A Functional Evaluation Model

COS 326
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A Functional Evaluation Model

In order to be able to write a program, you have to have a solid grasp of how a programming language works.

We often call the definition of "how a programming language works" its *semantics*.

There are many kinds of programming language semantics.

In this class, we will look at O'Caml's *call-by-value* evaluation:

- First, informally, giving *program rewrite rules by example*
- Second, using code, by specifying an *OCaml interpreter* in OCaml
- Third, more formally, using logical *inference rules*

In each case, we are specifying what is known as OCaml's *operational semantics*

O'CAML BASICS:
CORE EXPRESSION EVALUATION

Evaluation

- Execution of an OCaml expression
  - produces a value
  - and may have some effect (eg: it may raise an exception, print a string, read a file, or store a value in an array)

- A lot of OCaml expressions have no effect
  - they are pure
  - they produce a value and do nothing more
  - the pure expressions are the easiest kinds of expressions to reason about

- We will focus on evaluation of pure expressions

Evaluation of Pure Expressions

- Given an expression e, we write:

  e --> v

  to state that expression e evaluates to value v

- Note that "e --> v" is not itself a program -- it is some notation that we use talk about how programs work

Evaluation of Pure Expressions

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  to state that expression e evaluates to value v
- Some examples:

Evaluation of Pure Expressions

- Given an expression e, we write:

  e --> v

  to state that expression e evaluates to value v
- Some examples:

  1 + 2

Evaluation of Pure Expressions

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  1 + 2 --> 3

Evaluation of Pure Expressions

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  1 + 2 --> 3

  2

Evaluation of Pure Expressions

- Given an expression e, we write:

e --> v

to state that expression e evaluates to value v

- Some examples:

1 + 2 --> 3

2 --> 2

values step to values

Evaluation of Pure Expressions

- Given an expression e, we write:

e --> v

to state that expression e evaluates to value v

- Some examples:

1 + 2 --> 3

2 --> 2

int_to_string 5 --> "5"

Evaluation of Pure Expressions

More generally, we say expression e (partly) evaluates to expression e':

e --> e'

Evaluation of Pure Expressions

More generally, we say expression e (partly) evaluates to expression e':

```
e --> e'
```

Evaluation is *complete* when e' is a value

- In general, I'll use the letter "v" to represent an arbitrary value
- The letter "e" represents an arbitrary expression
- Concrete numbers, strings, characters, etc. are all values, as are:
  - tuples, where the fields are values
  - records, where the fields are values
  - datatype constructors applied to a value
  - *functions*

Evaluation of Pure Expressions

- Some expressions (all the interesting ones!) take many steps to evaluate them:

(2 * 3) + (7 * 5)

Evaluation of Pure Expressions

- Some expressions (all the interesting ones!) take many steps to evaluate them:

```
(2 * 3) + (7 * 5)
--> 6 + (7 * 5)
```

Evaluation of Pure Expressions

- Some expressions (all the interesting ones!) take many steps to evaluate them:

```
(2 * 3) + (7 * 5)
--> 6 + (7 * 5)
--> 6 + 35
```

Evaluation of Pure Expressions

- Some expressions (all the interesting ones!) take many steps to evaluate them:

```
(2 * 3) + (7 * 5)
--> 6 + (7 * 5)
--> 6 + 35
--> 41
```

Evaluation of Pure Expressions

- Some expressions do not compute a value and it is not obvious how to proceed:

```
"hello" + 1 -->   ????
```

- A *strongly typed language rules out a lot of nonsensical expressions that compute no value*, like the one above
- Other expressions compute no value but raise an exception:

```
7 / 0 --> raise Divide_by_zero
```

- Still others simply fail to terminate …

Let Expressions: Evaluate using Substitution

```
let x = 30 in
let y = 12 in
x+y
```

-->

```
let y = 12 in
30 + y
```

-->

```
30 + 12
```

-->

```
42
```

Informal Evaluation Model

To evaluate a function call "`f a`"

- first evaluate f until we get a function value (`fun x -> e`)
- then evaluate a until we get an argument value v
- then substitute `v` for `x` in `e`, the function body
- then evaluate the resulting expression.

this is why we say O'Caml is "call by value"

```
(let f = (fun x -> x + 1) in f) (30+11)
```

-->

```
(fun x -> x + 1) (30 + 11)
```

-->

```
(fun x -> x + 1) 41
```

-->

```
41 + 1
```

-->

```
42
```

Informal Evaluation Model

Another example:

```
let add x y = x+y in
let inc = add 1 in
let dec = add (-1) in
dec(inc 42)
```

Informal Evaluation Model

Recall the syntactic sugar:

```
let add = fun x -> (fun y -> x+y) in
let inc = add 1 in
let dec = add (-1) in
dec(inc 42)
```

Informal Evaluation Model

Then we use the let rule – we substitute the *value* for `add`:

```
let add = fun x -> (fun y -> x+y) in
let inc = add 1 in
let dec = add (-1) in
dec(inc 42)
```

functions are values

```
-->

let inc = (fun x -> (fun y -> x+y)) 1 in
let dec = (fun x -> (fun y -> x+y)) -1 in
dec(inc 42)
```

Informal Evaluation Model

```
let inc = (fun x -> (fun y -> x+y)) 1 in
let dec = (fun x -> (fun y -> x+y)) (-1) in
dec(inc 42)
```

not a value; must reduce before substituting for inc

```
-->
```

```
let inc = fun y -> 1+y in
let dec = (fun x -> (fun y -> x+y)) (-1) in
dec(inc 42)
```

Informal Evaluation Model

now a value

```
let inc = fun y -> 1+y in
let dec = (fun x -> (fun y -> x+y)) (-1) in
dec(inc 42)

-->

let dec = (fun x -> (fun y -> x+y)) (-1) in
dec((fun y -> 1+y) 42)
```

Informal Evaluation Model

Next: simplify dec's definition using the function-call rule.

```
let dec = (fun x -> (fun y -> x+y)) (-1) in
dec((fun y -> 1+y) 42)

-->

let dec = fun y -> -1+y in
dec((fun y -> 1+y) 42)
```

now a value

Informal Evaluation Model

And we can use the let-rule now to substitute `dec`:

```
let dec = fun y -> -1+y in
dec((fun y -> 1+y) 42)          -->


 (fun y -> -1+y) ((fun y -> 1+y) 42)
```

Informal Evaluation Model

Now we can’t yet apply the first function because the argument is not yet a value – it’s a function call. So we need to use the function-call rule to simplify it to a value:

```
(fun y -> -1+y) ((fun y -> 1+y) 42) -->

(fun y -> -1+y) (1+42) -->

(fun y -> -1+y) 43 -->

-1+43 -->

42
```

Variable Renaming

Consider the following OCaml code:

```
let x = 30 in
let y = 12 in
x+y;;
```

Does this evaluate any differently than the following?

```
let a = 30 in
let b = 12 in
a+b;;
```

Renaming

A basic principle of programs is that systematically changing the names of variables shouldn't cause the program to behave any differently – it should evaluate to the same thing.

```
let x = 30 in
let y = 12 in
x+y;;
```

But we do have to be careful about *systematic* change.

```
let a = 30 in
let a = 12 in
a+a;;
```

Systematic change of variable names is called *alpha-conversion*.

Substitution

Wait a minute, how do we evaluate this using the let-rule? If we substitute 30 for "a" naively, then we get:

```
let a = 30 in
let a = 12 in
a+a
```

-->

```
let 30 = 12 in
30+30
```

Which makes no sense at all!

Besides, Ocaml returns 24 not 60.

What went wrong with our informal model?

Scope and Modularity

- Lexically scoped (a.k.a. statically scoped) variables have a simple rule: the nearest enclosing "let" in the code defines the variable.
- So when we write:

```
let a = 30 in
let a = 12 in
a+a;;
```

- we know that the "a+a" corresponds to "12+12" as opposed to "30+30" or even weirder "30+12".

A Revised Let-Rule:

- To evaluate "**let** x = e_1 **in** e_2":
  - First, evaluate e_1 to a value v.
  - Then substitute v for the *corresponding uses* of x in e_2.
  - Then evaluate the resulting expression.

```
let a = 30 in
let a = 12 in
a+a
```

This "a" doesn't correspond to the uses of "a" below.

```
-->
```

```
let a = 12 in
a+a
```

So when we substitute 30 for it, it doesn't change anything.

```
-->
```

```
12+12
```

```
-->
```

```
24
```

Scope and Modularity

- But what does “corresponding uses” mean?

- Consider:

```
let a = 30 in
let a = (let a = 3 in a*4) in
a+a;;
```

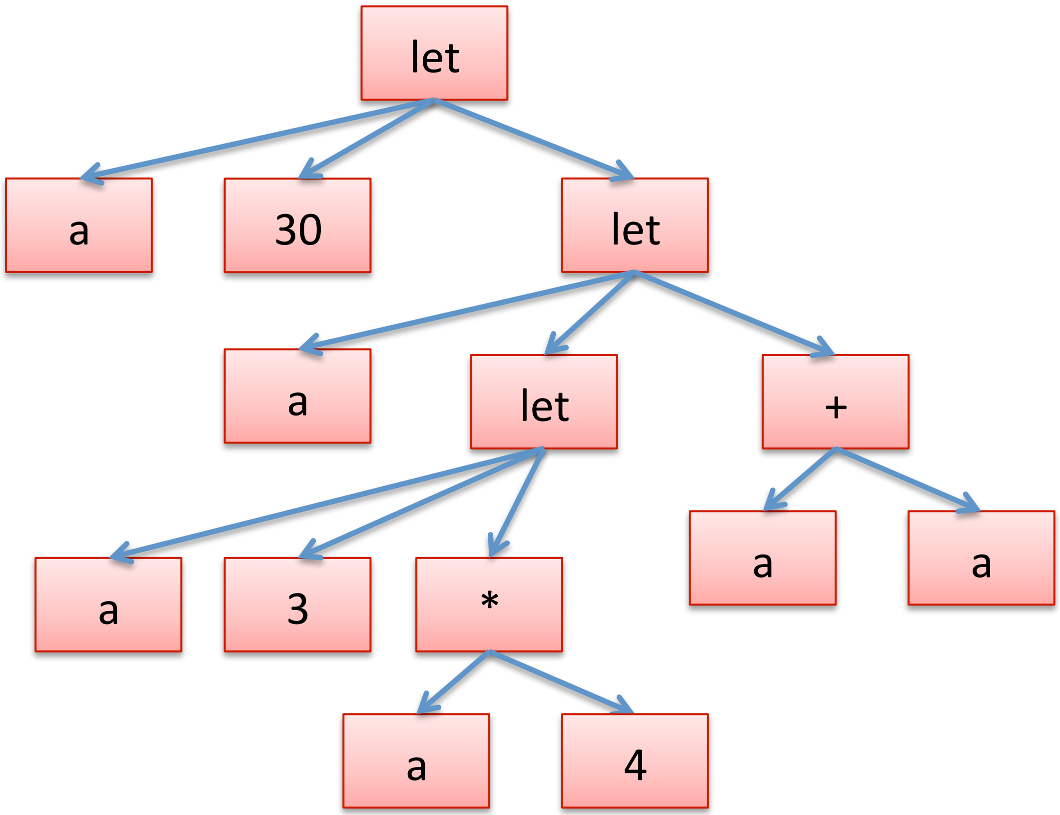
Abstract Syntax Trees

- We can view a program as a tree – the parentheses and precedence rules of the language help determine the structure of the tree.

```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```

==

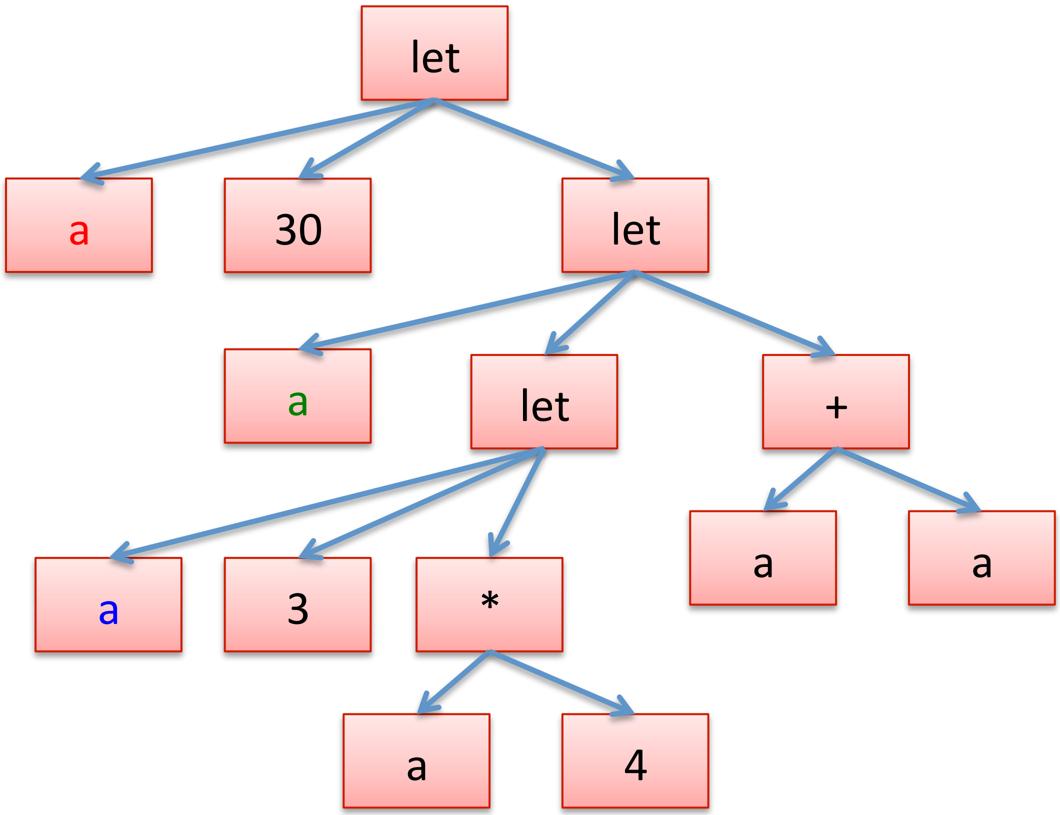
```
(let a = (30) in
 (let a =
    (let a = (3) in (a*4))
  in
  (a+a)))
```



Binding Occurrences

An occurrence of a variable where we are defining it via let is said to be a *binding occurrence* of the variable.

```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```

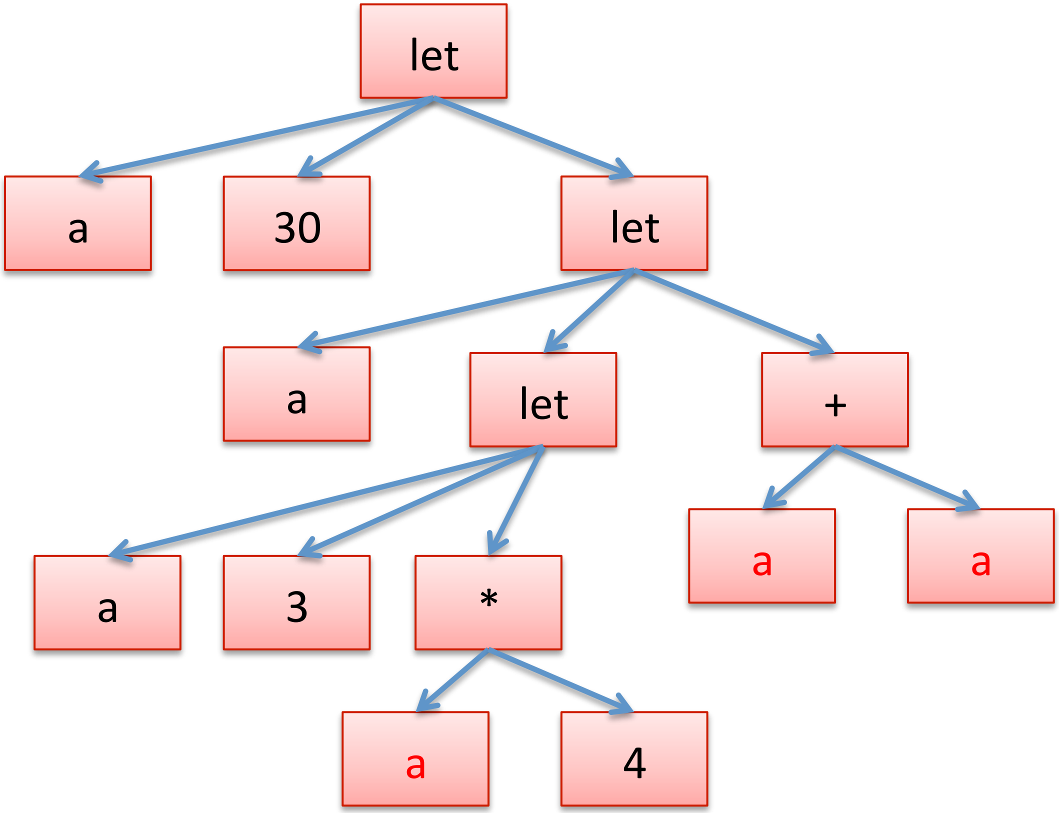


Free Occurrences

A non-binding occurrence of a variable is said to be a *free variable*.

That is a *use* of a variable as opposed to a definition.

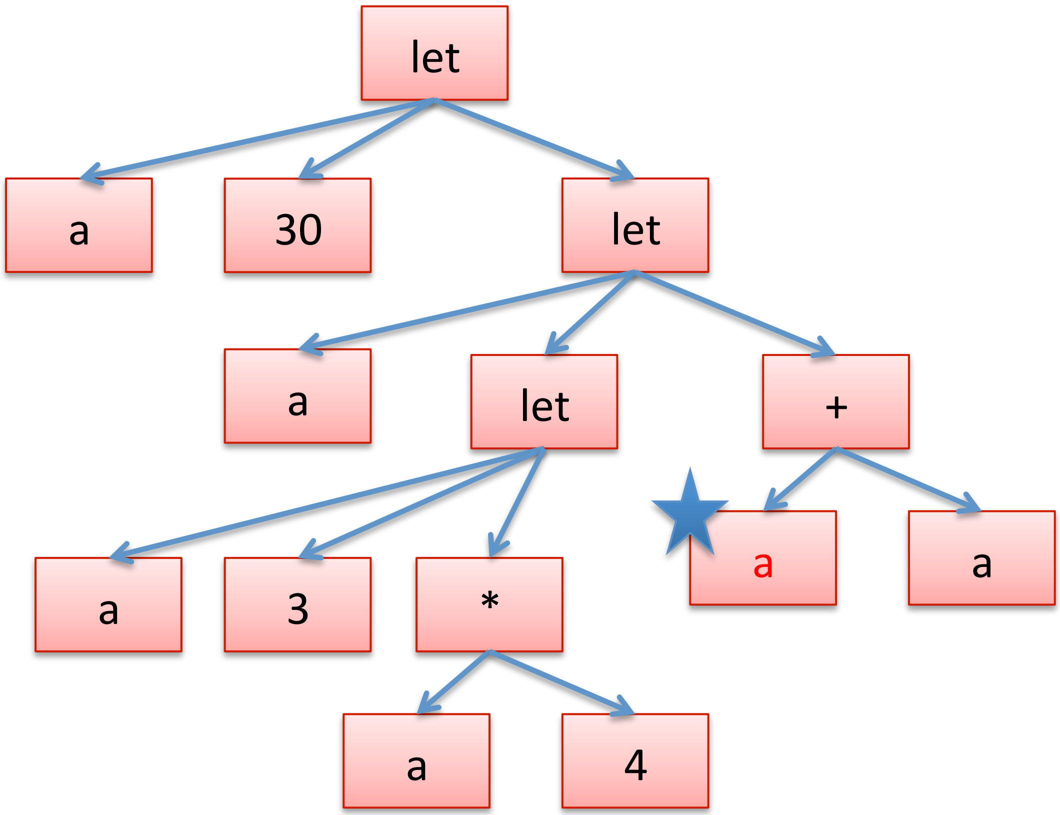
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- Given a free variable occurrence, we can find where it is bound by …

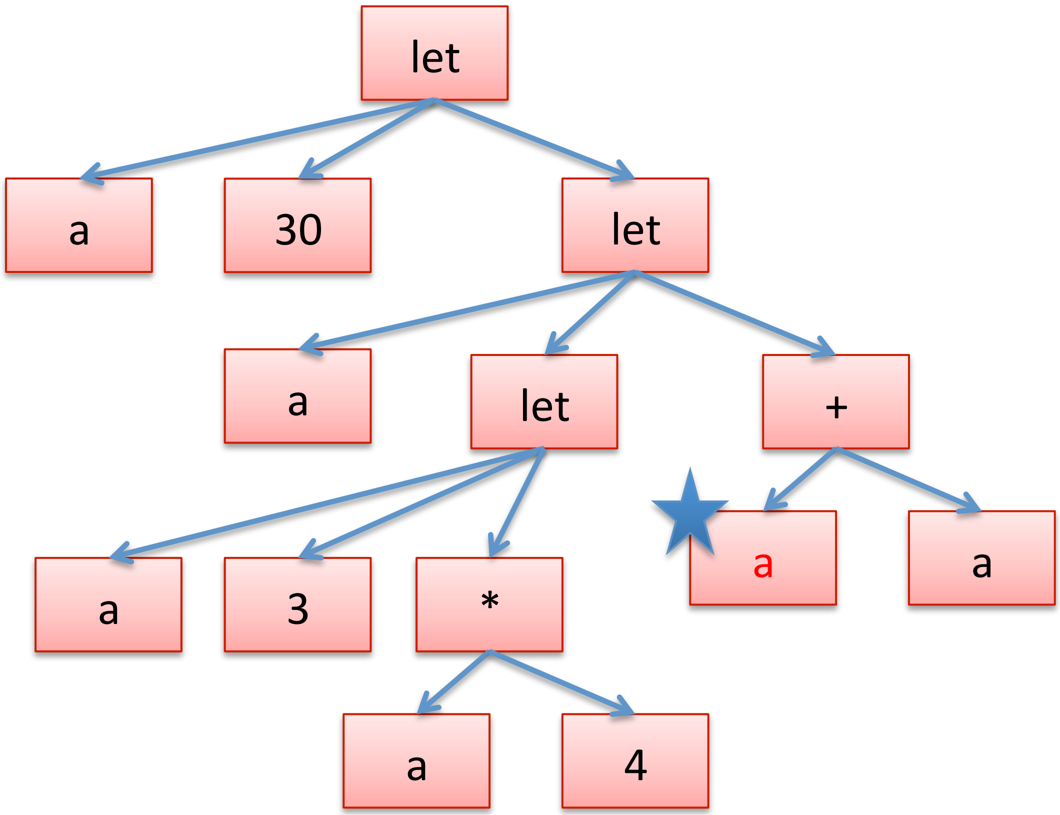
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- crawling up the tree to the nearest enclosing let…

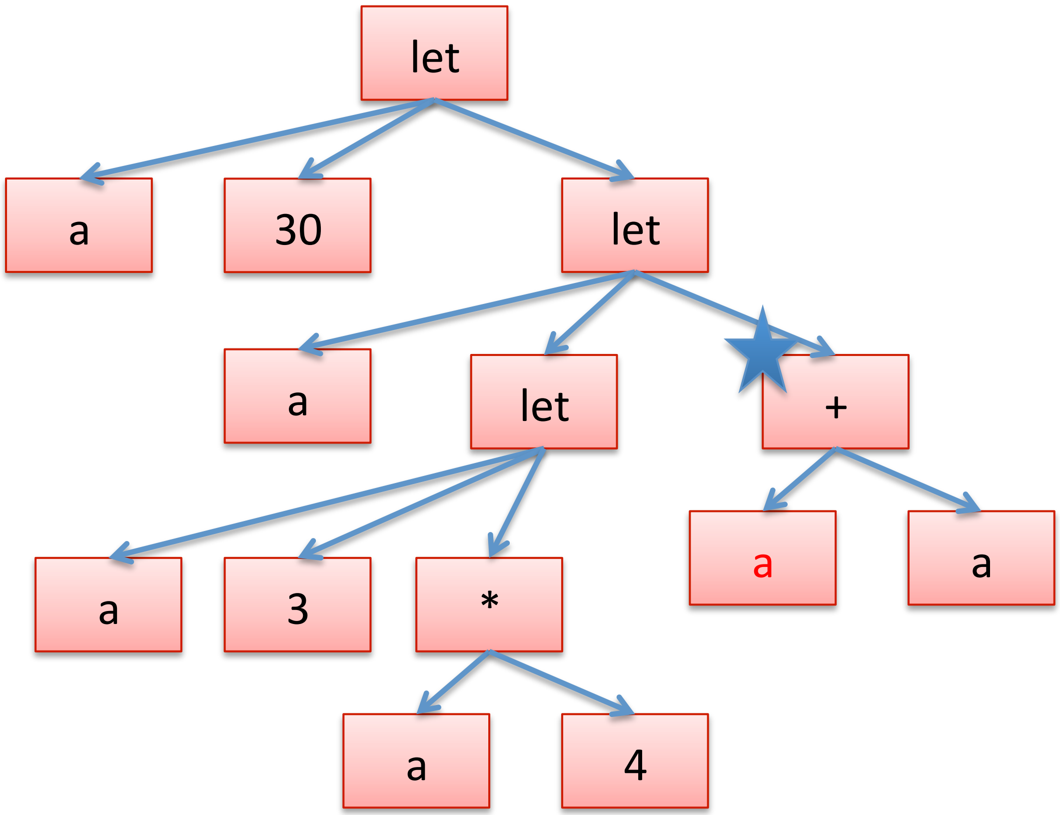
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- crawling up the tree to the nearest enclosing let…

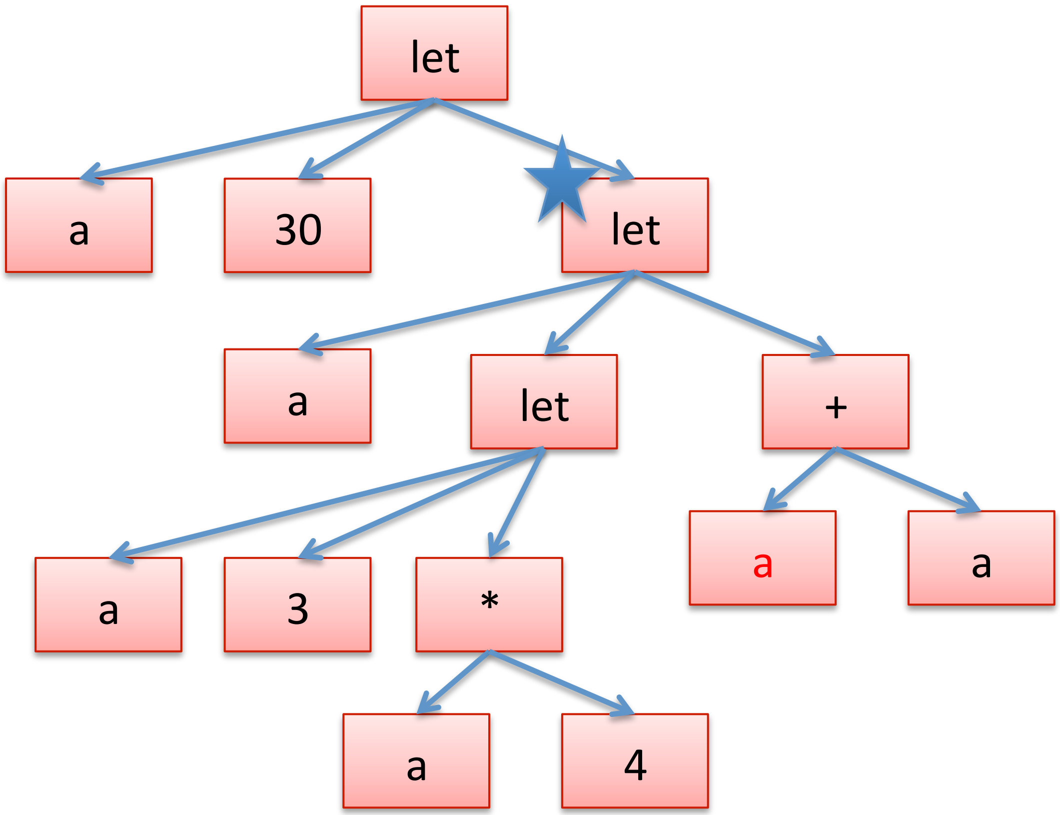
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- crawling up the tree to the nearest enclosing let…

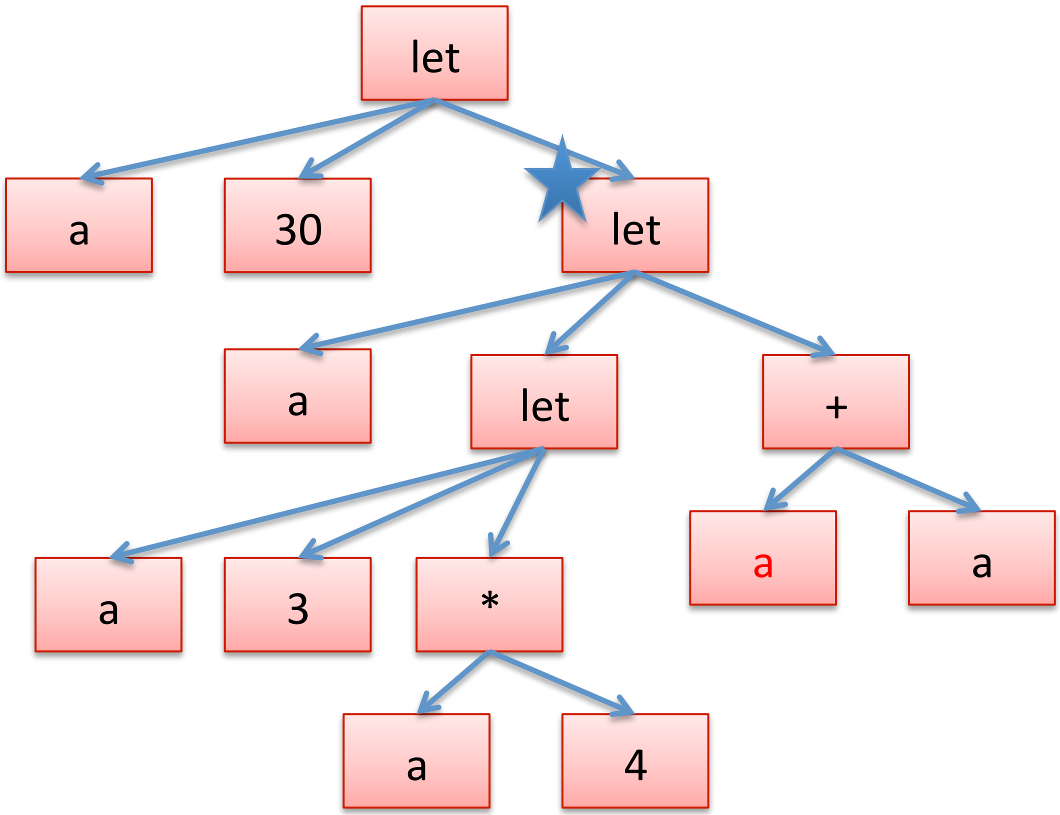
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

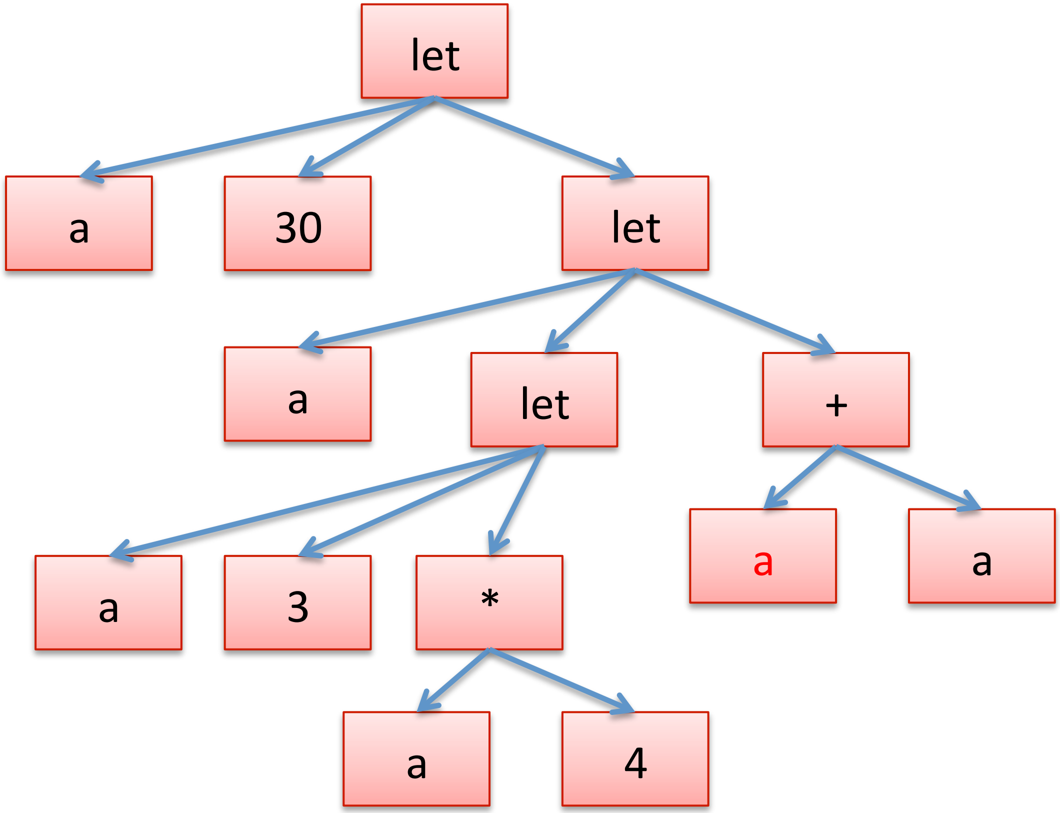
- and see if the "let" binds the variable – if so, we've found the nearest enclosing definition. If not, we keep going up.

```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- Now we can also systematically rename the variables so that it's not so confusing. Systematic renaming is called *alpha-conversion*

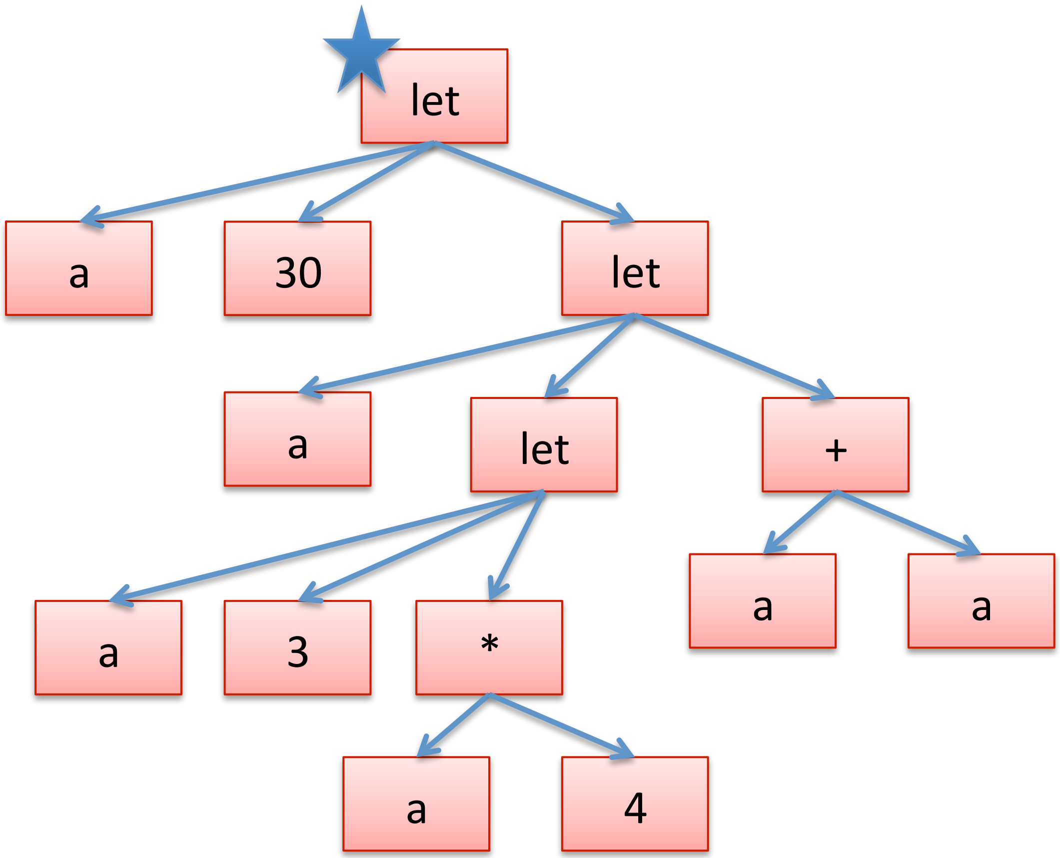



```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```

Abstract Syntax Trees

- Start with a let, and pick a fresh variable name, say “x”

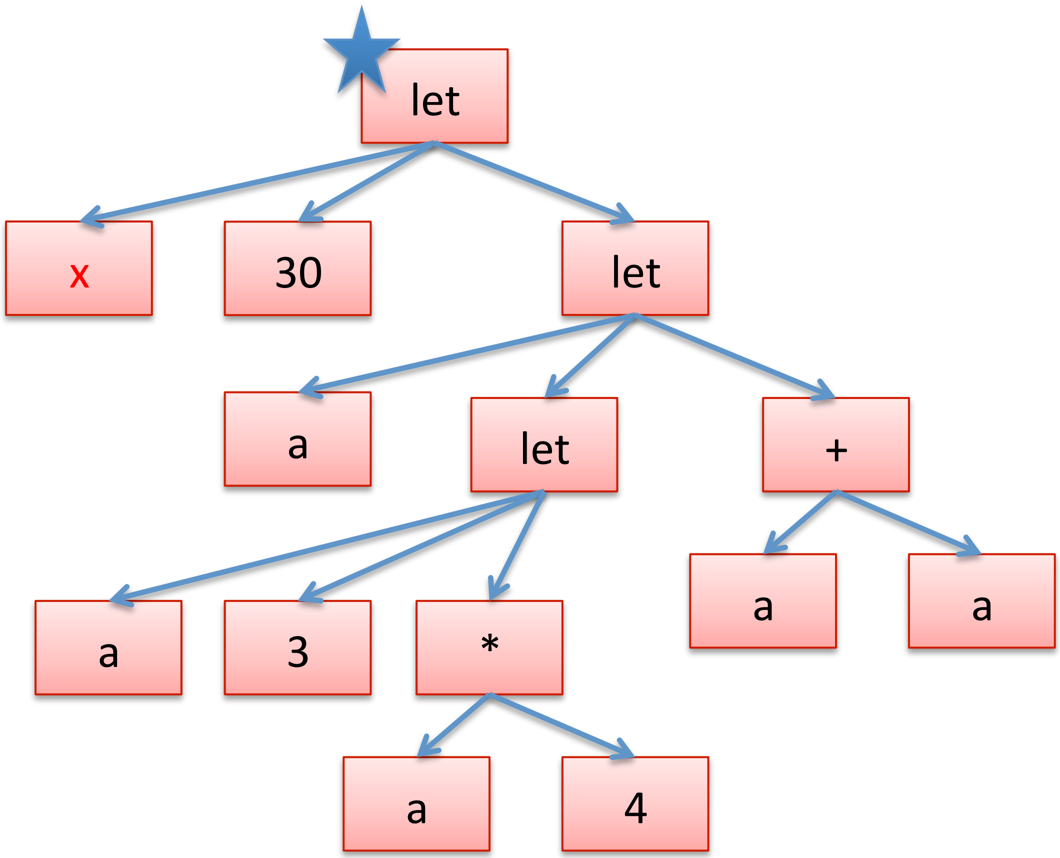
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- Rename the binding occurrence from “a” to “x”.

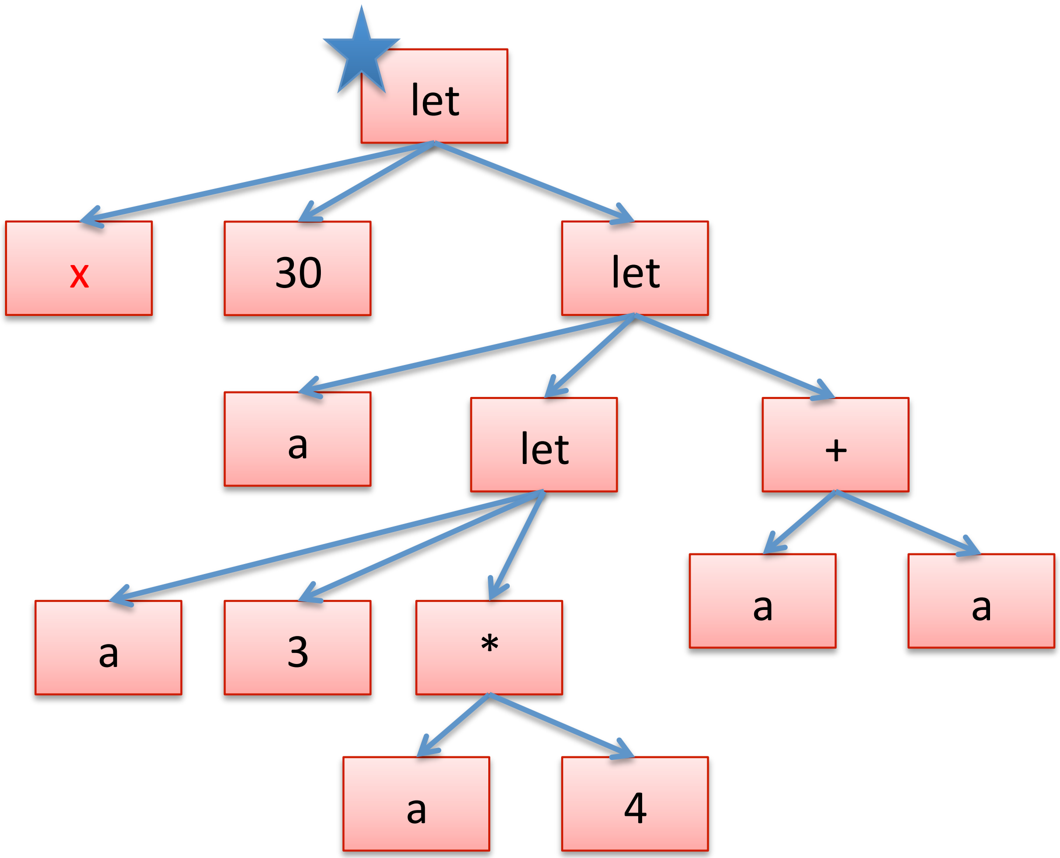
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- Then rename all of the free occurrences of the variables that this let binds.

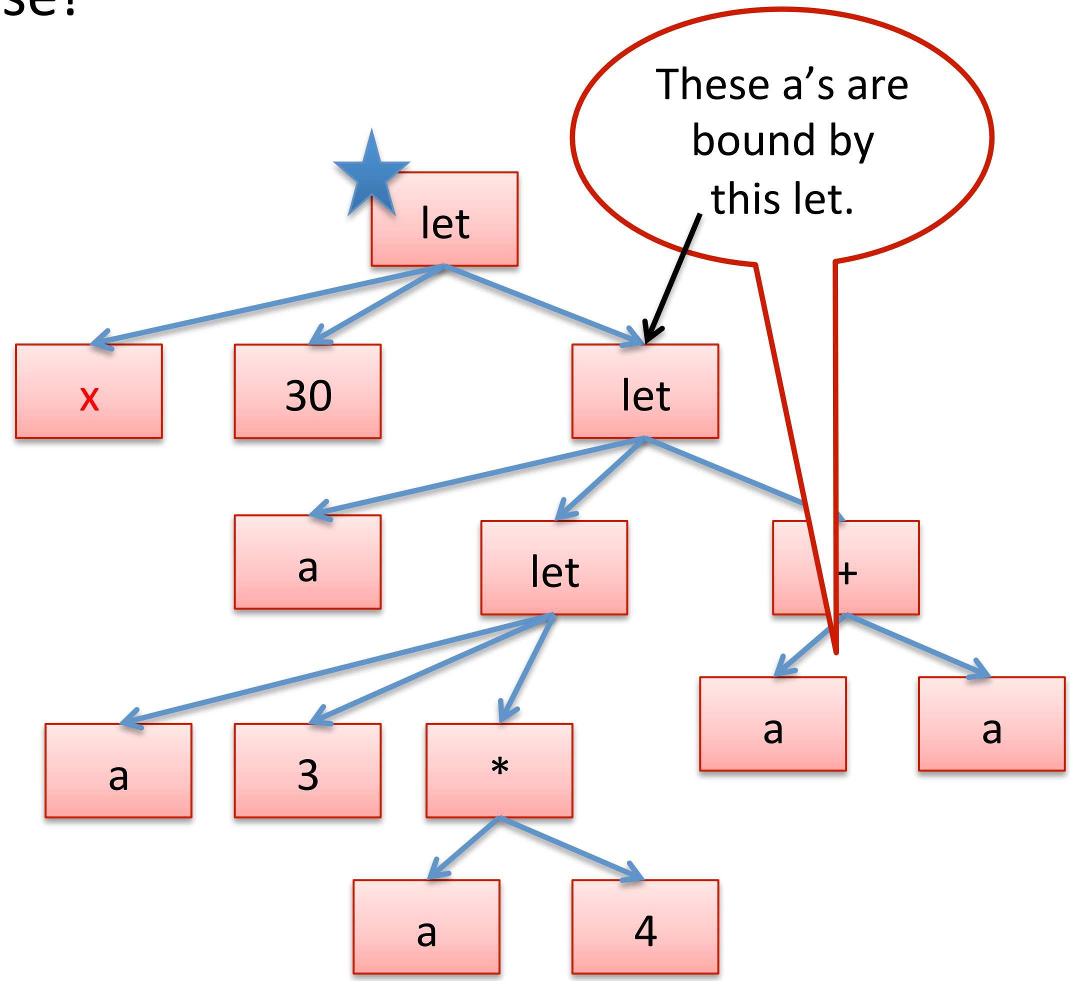
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- There are none in this case!

```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

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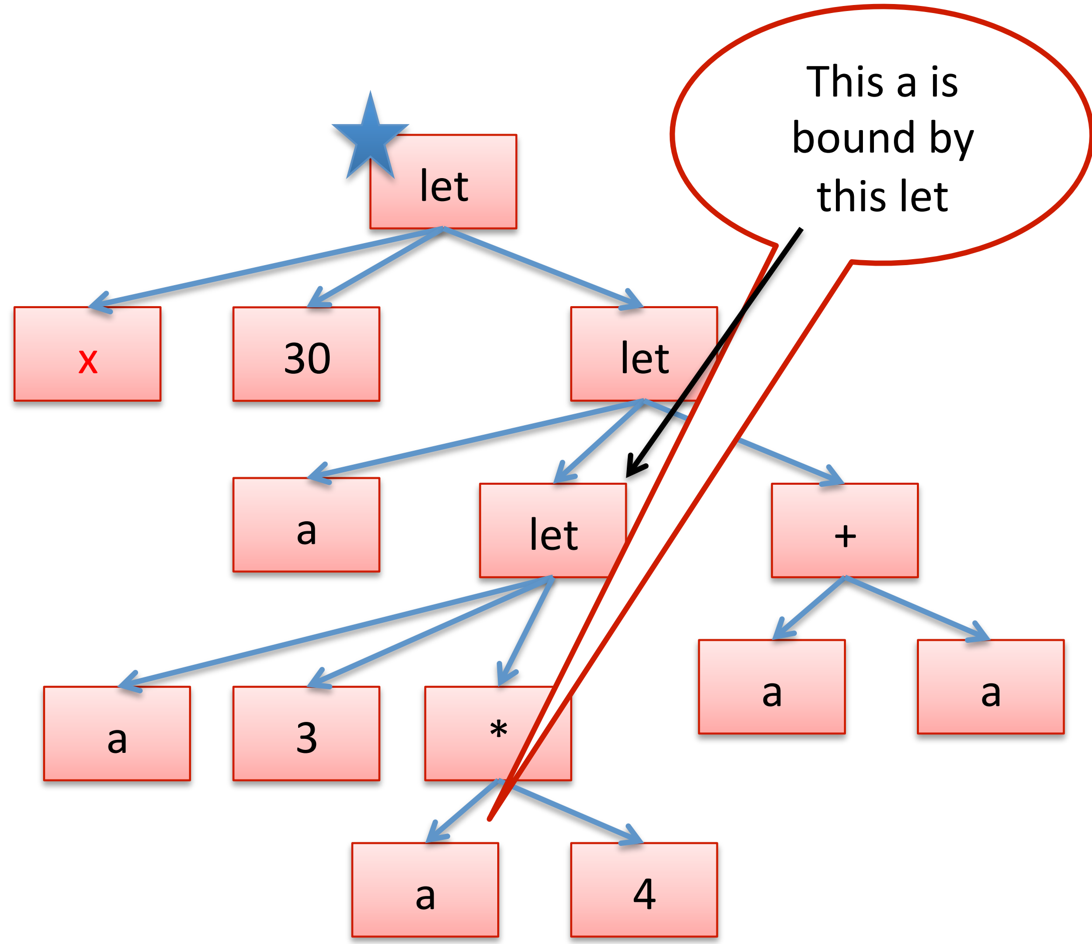
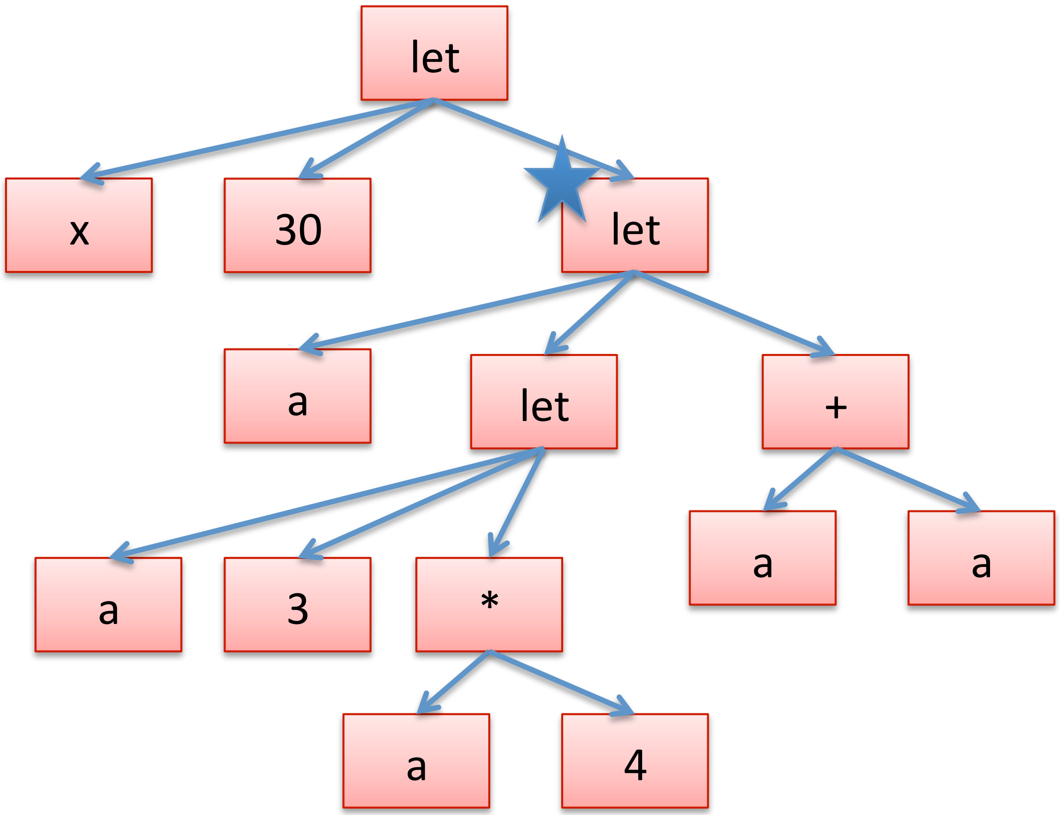
in
a+a;;
```



Abstract Syntax Trees

- Let's do another let, renaming "a" to "y".

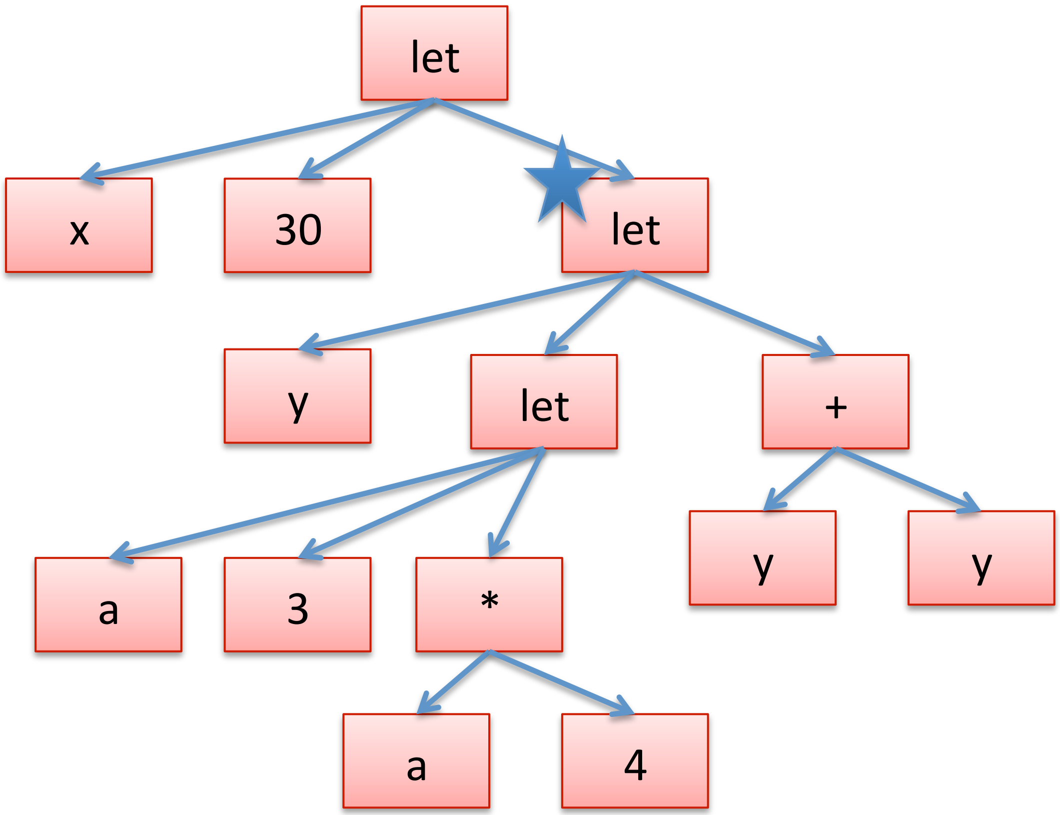
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a;;
```



Abstract Syntax Trees

- Let's do another let, renaming "a" to "y".

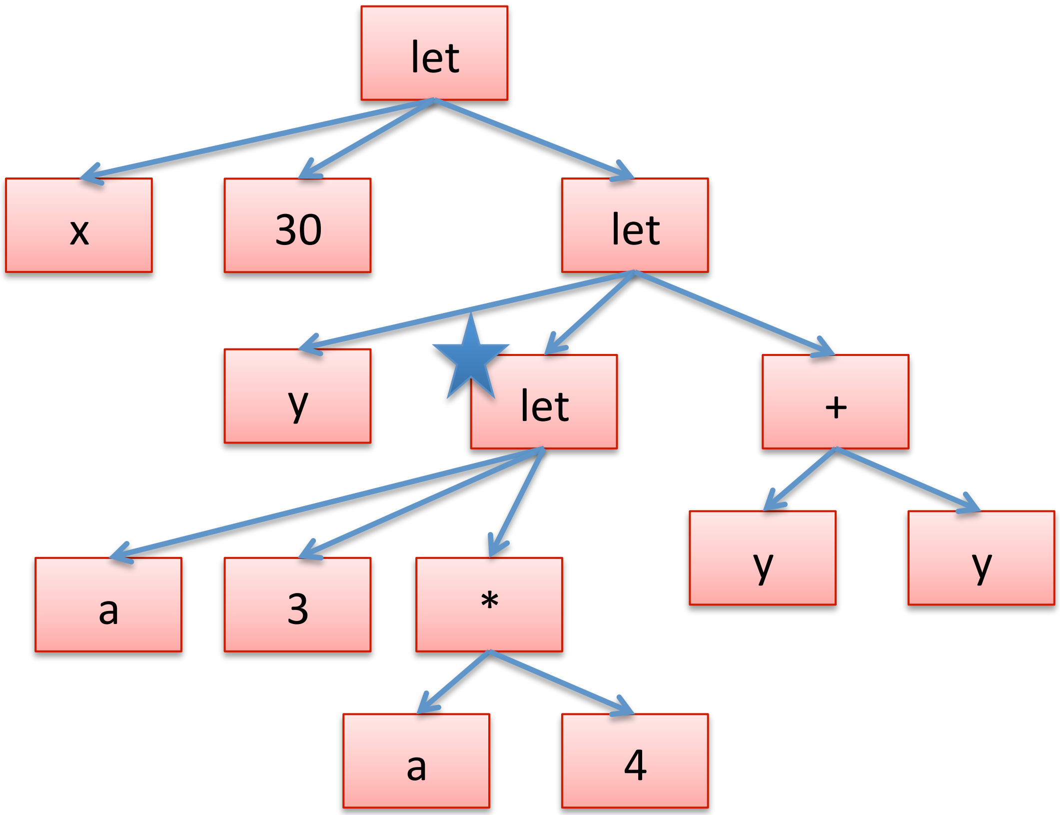
```
let x = 30 in
let y =
  (let a = 3 in a*4)
in
y+y;;
```



Abstract Syntax Trees

- And if we rename the other let to “z”:

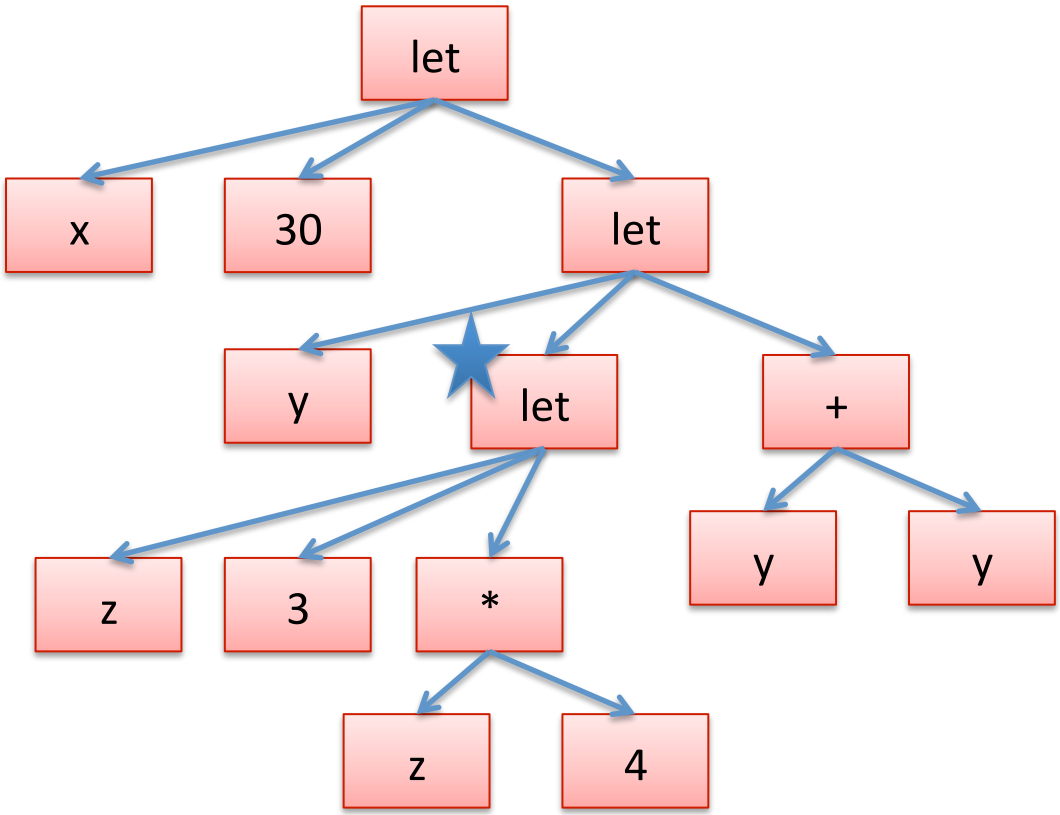
```
let x = 30 in
let y =
  (let z = 3 in z*4)
in
y+y;;
```



Abstract Syntax Trees

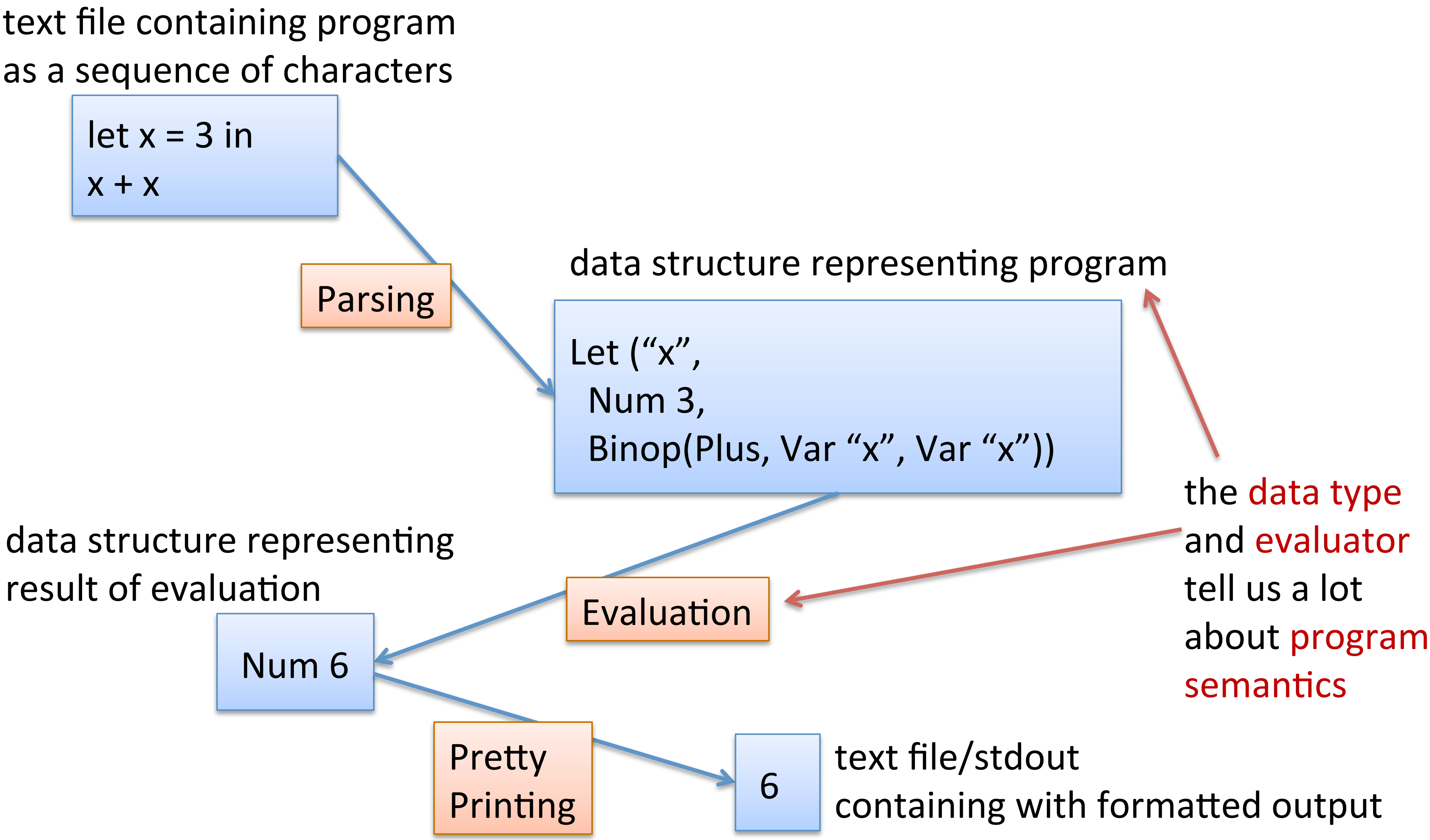
- And if we rename the other let to “z”:

```
let x = 30 in
let y =
  (let z = 3 in z*4)
in
y+y;;
```



AN O’CAML DEFINITION OF O’CAML EVALUATION

Implementing an Interpreter



Making These Ideas Precise

We can define a datatype for simple OCaml expressions:

```
type variable = string ;;
type op = Plus | Minus | Times | … ;;
type exp =
  | Int_e of int
  | Op_e of exp * op * exp
  | Var_e of variable
  | Let_e of variable * exp * exp ;;
```

Making These Ideas Precise

We can define a datatype for simple OCaml expressions:

```
type variable = string ;;
type op = Plus | Minus | Times | … ;;
type exp =
  | Int_e of int
  | Op_e of exp * op * exp
  | Var_e of variable
  | Let_e of variable * exp * exp ;;

let three = Int_e 3 ;;
let three_plus_one =
      Op_e (Int_e 1, Plus, Int_e 3) ;;
```

Making These Ideas Precise

We can represent the OCaml program:

```
let x = 30 in
let y =
   (let z = 3 in
    z*4)
in
y+y;;
```

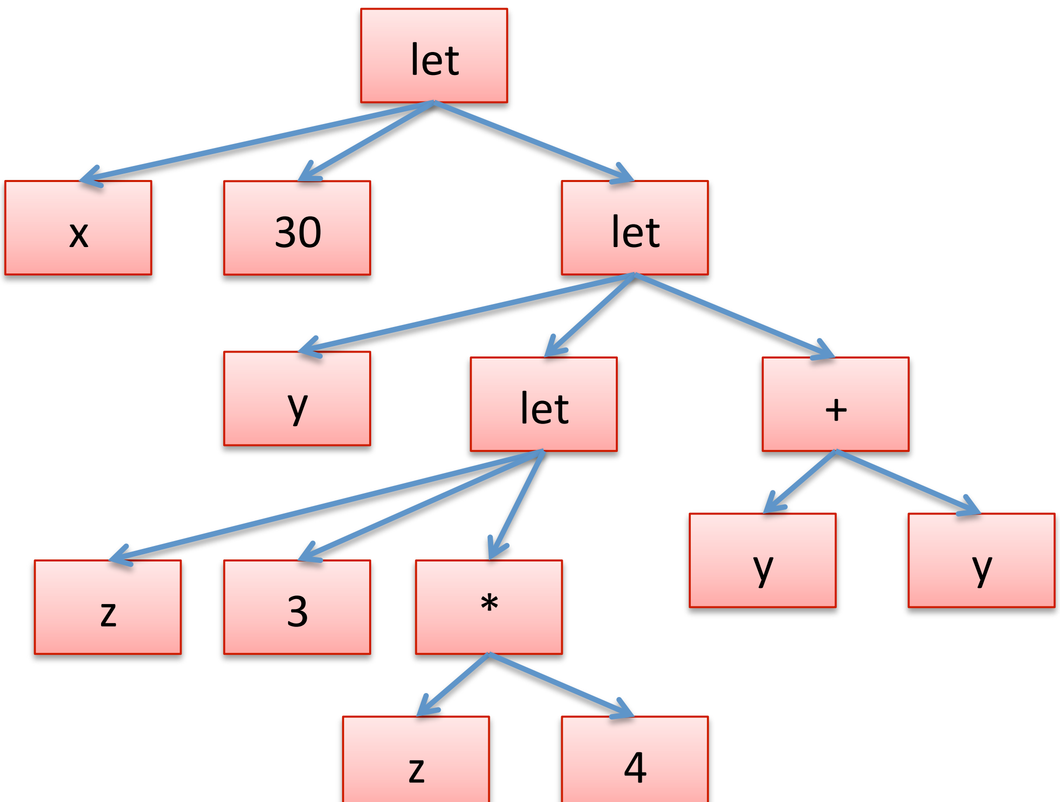
as an exp value:

```
Let_e("x", Int_e 30,
  Let_e("y",
    Let_e("z", Int_e 3,
      Op_e(Var_e "z", Times, Int_e 4)),
  Op_e(Var_e "y", Plus, Var_e "y")
```

Making These Ideas Precise

Notice how this reflects the "tree":

```
Let_e("x",Int_e 30,
     Let_e("y",Let_e("z",Int_e 3,
                    Op_e(Var_e "z", Times, Int_e 4)),
         Op_e(Var_e "y", Plus, Var_e "y")
```



Free versus Bound Variables

```
type exp =
   | Int_e of int
   | Op_e of exp * op * exp
   | Var_e of variable
   | Let_e of variable * exp * exp
```

This is a free occurrence of a variable

Free versus Bound Variables

```
type exp =
   | Int_e of int
   | Op_e of exp * op * exp
   | Var_e of variable
   | Let_e of variable * exp * exp
```

This is a free occurrence of a variable

This is a binding occurrence of a variable

Implementing a Simple Evaluator

A Simple Evaluator

```
let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | (Op_e (_,_,_) | Let_e(_,_,_) | Var_e _) -> false

let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
          let v1 = eval e1 in
          let v2 = eval e2 in
          eval_op v1 op v2
  | Let_e(x,e1,e2) ->
          let v1 = eval e1 in
          let e = substitute v1 x e2 in
          eval e
```

Even Simpler

```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
```

Oops! We Missed a Case:

```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> ???
```

We should never encounter a variable – they should have been substituted with a value! (This is a type-error.)

We Could Use Options:

```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp option =
  match e with
   | Int_e i -> Some(Int_e i)
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> None
```

But this isn't quite right – we need to match on the recursive calls to eval to make sure we get Some value!

Exceptions

```
exception UnboundVariable of variable ;;

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
```

Instead, we can throw an exception.

Exceptions

```
exception UnboundVariable of variable ;;

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
```

Note that an exception declaration is a lot like a datatype declaration. Really, we are extending one big datatype (exn) with a new constructor (UnboundVariable).

Exceptions

```
exception UnboundVariable of variable ;;

let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
```

Later on, we'll see how to catch an exception.

Back to our Evaluator

```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x) ;;
```

Evaluating the Primitive Operations

```
let eval_op (v1:exp) (op:operand) (v2:exp) : exp =
  match v1, op, v2 with
  | Int_e i, Plus, Int_e j -> Int_e (i+j)
  | Int_e i, Minus, Int_e j -> Int_e (i-j)
  | Int_e i, Times, Int_e j -> Int_e (i*j)
  …;;

let substitute v x e = …
```

```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x) ;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```

Substitution

We want to replace x (and only x) with v.

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
        match e with
        | Int_e _ -> e
        | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
        | Var_e y -> if x = y then v else e
        | Let_e (y,e1,e2) ->
             Let_e (y,
                    subst e1,
                    if x = y then e2 else subst e2)
  in
  subst e
;;
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
        match e with
        | Int_e _ -> e
        | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
        | Var_e y -> if x = y then v else e
        | Let_e (y,e1,e2) ->
             Let_e (y,
                    subst e1,
                    if x = y then e2 else subst e2)
  in
  subst e
;;
```

If x and y are the same variable, then y *shadows* x.

Let us Scale up the Language

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

Let us Scale up the Language

```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

(fun x -> e) is represented as Fun_e(x,e)

Let us Scale up the Language

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp ;;
```

A function call

fact 3 ==>
FunCall_e (Var_e "fact", Int_e 3)

Let us Scale up the Language:

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp;;

let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | (  Op_e (_,_,_)
      | Let_e (_,_,_)
      | Var_e _
      | FunCall_e (_,_) ) -> false ;;
```

Functions are values!

Let us Scale up the Language:

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp
   | Fun_e of variable * exp | FunCall_e of exp * exp;;

let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | (  Op_e (_,_,_)
      | Let_e (_,_,_)
      | Var_e _
      | FunCall_e (_,_) ) -> false ;;
```

Function calls are
not values.

Let us Scale up the Language:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
        | Fun_e (x,e), v2 -> eval (substitute v2 x e)
        | _ -> raise TypeError)
```

Let us Scale up the Language:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
        | Fun_e (x,e), v2 -> eval (substitute v2 x e)
        | _ -> raise TypeError)
```

values (including functions) always evaluate to themselves.

Let us Scale up the Language:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

To evaluate a function call, we first evaluate both e1 and e2 to values.

Let us Scale up the Language

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
        | Fun_e (x,e), v2 -> eval (substitute v2 x e)
        | _ -> raise TypeError)
```

e1 had better evaluate to a function value, else we have a type error.

Let us Scale up the Language

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1, eval e2 with
         | Fun_e (x,e), v2 -> eval (substitute v2 x e)
         | _ -> raise TypeError)
```

Then we substitute e2's value (v2) for x in e and evaluate the resulting expression.

Simplifying a little

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
         | Fun_e (x,e) -> eval (substitute (eval e2) x e)
         | _ -> raise TypeError)
```

We don't really need
to pattern-match on e2.
Just evaluate here

Simplifying a little

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (ef,e1) ->
       (match eval ef with
         | Fun_e (x,e2) -> eval (substitute (eval e1) x e2)
         | _ -> raise TypeError)
```

This looks like the case for let!

Let and Lambda

```
let x = 1 in x+41
-->
1+41
-->
42
```

```
(fun x -> x+41) 1
-->
1+41
-->
42
```

So we could write:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (FunCall (Fun_e (x,e2), e1))
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (ef,e2) ->
        (match eval ef with
          | Fun_e (x,e1) -> eval (substitute (eval e1) x e2)
          | _ -> raise TypeError)
```

In programming-languages speak: "Let is *syntactic sugar* for a function call"

Syntactic sugar: A new feature defined by a simple, local transformation.

Recursive definitions

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp ;;
```

```
let rec f x = f (x+1) in f 3
```

(rewrite)

```
let f = rec f x -> f (x+1)) in
f 3
```

(alpha-convert)

```
let g = rec f x -> f (x+1)) in
g 3
```

(implement)

```
Let_e ("g,
  Rec_e ("f", "x",
    FunCall_e (Var_e "f", Op_e (Var_e "x", Plus, Int_e 1))
 ),
 FunCall (Var_e "g", Int_e 3)
)
```

Recursive definitions

```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp ;;
```

```
let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | Rec_e of (_,_,_) -> true
   | (Op_e (_,_,_) | Let_e (_,_,_) |
      Var_e _ | FunCall_e (_,_) ) -> false ;;
```

Before Evaluation: Notation for Substitution

"Substitute value v for variable x in expression e:" e [v / x]

examples of substitution:

(x + y) [7/y]	is	(x + 7)
(let x =30 in let y=40 in x + y) [7/y]	is	(let x =30 in let y=40 in x + y)
(let y = y in let y = y in y + y) [7/y]	is	(let y = 7 in let y = y in y + y)

Evaluating Recursive Functions

Basic evaluation rule for recursive functions:

```
(rec f x = body) arg      -->      body [arg/x] [rec f x = body/f]
```

argument substituted for parameter

entire function substituted for function name

Evaluating Recursive Functions

Start out with
a let bound to
a recursive function:

```
let g =
  rec f x ->
    if x <= 0 then x
    else x + f (x-1)
in g 3
```

The Substitution:

```
g 3 [rec f x ->
      if x <= 0 then x
      else x + f (x-1)  / g]
```

The Result:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

Evaluating Recursive Functions

Recursive Function Call:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

The Substitution:

```
(if x <= 0 then x else x + f (x-1))
  [ rec f x ->
        if x <= 0 then x
        else x + f (x-1)  / f ]
  [ 3 / x ]
```

Substitute argument for parameter

Substitute entire function for function name

The Result:

```
(if 3 <= 0 then 3 else 3 +
   (rec f x ->
        if x <= 0 then x
        else x + f (x-1)) (3-1))
```

Evaluating Recursive Functions

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1 with
        | Fun_e (x,e) ->
            let v = eval e2 in
            substitute e x v

        | (Rec_e (f,x,e)) as g ->
            let v = eval e2 in
            substitute (substitute e x v) f g

        | _ -> raise TypeError)
```

More Evaluation

```
(rec fact n = if n <= 1 then 1 else n * fact(n-1)) 3
-->
if 3 < 1 then 1 else
  3 * (rec fact n = if ... then ... else ...) (3-1)
-->
3 * (rec fact n = if … ) (3-1)
-->
3 * (rec fact n = if … ) 2
-->
3 * (if 2 <= 1 then 1 else 2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(1))
-->
3 * 2 * if 1 <= 1 then 1 else 1 * (rec fact ...)(1-1)
-->
3 * 2 * 1
```

A MATHEMATICAL DEFINITION* OF O'CAML EVALUATION

* it's a partial definition and this is a big topic; for more, see COS 441

From Code to Abstract Specification

- OCaml code can give a language semantics
  - advantage: it can be executed, so we can try it out
  - advantage: it is amazingly concise
    - especially compared to what you would have written in Java
  - disadvantage: it is a little ugly to operate over concrete ML datatypes like "Op_e(e1,Plus,e2)" as opposed to "e1 + e2"
- PL researchers have developed their own, relatively standard notation for writing down how programs execute
  - it has a mathematical "feel" that makes PL researchers feel special and gives us goosebumps inside
  - it operates over abstract expression syntax like "e1 + e2"
  - it is useful to know this notation if you want to read specifications of programming language semantics
    - eg: Standard ML (of which OCaml is a descendent) has a formal definition given in this notation

Rules

- Our goal is to explain how an expression e evaluates to a value v.

- We are going to do so using a set of (inductive) rules
- A rule looks like this:

$$\frac{\text{premise 1} \quad \text{premise 2} \quad \ldots \quad \text{premise 3}}{\text{conclusion}}$$

- You read a rule like this:
  - “if premise 1 can be proven and premise 2 can be proven and ... and premise n can be proven then conclusion can be proven”

- Some rules have no premises -- this means their conclusions are always true
  - we call such rules “axioms” or “base cases”

An example rule concerning evaluation

As a rule:

$$\frac{e1 \dashrightarrow v1 \qquad e2 \dashrightarrow v2 \qquad eval\_op\ (v1, op, v2) == v'}{e1\ op\ e2 \dashrightarrow v'}$$

e1 --> v1   e2 --> v2   eval_op (v1, op, v2) == v’

e1 op e2 --> v’

In English:

“If e1 evaluates to v1
and e2 evaluates to v2
and eval_op (v1, op, v2) is equal to v’
then
e1 op e2 evaluates to v’

In code:

```
let rec eval (e:exp) : exp =
  match e with
  ...
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
```

An example rule concerning evaluation

As a rule:

$$\frac{i \in Z}{i \dashrightarrow i}$$

asserts i is an integer

In English:

> "If the expression is an integer, it evaluates to itself."

In code:

```
let rec eval (e:exp) : exp =
  match e with
 | Int_e i -> Int_e i
 ...
```

An example rule concerning evaluation

As a rule:

$$\frac{e1 \dashrightarrow v1 \qquad e2\,[v1/x] \dashrightarrow v2}{\text{let } x = e1 \text{ in } e2 \dashrightarrow v2}$$

In English:

> "If e1 evaluates to v1
> and e2 with v1 substituted for x evaluates to v2
> then let x=e1 in e2 evaluates to v2."

In code:

```
let rec eval (e:exp) : exp =
  match e with
 ...
 | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
 ...
```

An example rule concerning evaluation

As a rule:

$$\frac{}{\lambda x.e \dashrightarrow \lambda x.e}$$

typical “lambda” notation for a function with argument x, body e

In English:

> “A function evaluates to itself.”

In code:

```
let rec eval (e:exp) : exp =
  match e with
  ...
  | Fun_e (x,e) -> Fun_e (x,e)
  ...
```

An example rule concerning evaluation

As a rule:

$$\frac{e1 \dashrightarrow \lambda x.e \qquad e2 \dashrightarrow v2 \qquad e[v2/x] \dashrightarrow v}{e1\ e2 \dashrightarrow v}$$

In English:

"if e1 evaluates to a function with argument x and body e
and e2 evaluates to a value v2
and e with v2 substituted for x evaluates to v
then e1 applied to e2 evaluates to v"

In code:

```
let rec eval (e:exp) : exp =
  match e with
 ..
| FunCall_e (e1,e2) ->
         (match eval e1 with
          | Fun_e (x,e) -> eval (substitute e x (eval e2))
          | ...)
...
```

An example rule concerning evaluation

As a rule:

$$\frac{e1 \dashrightarrow \text{rec } f\ x = e \qquad e2 \dashrightarrow v \qquad e[\text{rec } f\ x = e/f][v/x] \dashrightarrow v2}{e1\ e2 \dashrightarrow v2}$$

In English:

“uggh”

In code:

```
let rec eval (e:exp) : exp =
  match e with
     ...
   | (Rec_e (f,x,e)) as g ->
        let v = eval e2 in
        substitute (substitute e x v) f g
```

Comparison: Code vs. Rules

complete eval code:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
        | Fun_e (x,e) -> eval (Let_e (x,e2,e))
        | _ -> raise TypeError)
   | LetRec_e (x,e1,e2) ->
      (Rec_e (f,x,e)) as g ->
        let v = eval e2 in
        substitute (substitute e x v) f g
```

complete set of rules:

$$\frac{i \in Z}{i \dashrightarrow i}$$

$$\frac{e1 \dashrightarrow v1 \qquad e2 \dashrightarrow v2 \qquad eval\_op\ (v1, op, v2) == v}{e1\ op\ e2 \dashrightarrow v}$$

$$\frac{e1 \dashrightarrow v1 \qquad e2\ [v1/x] \dashrightarrow v2}{let\ x = e1\ in\ e2 \dashrightarrow v2}$$

$$\frac{}{\lambda x.e \dashrightarrow \lambda x.e}$$

$$\frac{e1 \dashrightarrow \lambda x.e \qquad e2 \dashrightarrow v2 \qquad e[v2/x] \dashrightarrow v}{e1\ e2 \dashrightarrow v}$$

$$\frac{e1 \dashrightarrow rec\ f\ x = e \qquad e2 \dashrightarrow v2 \qquad e[rec\ f\ x = e/f][v2/x] \dashrightarrow v3}{e1\ e2 \dashrightarrow v3}$$

Almost isomorphic:

– one rule per pattern-matching clause

– recursive call to eval whenever there is a --> premise in a rule

– what's the main difference?

Comparison: Code vs. Rules

complete eval code:

```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (e1,e2) ->
       (match eval e1
         | Fun_e (x,e) -> eval (Let_e (x,e2,e))
         | _ -> raise TypeError)
   | LetRec_e (x,e1,e2) ->
      (Rec_e (f,x,e)) as g ->
        let v = eval e2 in
        substitute (substitute e x v) f g
```

complete set of rules:

$$\frac{i \in \mathbb{Z}}{i \text{ --> } i}$$

$$\frac{e1 \text{ --> } v1 \qquad e2 \text{ --> } v2 \qquad \text{eval\_op}\ (v1, op, v2) == v}{e1\ op\ e2 \text{ --> } v}$$

$$\frac{e1 \text{ --> } v1 \qquad e2\ [v1/x] \text{ --> } v2}{\text{let } x = e1 \text{ in } e2 \text{ --> } v2}$$

$$\frac{}{\lambda x.e \text{ --> } \lambda x.e}$$

$$\frac{e1 \text{ --> } \lambda x.e \qquad e2 \text{ --> } v2 \qquad e[v2/x] \text{ --> } v}{e1\ e2 \text{ --> } v}$$

$$\frac{e1 \text{ --> rec } f\ x = e \qquad e2 \text{ --> } v2 \qquad e[\text{rec } f\ x = e/f][v2/x] \text{ --> } v3}{e1\ e2 \text{ --> } v3}$$

- There's no formal rule for handling free variables
- No rule for evaluating function calls when a non-function in the caller position
- In general, *no rule when further evaluation is impossible*
  - the rules express the legal evaluations and say nothing about what to do in error situations
  - the code handles the error situations by raising exceptions

Summary

- We can reason about OCaml programs using a *substitution model*.
  - integers, bools, strings, chars, and *functions* are values
  - value rule: values evaluate to themselves
  - let rule: "let x = e1 in e2" : substitute e1's value for x into e2
  - fun call rule: "(fun x -> e2) e1": substitute e1's value for x into e2
  - rec call rule: "(rec x = e1) e2" : like fun call rule, but also substitute recursive function for name of function
    - To unwind: substitute (rec x = e1) for x in e1

- We can make the evaluation model precise by building an interpreter and using that interpreter as a specification of the language semantics.

- We can also specify the evaluation model using a set of *inference rules*
  - more on this in COS 441

Some Final Words

- The substitution model is only a model.
  - it does not accurately model all of OCaml's features
    - I/O, exceptions, mutation, concurrency, …
    - we can build models of these things, but they aren't as simple.
    - even substitution was tricky to formalize!
- It's useful for reasoning about higher-order functions, correctness of algorithms, and optimizations.
  - we can use it to formally prove that, for instance:
    - map f (map g xs) == map (comp f g) xs
    - proof: by induction on the length of the list xs, using the definitions of the substitution model.
  - we often model complicated systems (e.g., protocols) using a small functional language and substitution-based evaluation.
- It is *not* useful for reasoning about execution time or space

Some Exercises

Complete the following expressions so they evaluate to 42 or explain why this is impossible, appealing to the substitution model.

```
let x = ??? in
let x = 43 in
x ;;
```

```
let x = fun x -> x*2 in
    let x = ??? 21 in
        x ;;
```

```
let x = ??? in
let y = (let x = 21 in x+x) in
x ;;
```

```
let x = ??? in
let y = [42] in
x y ;;
```

END