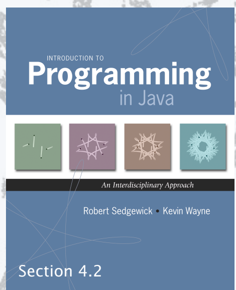
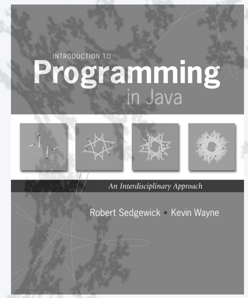
COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching

COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching

- **A typical client**
- Binary search
- Insertion sort
- Mergesort
- Longest repeated substring

A typical client: Whitelist filter

A blacklist is a list of entities to be *rejected* for service. ← Examples: Overdrawn account, Spammers

A whitelist is a list of entities to be *accepted* for service. ← Examples: Account in good standing, Friends and relatives

Whitelist filter
- Read a list of strings from a *whitelist* file.
- Read strings from StdIn and write to StdOut only those in the whitelist.

Example. Email spam filter
(message contents omitted)

whitelist
```
alice@home
bob@office
carl@beach
dave@boat
```

StdIn
```
bob@office   ✓
carl@beach   ✓
marvin@spam
bob@office   ✓
bob@office   ✓
mallory@spam
dave@boat    ✓
eve@airport
alice@home   ✓
...
```

StdOut
```
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
...
```


Search client: Whitelist filter

```
public class WhiteFilter
{
   public static int search(String key, String[] a)
   // Search method (stay tuned).

   public static void main(String[] args)
   {
      In in = new In(args[0]);
      String[] words = in.readAllStrings();

      while (!StdIn.isEmpty())
      {
         String key = StdIn.readString();
         if (search(key, words) != -1)
            StdOut.println(key);
      }
   }
}
```

```
% more white4.txt
alice@home
bob@office
carl@beach
dave@boat

% more test.txt
bob@office
carl@beach
marvin@spam
bob@office
bob@office
mallory@spam
dave@boat
eve@airport
alice@home

% java WhiteFilter white4.txt < test.txt
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
```

Alice and Bob

Hey, Alice. I think I'm going to start an internet company.

Me too. I'm thinking about having 1 thousand customers next month and 1 million next year.

We're hoping to grow even faster than that.

Good luck! BTW, you're going to need a whitelist filter.

Yes, I know. I'm going to a hackathon to knock it out.

I'm going to take a few CS courses first.

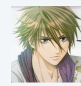
Strawman implementation: Sequential search (first try)

Sequential search

- Check each array entry 0, 1, 2, 3, ... for match with search string.
- If match found, return index of matching string.
- If not, return −1.

```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if (a[i] == key) return i;
   return -1;
}
```

✘ Compares references, not strings!



@#$%$#@@%#!!

i	a[i]
0	alice
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

oscar?

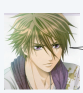
Strawman implementation: Sequential search

Sequential search

- Check each array entry 0, 1, 2, 3, ... for match with search string.
- If match found, return index of matching string.
- If not, return −1.

```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if (a[i].compareTo(key) == 0) return i;
   return -1;
}
```

Match found. Return 10



Still, this was even easier than I thought!

i	a[i]
0	alice
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

oscar?

Mathematical analysis of whitelist filter using sequential search

Model

- N strings on the whitelist.
- cN transactions for c not large.
- String length not long.

Analysis

- A random search hit checks about half of the N strings on the whitelist, on average.
- A random search miss checks all of the N strings on the whitelist, on average.
- Expected order of growth of running time: N^2.

whitelist
dobqi
xwnzb
dqwak
lnuqv
czpwx
bshla
idhld
utfyw
hafah
tsirv

transactions
xwnzb
lnuqv
lnuqv
czpwx
czpwx
dqwak
idhld
dobqi
dobqi
tsirv
dqwak
dobqi
idhld
dqwak
dobqi
lnuqv
xwnzb
idhld
bshla
xwnzb

Random representative inputs for searching and sorting

Generate N random strings of length L from a given alphabet

```
public class Generator
{
   public static String randomString(int L, String alpha)
   {
      char[] a = new char[L];
      for (int i = 0; i < L; i++)
      {
         int t = StdRandom.uniform(alpha.length());
         a[i] = alpha.charAt(t);
      }
      return new String(a);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      int L = Integer.parseInt(args[1]);
      String alpha = args[2];
      for (int i = 0; i < N; i++)
         StdOut.println(randomString(L, alpha));
   }
}
```

```
% java Generator 10 3 abc
bab
bab
bbb
cac
aba
abb
bab
ccb
cbc
bab
```

good chance of duplicates

```
% java Generator 15 8 0123456789
62855405
83179069
79061047
27258805
54441080
76592141
95956542
19442316
75032539
10528640
42496398
34226197
10320073
80072566
87979201
```

not much chance of duplicates

```
% java Generator 1 60 actg
tctatagggtcgtttgcgaagcctacacaaaagtagttgttggacaacgattgacaaaca
```


Empirical tests of sequential search

Whitelist filter scenario
- Whitelist of size N.
- $10N$ transactions.

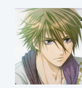
N	T_N (seconds)	$T_N/T_{N/2}$	*transactions per second*
10,000	3		3,333
20,000	9		2,222
40,000	35	3.9	1,143
80,000	149	4.3	536
...			
1.28 million		4	34

```
% java Generator 10000 ...
3 seconds
% java Generator 20000 ...
9 seconds
% java Generator 40000 ...
35 seconds
% java Generator 80000 ...
149 seconds
```

```
... = 10 a-z | java TestSS
a-z = abcdefghijklmnopqrstuvwxyz
```

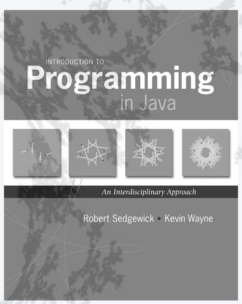
12.8 million transactions at a rate of 34 per second and dropping

Hmmm. That doesn't seem too good.



Hypothesis. Order of growth is N^2. ← Does NOT scale.

COMPUTER SCIENCE
SEDGEWICK / WAYNE

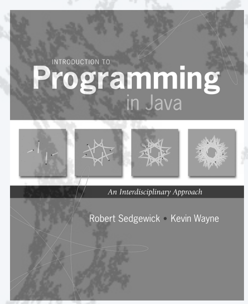


http://introcs.cs.princeton.edu

13. Sorting and Searching

- **A typical client**
- Binary search
- Insertion sort
- Mergesort
- Longest repeated substring

COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- **Binary search**
- Insertion sort
- Mergesort
- Longest repeated substring

Binary search

Binary search
- Keep the array in sorted order (stay tuned).
- Examine the middle key.
- If it matches, return its index.
- If it is larger, search the half with lower indices.
- If it is smaller, search the half with upper indices.

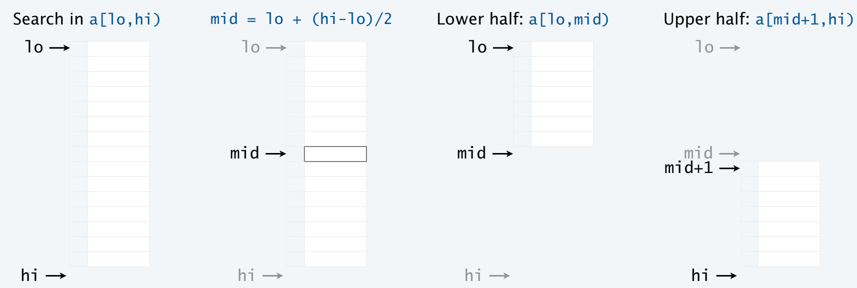
i	a[i]
0	alice
1	bob
2	carlos
3	carol
4	craig
5	dave
6	erin
7	eve
8	frank
9	mallory
10	oscar
11	peggy
12	trent
13	walter
14	wendy

oscar?

Match found.
Return 10

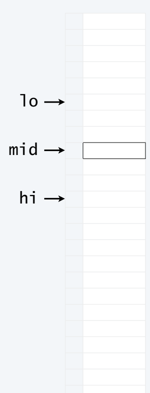
Binary search arithmetic

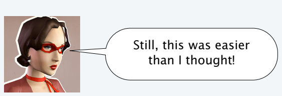
Notation. a[lo,hi] means a[lo], a[lo+1] ... a[hi-1] (does not include a[hi]).



Binary search: Java implementation

```java
public static int search(String key, String[] a)
{  return search(key, a, 0, a.length); }

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```





Recursion trace for binary search

```java
public static int search(String key, String[] a)
{  return search(key, a, 0, a.length); }

public static int search(String key, String[] a,
                                        int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

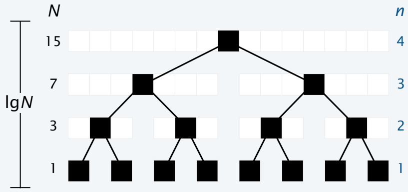
```
search("oscar")
   return              10

   search("oscar", a, 0, 15)
      mid = 7;
      > "eve"
      return           10

   search("oscar", a, 8, 15)
      mid = 11;
      < "peggy"
      return           10

   search("oscar", a, 8, 10)
      mid = 9;
      > "mallory"
      return           10

   search("oscar", a, 10, 10)
      mid = 10;
      == "oscar"
      return 10;
```

10 oscar

Mathematical analysis of binary search

Exact analysis for search miss for $N = 2^n - 1$

- Note that $n = \lg(N+1) \sim \lg N$.
- Subarray size for 1st call is $2^n - 1$.
- Subarray size for 2nd call is $2^{n-1} - 1$.
- Subarray size for 3rd call is $2^{n-2} - 1$.
- …
- Subarray size for nth call is 1.
- Total # compares (one per call): $n \sim \lg N$.



Every search miss is a top-to-bottom path in this tree.

Proposition. Binary search uses $\sim\lg N$ compares for a search miss.

Proof. An (easy) exercise in discrete math.

Interested in details? Take a course in algorithms.

Proposition. Binary search uses $\sim\lg N$ compares for a random search hit.

Proof. A slightly more difficult exercise in discrete math.

Empirical tests of binary search

Whitelist filter scenario

- Whitelist of size N.
- $10N$ transactions.

N	T_N (seconds)	$T_N/T_{N/2}$	transactions per second (thousands)
100,000	1		
200,000	3		
400,000	6	2	67
800,000	14	2.35	57
1,600,000	33	2.33	48
10.28 million	264	2	48

```
% java Generator 100000 ...
1 seconds
% java Generator 200000 ...
3 seconds
% java Generator 400000 ...
6 seconds
% java Generator 800000 ...
14 seconds
% java Generator 1600000 ...
33 seconds
```

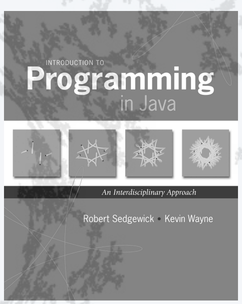
```
... = 10 a-z | java TestBS
a-z = abcdefghijklmnopqrstuvwxyz
```

nearly 50,000 transactions per second, and holding

Confirms hypothesis that order of growth is $N\log N$.

Will scale.

COMPUTER SCIENCE
SEDGEWICK / WAYNE

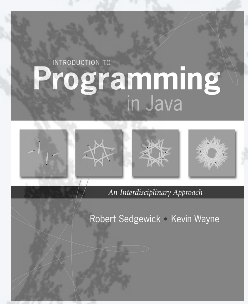


http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- **Binary search**
- Insertion sort
- Mergesort
- Longest repeated substring

COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- **Insertion sort**
- Mergesort
- Longest repeated substring

Sorting: Rearrange N items to put them in ascending order

Applications

- Binary search
- Statistics
- Databases
- Data compression
- Bioinformatics
- Computer graphics
- Scientific computing
- ...
- [Too numerous to list]

		→		
0	wendy		0	alice
1	alice		1	bob
2	dave		2	carlos
3	walter		3	carol
4	carlos		4	craig
5	carol		5	dave
6	erin		6	erin
7	oscar		7	eve
8	peggy		8	frank
9	trudy		9	oscar
10	eve		10	peggy
11	trent		11	trent
12	bob		12	trudy
13	craig		13	victor
14	frank		14	walter
15	victor		15	wendy

TEQ 0 on sorting

Q. What's the most efficient way to sort 1 million 32-bit integers?

Insertion sort algorithm

Insertion sort

- Move down through the array.
- Each item *bubbles up* above the larger ones above it.
- Everything above the current item is in order.
- Everything below the current item is untouched.

Like bubble sort, but not bubble sort.
We don't teach bubble sort any more because this is simpler and faster.

0	wendy
1	alice
2	dave
3	walter
4	carlos
5	carol
6	erin
7	oscar
8	peggy
9	trudy
10	eve
11	trent
12	bob
13	craig
14	frank
15	victor

Insertion sort trace

0	wendy	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice	alice
1	alice	wendy	dave	dave	carlos	carlos	carlos	carlos	carlos	carlos	carlos	carlos	bob	bob	bob	bob
2	dave	dave	wendy	walter	dave	carol	carol	carol	carol	carol	carol	carol	carlos	carlos	carlos	carlos
3	walter	walter	walter	wendy	walter	dave	dave	dave	dave	dave	dave	dave	carol	carol	carol	carol
4	carlos	carlos	carlos	carlos	wendy	walter	erin	erin	erin	erin	erin	erin	dave	craig	craig	craig
5	carol	carol	carol	carol	carol	wendy	walter	oscar	oscar	oscar	eve	eve	erin	dave	dave	dave
6	erin	erin	erin	erin	erin	erin	wendy	walter	peggy	peggy	oscar	oscar	eve	erin	erin	erin
7	oscar	oscar	oscar	oscar	oscar	oscar	oscar	wendy	walter	trudy	peggy	peggy	oscar	eve	eve	eve
8	peggy	peggy	peggy	peggy	peggy	peggy	peggy	peggy	wendy	walter	trudy	trent	peggy	oscar	frank	frank
9	trudy	trudy	trudy	trudy	trudy	trudy	trudy	trudy	trudy	wendy	walter	trudy	trent	peggy	oscar	oscar
10	eve	eve	eve	eve	eve	eve	eve	eve	eve	eve	wendy	walter	trudy	trent	peggy	peggy
11	trent	trent	trent	trent	trent	trent	trent	trent	trent	trent	trent	wendy	walter	trudy	trent	trent
12	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	bob	wendy	walter	trudy	trudy
13	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	craig	wendy	walter	victor
14	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	frank	wendy	walter
15	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	victor	wendy

Insertion sort: Java implementation

```
public class Insertion
{
   public static void sort(String[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1].compareTo(a[j]) > 0)
               exch(a, j-1, j);
            else break;
   }

   private static void exch(String[] a, int i, int j)
   {  String t = a[i]; a[i] = a[j]; a[j] = t;  }

   public static void main(String[] args)
   {
      String[] a = StdIn.readAllStrings();
      sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a);
   }
}
```

```
% more names16.txt
wendy
alice
dave
walter
carlos
carol
erin
oscar
peggy
trudy
eve
trent
bob
craig
frank
victor
```

```
% java Insertion < names16.txt
alice
bob
carlos
carol
craig
dave
erin
eve
frank
oscar
peggy
trent
trudy
victor
walter
wendy
```


Empirical tests of insertion sort

Sort random strings
- Array of length *N*.
- 10-character strings.

N	T_N (seconds)	$T_N / T_{N/2}$
20,000	1	
40,000	4	
80,000	35	9
160,000	225	6.4
320,000	1019	4.5
...		
1.28 million	14400	4

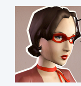
← 4 hours

```
% java Generator 20000 ...
1 seconds
% java Generator 40000 ...
4 seconds
% java Generator 80000 ...
35 seconds
% java Generator 160000 ...
225 seconds
% java Generator 320000 ...
1019 seconds
```

```
... = 10 a-z | java Insertion
a-z = abcdefghijklmnopqrstuvwxyz
```

Confirms hypothesis that order of growth is N^2.

will NOT scale



And 4x64/24 = 10+ *days* to sort 10 million? Sounds bad.

Do you have anything better?

A rule of thumb

Moore's law. The number of transistors in an integrated circuit doubles about every 2 years.

Implications
- Memory size doubles every two years.
- Processor speed doubles every two years.

Gordon Moore
Founder of Intel
1929 –

Sedgewick's rule of thumb.
It takes *seconds* to access every word in a computer.

computer	*instructions per second*	*words of memory*
PDP-9	tens of thousands	tens of thousands
VAX 11-780	millions	millions
CRAY 1	tens of millions	tens of millions
MacBook Air	billions	billions

Scalability

An algorithm *scales* if its running time doubles when the problem size doubles.

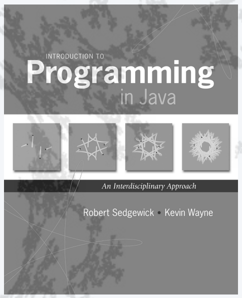
2x faster computer with 2x memory using an alg that scales?
- Can solve problems we're solving now in half the time.
- Can solve a 2x-sized problem in the *same* time it took to solve an x-sized problem.
- Progress.

2x faster computer with 2x memory using quadratic alg?
- Can solve problems we're solving now in half the time.
- Takes *twice* as long solve a 2x-sized problem as it took to solve an x-sized problem.
- Frustration.

order of growth	*scales?*
N	✓
$N \log N$	✓
N^2	✗
N^3	✗

Bottom line. Need algorithms that scale to keep pace with Moore's law.

COMPUTER SCIENCE
SEDGEWICK / WAYNE

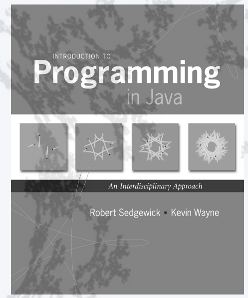


http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- **Insertion sort**
- Mergesort
- Longest repeated substring

COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- Insertion sort
- **Mergesort**
- Longest repeated substring

Mergesort algorithm

Merge sort

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

Divide	Sort halves	Merge
wendy	alice	
alice	carlos	
dave	carol	
walter	dave	
carlos	erin	
carol	oscar	
erin	walter	
oscar	wendy	
peggy	bob	
trudy	craig	
eve	eve	
trent	frank	
bob	peggy	
craig	trent	
frank	trudy	
victor	victor	

Merge: Java implementation

Abstract inplace merge

- Merge a[lo, mid) with a[mid, hi).
- Use auxiliary array for result.
- Copy back when sort complete.

```
private static String[] aux;
public static void merge(String[] a, int lo, int mid, int hi)
{
   // Merge a[lo, mid) with a[mid, hi) into aux[0, hi-lo).
   int i = lo, j = mid, N = hi - lo;
   for (int k = 0; k < N; k++)
   {
      if      (i == mid) aux[k] = a[j++];
      else if (j == hi)  aux[k] = a[i++];
      else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
      else                               aux[k] = a[i++];
   }
   // Copy back into a[lo, hi)
   for (int k = 0; k < N; k++)
      a[lo + k] = aux[k];
}
```

	a		aux
i →	alice	lo	k →
	carlos		
	carol		
	dave		
	erin		
	oscar		
	walter		
	wendy		
j →	bob	mid	
	craig		
	eve		
	frank		
	peggy		
	trent		
	trudy		
	victor		
		hi	

Merge sort: Java implementation

Merge sort
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

```
public class Merge
{
   private static String[] aux;
   public static void merge(String[] a, int lo, int mid, int hi)
   { // See previous slide.  }
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }
   public static void sort(String[] a, int lo, int hi)
   {  // Sort a[lo, hi).
      int N = hi - lo;
      if (N <= 1) return;
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);
      merge(a, lo, mid, hi);
   }
}
```

```
% more names16.txt
wendy
alice
dave
walter
carlos
carol
erin
oscar
peggy
trudy
eve
trent
bob
craig
frank
victor
```

```
% java Merge < names16.txt
alice
bob
carlos
carol
craig
dave
erin
eve
frank
oscar
peggy
trent
trudy
victor
walter
wendy
```


Mergesort trace

Merge sort
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

alice	alice	alice	alice	wendy
bob	carlos	dave	wendy	alice
carlos	carol	walter	dave	dave
carol	dave	wendy	walter	walter
craig	erin	carlos	carlos	carlos
dave	oscar	carol	carol	carol
erin	walter	erin	erin	erin
eve	wendy	oscar	oscar	oscar
frank	bob	eve	peggy	peggy
oscar	craig	peggy	trudy	trudy
peggy	eve	trent	eve	eve
trent	frank	trudy	trent	trent
trudy	peggy	bob	bob	bob
victor	trent	craig	craig	craig
walter	trudy	frank	frank	frank
wendy	victor	victor	victor	victor

Mergesort analysis

Cost model. Count *data moves*.

# of times a string moves from one array to another

Exact analysis for $N = 2^n$.
- Note that $n = \lg N$.
- 1 subarray of size 2^n.
- 2 subarrays of size 2^{n-1}.
- 4 subarrays of size 2^{n-2}.
- ...
- 2^n subarrays of size 1.
- Total # data moves: $2N \lg N$.

$\lg N$

1 subarray of size N	2 subarrays of size $N/2$	4 subarrays of size $N/4$	8 subarrays of size $N/8$	16 subarrays of size $N/16$
alice	alice	alice	alice	wendy
bob	carlos	dave	wendy	alice
carlos	carol	walter	dave	dave
carol	dave	wendy	walter	walter
craig	erin	carlos	carlos	carlos
dave	oscar	carol	carol	carol
erin	walter	erin	erin	erin
eve	wendy	oscar	oscar	oscar
frank	bob	eve	peggy	peggy
oscar	craig	peggy	trudy	trudy
peggy	eve	trent	eve	eve
trent	frank	trudy	trent	trent
trudy	peggy	bob	bob	bob
victor	trent	craig	craig	craig
walter	trudy	frank	frank	frank
wendy	victor	victor	victor	victor

$2N$ data moves $2N$ data moves $2N$ data moves $2N$ data moves

Interested in details? Take a course in algorithms.

Empirical tests of mergesort

Sort random strings
- Array of length *N*.
- 10-character strings.

N	T_N (seconds)	$T_N/T_{N/2}$
1 million	1	
2 million	2	
4 million	5	2.5
8 million	10	2
16 million	20	2.5
...		
1.02 billion	1280	2

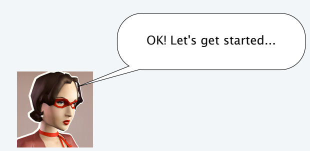
20 minutes

```
% java Generator 1000000 ...
1 seconds
% java Generator 2000000 ...
2 seconds
% java Generator 4000000 ...
5 seconds
% java Generator 8000000 ...
10 seconds
% java Generator 16000000 ...
20 seconds
```

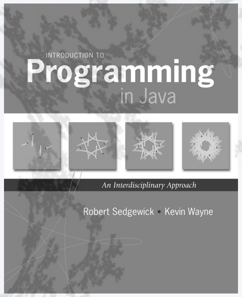
```
... = 10 a-z | java Merge
a-z = abcdefghijklmnopqrstuvwxyz
```

Confirms hypothesis that order of growth is $N \log N$

WILL scale



COMPUTER SCIENCE
SEDGEWICK / WAYNE

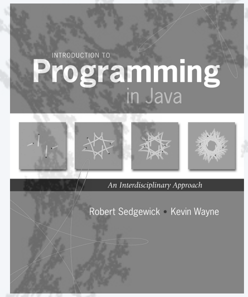


http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- Insertion sort
- **Mergesort**
- Longest repeated substring

COMPUTER SCIENCE
SEDGEWICK / WAYNE



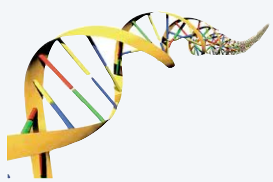
http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- Insertion sort
- Mergesort
- **Longest repeated substring**

Detecting repeats in a string

Longest repeated substring
- Given: A string s.
- Task: Find the longest substring in s that appears at least twice.



Example 1. a a c a a g t t t a c a a g c

Example 2. a a c a a g t t t a c a a g t t t a c a a g c t a g c

Example 3 (first 100 digits of π).

3	.	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4
	6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8	4	1	9	7
	1	6	9	3	9	9	3	7	5	1	0	5	8	2	0	9	7	4	9	4
	4	5	9	2	3	0	7	8	1	6	4	0	6	2	8	6	2	0	8	9
	9	8	6	2	8	0	3	4	8	2	5	3	4	2	1	1	7	0	6	9

LRS example: repetitive structure in music

Mary had a little lamb

Für Elise

source: http://www.bewitched.com/match/

LRS applications

Analysts seek repeated sequences in real-world data because they are causal.

Example 1: Digits of π
- Q. Are they "random" ?
- A. No, but we can't tell the difference.
- Ex. Length of LRS in first 10 million digits is 14.

```
3.141592653589793238462643383279502884
19716939937510582097494459230781640628
62089986280348253421170679821480865132
82306647093844609550582231725359408128
48111745028410270193852110555964462294
89549303819644288109756659334461284756
48233786783165271201909145648566923460
34861045432664821339360726024914127372
45870066063155881748815209209628292540
```

Example 2: Cryptography
- Find LRS.
- Check for "known" message header information.
- Break code.

```
11001001001111011011100101101011100110
00101111110100100001001101001011110011
0010011111101110000010101100010000111
01010011010000111100100110011101111111
010100000010000100010100101010001100000
10111100010010011010110111100011010011
01110011110101111001000100111010101110
10000010100100010001101010101110000000
101100000010011100010111011010010101100
```

Example 3: DNA
- Find LRS
- Look somewhere else for causal mechanisms
- Ex. Chromosome 11 has 7.1 million nucleotides

```
tgactaatccagtatccagggcaaattaggttacccac
gtgattacgagaggttccgccgctaatcgggtgcgtcc
gaaacgtatgccctcttctgctcgatgtgattggccgg
cctgtgtcatgccggcacttaaacgatcaaatagtgaa
aatcaaaatcgccggtctgtgagcctagcggatgcaag
atgggcgtacatgcccagcccaccttcggaccgagctg
cgcgtagggccgtagtgctaaagtctgagaatacccca
gtcgttcgttgaggcgcacgtctatgcataatttatgg
aggtcagtgctcttcagaggttgcagtttactctattc
```


Warmup: Longest common prefix

Longest common prefix
- Given: Two strings string s and t.
- Task: Find the longest substring that appears at the beginning of both strings.

Example.

```
a a c a a g t t t a c a a g c
a a c a a g t t t a c a a g t t t a c a a g c t a g c
```

Implementation (easy)

```
private static String lcp(String s, String t)
{
   int N = Math.min(s.length(), t.length())
   for ( int i = 0; i < N; i++)
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   return s.substring(0, N);
}
```


LRS: Brute-force implementation

```
public class LRS
{
   public static String lcp(String s)
   { // See previous slide. }

   public static String lrs(String s)
   {
      int N = s.length();
      String lrs = "";
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
         {
            String x = lcp(s.substring(i, N), s.substring(j, N));
            if (x.length() > lrs.length()) lrs = x;
         }
      return lrs;
   }
   public static void main(String[] args)
   {
      String s= StdIn.readAll();
      StdOut.println(lrs(s));
   }
}
```

```
% more tiny.txt
aacaagtttacaagc

% java LRS
acaag
```

Analysis
- $\sim N^2/2$ calls on `lcp()`.
- Obviously does not scale.

LRS: An efficient solution that uses sorting

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
a a c a a g t t t a c a a g c
```

1. Form suffix strings

```
 0  a a c a a g t t t a c a a g c
 1  a c a a g t t t a c a a g c
 2  c a a g t t t a c a a g c
 3  a a g t t t a c a a g c
 4  a g t t t a c a a g c
 5  g t t t a c a a g c
 6  t t t a c a a g c
 7  t t a c a a g c
 8  t a c a a g c
 9  a c a a g c
10  c a a g c
11  a a g c
12  a g c
13  g c
14  c
```

2. Sort suffix strings

```
 0  a a c a a g t t t a c a a g c
11  a a g c
 3  a a g t t t a c a a g c
 9  a c a a g c
 1  a c a a g t t t a c a a g c
12  a g c
 4  a g t t t a c a a g c
14  c
10  c a a g c
 2  c a a g t t t a c a a g c
13  g c
 5  g t t t a c a a g c
 8  t a c a a g c
 7  t t a c a a g c
 6  t t t a c a a g c
```

3. Find longest LCP among adjacent entries.

LRS implementation

```
public class LRS
{
   public static String lcp(String s)
   { // See previous slide. }

   int N = s.length();
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);

   Merge.sort(suffixes);

   String lrs = "";
   for (int i = 0; i < N-1; i++)
   {
      String x = lcp(suffixes[i], suffixes[i+1]);
      if (x.length() > lrs.length()) lrs = x;
   }
   return lrs;
}
```

Form suffix strings

Sort suffix strings

Find longest LCP among adjacent entries.

```
% more example.txt
aacaagtttacaagc

% java LRS < tiny.txt
acaag
```

```
% more moby.txt
moby dick
herman melville
call me ishmael some years ago never
mind how long precisely having
little or no money
...
% java LRS < moby.txt
 such a funny sporty gamy jesty joky
hoky poky lad is the ocean oh th
```

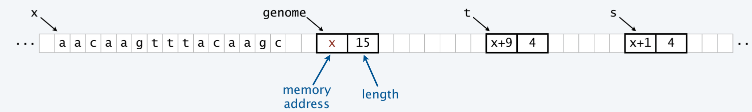
Important note

- Efficiency depends on constant-time substring operation.
- Forming suffix string array takes quadratic time and space if substring operation copies the substring to make a new string.
- [see next slide]

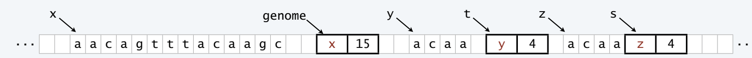
Two alternatives for implementing substrings

1. Refer to original string.
   - No need to copy characters.
   - *Constant* time.

```
String genome = "aacaagtttacaagc";
String s = genome.substring(1, 5);
String t = genome.substring(9, 13);
```



2. Copy the characters to make a new string.
   - Allows potential to free up memory when the original string is no longer needed.
   - *Linear* time (in the length of the substring).



LRS: Empirical analysis

Model

- Alphabet: actg.
- *N*-character random strings.

```
% java Generator 1000000 1 actg | java LRS
2 seconds
% java Generator 10000000 1 actg | java LRS
21 seconds
```

Doubling

N	T_N	$T_N/T_{N/2}$
2,000,000	3	
4,000,000	7	2.3
8,000,000	16	2.3
16,000,000	39	2.4

x10

N	T_N	$T_N/T_{N/10}$
1,000,000	2	
10,000,000	21	10

Confirms hypothesis that the order of growth is $N \log N$ (for the sort).

Bottom line. Scales with the size of the input and enables new research and development.

Important notes on LRS implementation

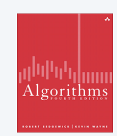
Long repeats

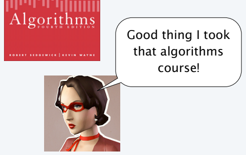
- More precise analysis reveals that running time is *quadratic* in the length of the longest repeat.
- Model has no long repeats.
- Real data may have long repeats.
- Linear time algorithm (guarantee) is known.

Example: Chromosome 11 has a repeat of length 12,567.

String representation

- Efficiency depends on constant-time substring operation.
- 1995–2012: Java substring is constant-time.
- 2013: Java 7 changes to linear-time substring operation! (breaks this and many classic algorithms).
- Need to implement our own constant-time-substring.





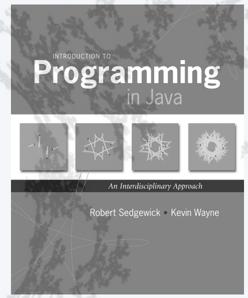
Summary

Binary search. Efficient algorithm to search a sorted array.
Merge sort. Efficient algorithm to sort an array.
Applications. Many, many, many things are enabled by fast sort and search.

Hey, Bob. Our IPO is next week!

I think I'll take a few CS courses.

COMPUTER SCIENCE
SEDGEWICK / WAYNE

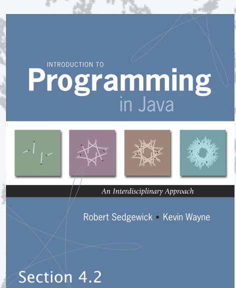


http://introcs.cs.princeton.edu

13. Sorting and Searching

- A typical client
- Binary search
- Insertion sort
- Mergesort
- **Longest repeated substring**

COMPUTER SCIENCE
SEDGEWICK / WAYNE



http://introcs.cs.princeton.edu

13. Sorting and Searching