COS 487 Final Problem Set Due Monday January 21, 2013
Fall 2012 Turn in by 1:30 PM to Room 324
Tarjan

No Collaboration

1. A DFA is *synchronizable* if it has a state *h* and an input string *s* such that if the machine starts in any state and reads *s*, it will end up in state *h*. We call *h* a *home state* and *s* a *synchronizing string* for *h*.
(a) (2 points) Give an example of a DFA that is synchronizable and one that is not.
(b) (8 points) Give a polynomial-time algorithm (in the number of states and symbols) to determine if a DFA is synchronizable, and, if it is, to produce a home state *h* and a synchronizing string *s* for *h*. Hint: show that if there is a synchronizing string, there is always one of length polynomial in the number of states.

2. (10 points) Let $A = \{wtw^R | \ w, t \text{ in } \{0, 1\}^* \text{ and } |w| = |t|\}$. Prove that *A* is not a CFL but it is in L (languages recognizable in deterministic log space).

3. (10 points) Consider deterministic Turing machines whose alphabet is {0, 1, blank}, and that have a single two-way infinite read/write tape. For such a machine M, B(M) = k if M on a blank input tape takes k steps and halts; if M does not halt on a blank tape, B(M) = 0. The busy beaver function B(k) = max{B(M)| M has k states}. Prove that B is uncomputable: there is no Turing machine M that on every input k, outputs B(k). (Assume a binary encoding of k and B(k), although the result is the same for any computable encoding.)

4. (10 points) Consider the following game on sets. Given a collection of sets S_1, $S_2, \dots, S_m$, two players alternate choosing sets. When a set is chosen, all its elements are deleted from all sets. The last player to choose a non-empty set wins. Prove that the problem of determining who wins a given instance of this game is complete in polynomial space. Hint: To show that the problem is PSPACE hard, reduce an instance of QCNF ("yes" instances are true quantified CNF Boolean formulas, with existential and universal quantifiers strictly alternating) to a set game. Construct sets so that "normal" game play corresponds to choosing truth assignments of the variables in the order of their quantification, and the first player wins if and only the assignment makes the formula true. Add sets to guarantee that if one player makes an "abnormal" move, the other player can win immediately. (This is the interesting part of this construction.)

Additional hint: Consider the following related problem. Given is an undirected graph. Players alternate choosing vertices. When a player chooses a vertex, it and all adjacent vertices are deleted. The last player to move wins. (That is, when there are no vertices left, the next player loses.) Prove that this *vertex selection game* is complete in polynomial space. You will get up to 5 points (out of a possible 10) if you prove that the vertex selection game is complete in polynomial space. (If you can prove directly that the set game is complete in polynomial space, you will earn the full 10 points; you do not need to use the hint if you can solve the original problem without it.)

5. Consider a Boolean formula in conjunctive normal form with exactly three literals per clause, all different, and none the negation of another.
(a) (8 points) Describe a deterministic polynomial-time algorithm to find a truth assignment that satisfies at least 7/8 of the clauses. (Thus all such formulas, even unsatisfiable ones, are "almost" satisfiable.) Hint: Consider an arbitrary Boolean formula in conjunctive normal form, each clause having at most three different literals, none the negation of another. Let the formula have m_k clauses with k literals, for k = 1, 2, 3. Suppose that each variable is set true or false with probability ½, independently of all the other variables. Show that the expected number of true clauses is $m_1/2 + 3m_2/4 + 7m_3/8$. Consider an algorithm that chooses the truth value of one variable at a time so as to maximize the value of this formula after each choice. Prove that the algorithm makes 7/8 of the original clauses true.
(b) (2 points) Show that 7/8 is the best possible ratio that one can achieve. Hint: consider unsatisfiable formulas.