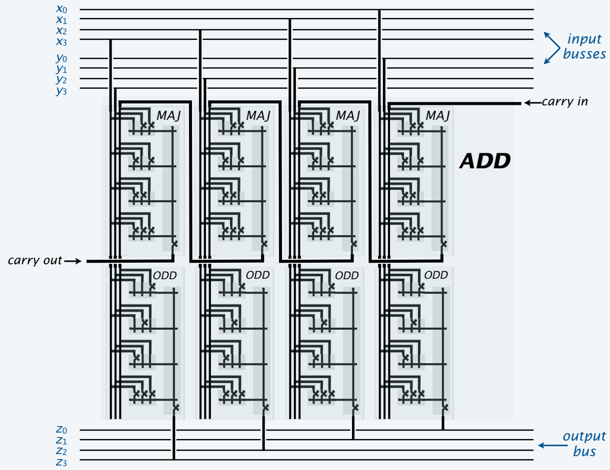
Adder component-level view (4-bit)

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0



Adder switch-level view (4-bit)

c_4	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0



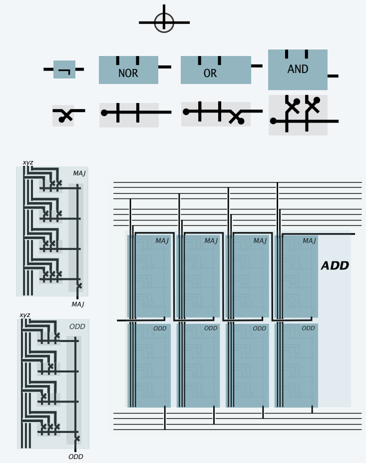
Summary

Lessons for software design apply to hardware!

- Interface describes behavior of circuit.
- Implementation gives details of how to build it.
- Boolean logic gives understanding of behavior.

Layers of abstraction apply with a vengeance!

- On/off.
- Controlled switch. [relay, pass transistor]
- Gates. [NOT, NOR, OR, AND]
- Boolean functions. [MAJ, ODD]
- Adder.
- ...
- Arithmetic/Logic unit (ALU).
- ...
- TOY machine (stay tuned).
- Your computer.



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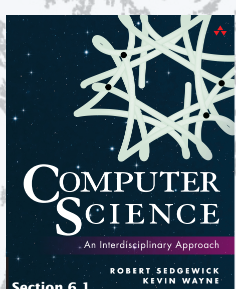

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20. Combinational Circuits