COS 226, FALL 2012

ALGORITHMS AND DATA STRUCTURES

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PRINCETON UNIVERSITY

http://www.princeton.edu/~cos226

COS 226 course overview

What is COS 226?

- Intermediate-level survey course.
- Programming and problem solving, with applications.
- Algorithm: method for solving a problem.
- Data structure: method to store information.

topic	data structures and algorithms
data types	stack, queue, bag, union-find, priority queue
sorting	quicksort, mergesort, heapsort, radix sorts
searching	BST, red-black BST, hash table
graphs	BFS, DFS, Prim, Kruskal, Dijkstra
strings	KMP, regular expressions, tries, data compression
advanced	B-tree, suffix array, maxflow, simplex

Why study algorithms?

Their impact is broad and far-reaching.

Internet. Web search, packet routing, distributed file sharing, ...

Biology. Human genome project, protein folding, ...

Computers. Circuit layout, file system, compilers, ...

Computer graphics. Movies, video games, virtual reality, ...

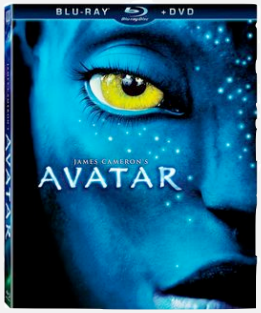
Security. Cell phones, e-commerce, voting machines, ...

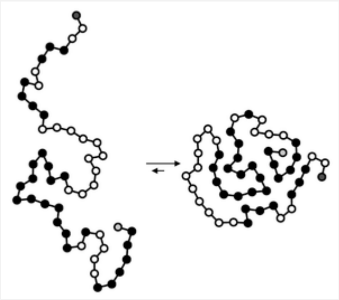
Multimedia. MP3, JPG, DivX, HDTV, face recognition, ...

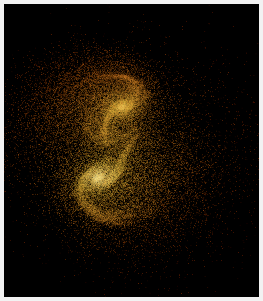
Social networks. Recommendations, news feeds, advertisements, ...

Physics. N-body simulation, particle collision simulation, ...

⋮



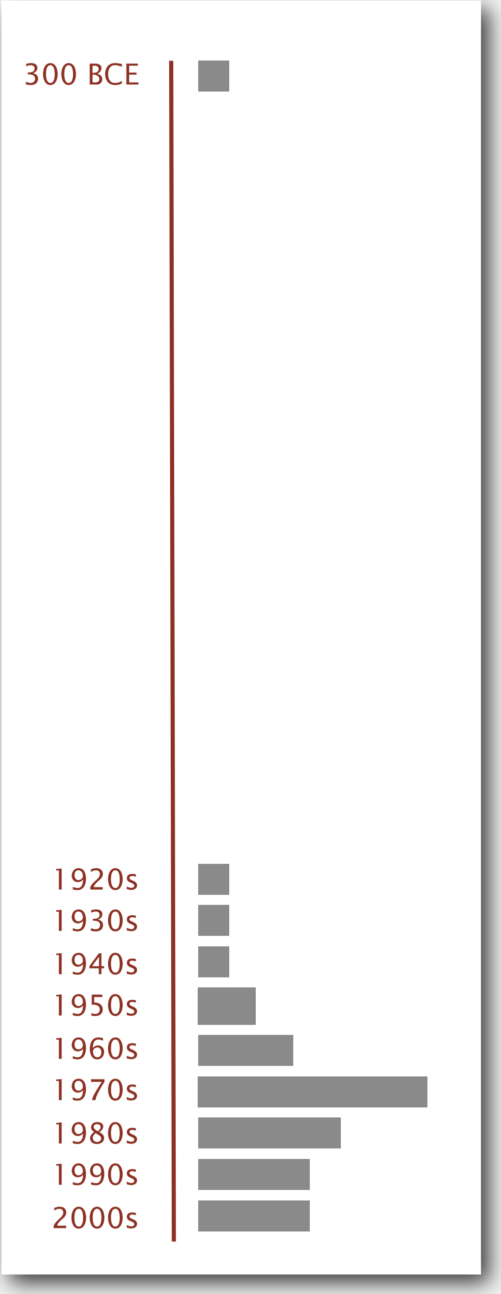




Why study algorithms?

Old roots, new opportunities.

- Study of algorithms dates at least to Euclid.
- Formalized by Church and Turing in 1930s.
- Some important algorithms were discovered by undergraduates in a course like this!



Why study algorithms?

For intellectual stimulation.

> " For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. " — Francis Sullivan

FROM THE EDITORS

THE JOY OF ALGORITHMS

Francis Sullivan, Associate Editor-in-Chief

THE THEME OF THIS FIRST-OF-THE-CENTURY ISSUE OF *COMPUTING IN SCIENCE & ENGINEERING* IS ALGORITHMS. IN FACT, WE WERE BOLD ENOUGH—AND PERHAPS FOOLISH ENOUGH—TO CALL THE 10 EXAMPLES WE'VE SELECTED "THE TOP 10 ALGORITHMS OF THE CENTURY."

Computational algorithms are probably as old as civilization. Sumerian cuneiform, one of the most ancient written records, consists partly of algorithm descriptions for reckoning in base 60. And I suppose we could claim that the Druid algorithm for estimating the start of summer is embodied in Stonehenge. (That's really hard hardware!)

Like so many other things that technology affects, algorithms have advanced in startling and unexpected ways in the 20th century—at least it looks that way to us now. The algorithms we chose for this issue have been essential for progress in communications, health care, manufacturing, economics, weather prediction, defense, and fundamental science. Conversely, progress in these areas has stimulated the search for ever-better algorithms. I recall one late-night bull session on the Maryland Shore when someone asked, "Who first ate a crab? After all, they don't look very appetizing." After the usual speculations about the observed behavior of sea gulls, someone gave what must be the right answer—namely, "A very hungry person first ate a crab."

The flip side to "necessity is the mother of invention" is "invention creates its own necessity." Our need for powerful machines always exceeds their availability. Each significant computation brings insights that suggest the next, usually much larger, computation to be done. New algorithms are an attempt to bridge the gap between the demand for cycles and the available supply of them. We've become accustomed to gaining the Moore's Law factor of two every 18 months. In effect, Moore's Law changes the constant in front of the estimate of running time as a function of problem size. Important new algorithms do not come along every 1.5 years, but when they do, they can change the exponent of the complexity!

For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. A colleague recently claimed that he'd done only 15 minutes of productive work in his whole life. He wasn't joking, because he was referring to the 15 minutes during which he'd sketched out a fundamental optimization algorithm. He regarded the previous years of thought and investigation as a sunk cost that might or might not have paid off.

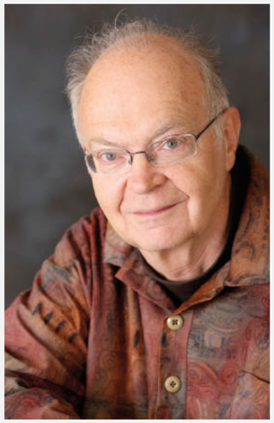
Researchers have cracked many hard problems since 1 January 1900, but we are passing some even harder ones on to the next century. In spite of a lot of good work, the question of how to extract information from extremely large masses of data is still almost untouched. There are still very big challenges coming from more "traditional" tasks, too. For example, we need efficient methods to tell when the result of a large floating-point calculation is likely to be correct. Think of the way that check sums function. The added computational cost is very small, but the added confidence in the answer is large. Is there an analog for things such as huge, multidisciplinary optimizations? At an even deeper level is the issue of reasonable methods for solving specific cases of "impossible" problems. Instances of NP-complete problems crop up in attempting to answer many practical questions. Are there efficient ways to attack them?

I suspect that in the 21st century, things will be ripe for another revolution in our understanding of the foundations of computational theory. Questions already arising from quantum computing and problems associated with the generation of random numbers seem to require that we somehow tie together theories of computing, logic, and the nature of the physical world.

The new century is not going to be very restful for us, but it is not going to be dull either! CiSE

2 COMPUTING IN SCIENCE & ENGINEERING

> " An algorithm must be seen to be believed. " — Donald Knuth

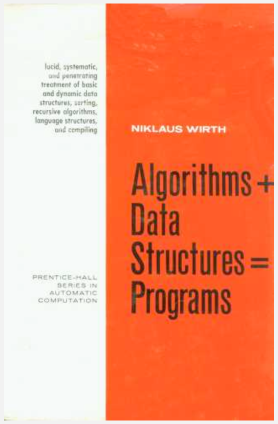


Why study algorithms?

To become a proficient programmer.

> *" I will, in fact, claim that the difference between a bad programmer and a good one is whether he considers his code or his data structures more important. Bad programmers worry about the code. Good programmers worry about data structures and their relationships. "*
>
> *— Linus Torvalds (creator of Linux)*

> *" Algorithms + Data Structures = Programs. "* *— Niklaus Wirth*


lucid, systematic, and penetrating treatment of basic and dynamic data structures, sorting, recursive algorithms, language structures, and compiling
NIKLAUS WIRTH
Algorithms + Data Structures = Programs
PRENTICE-HALL SERIES IN AUTOMATIC COMPUTATION

Why study algorithms?

They may unlock the secrets of life and of the universe.

Scientists are replacing mathematical models with computational models.

$$E = mc^2$$

$$F = ma$$

$$F = \frac{G m_1 m_2}{r^2}$$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi(r) = E \, \Psi(r)$$

20th century science
(formula based)

```
for (double t = 0.0; true; t = t + dt)
   for (int i = 0; i < N; i++)
   {
      bodies[i].resetForce();
      for (int j = 0; j < N; j++)
         if (i != j)
            bodies[i].addForce(bodies[j]);
   }
```

21st century science
(algorithm based)

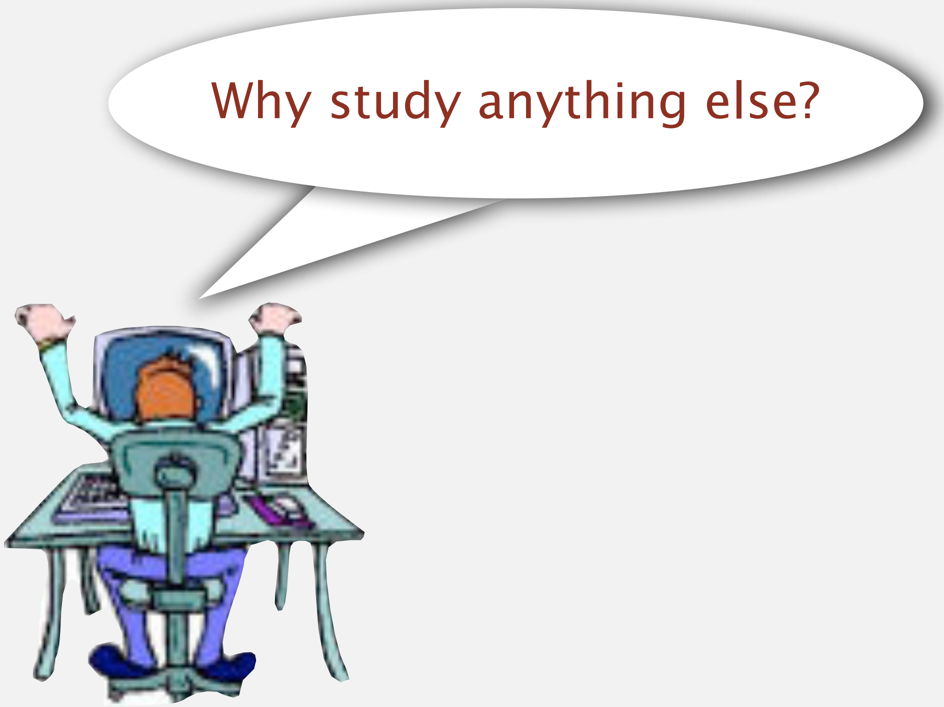
> *“ Algorithms: a common language for nature, human, and computer. ”* — *Avi Wigderson*

Why study algorithms?

For fun and profit.

Why study algorithms?

- Their impact is broad and far-reaching.
- Old roots, new opportunities.
- To solve problems that could not otherwise be addressed.
- For intellectual stimulation.
- To become a proficient programmer.
- They may unlock the secrets of life and of the universe.
- For fun and profit.



The usual suspects

Lectures. Introduce new material.

Precepts. Discussion, problem-solving, background for assignments.

What	When	Where	Who	Office Hours
L01	TTh 11–12:20	Frist 302	Kevin Wayne	see web
P01	F 11–11:50	Friend 109	Maia Ginsburg †	see web
P02	F 12:30–1:20	Friend 109	Diego Perez Botero	see web
P03	F 1:30–2:20	Friend 109	Diego Perez Botero	see web
P03B	F 1:30–2:20	Friend 110	Dushyant Arora	see web
P04	Th 2:30-3:20	Friend 109	Maia Ginsburg †	see web
P04A	Th 2:30-3:20	Friend 112	Dan Larkin	see web

† lead preceptor

Where to get help?

Piazza. Online discussion forum.

- Low latency, low bandwidth.
- Mark solution-revealing questions as private.

piazza

http://www.piazza.com/class#fall2012/cos226

Office hours.

- High bandwidth, high latency.
- See web for schedule.

http://www.princeton.edu/~cos226

Computing laboratory.

- Undergrad lab TAs in Friend 017.
- For help with debugging.
- See web for schedule.

http://www.princeton.edu/~cos226

Coursework and grading

Programming assignments. 45%

- Due on Tuesdays at 11pm via electronic submission.
- Collaboration/lateness policies: see web.

Exercises. 10%

- Due on Mondays at 11pm in Blackboard.
- Collaboration/lateness policies: see web.

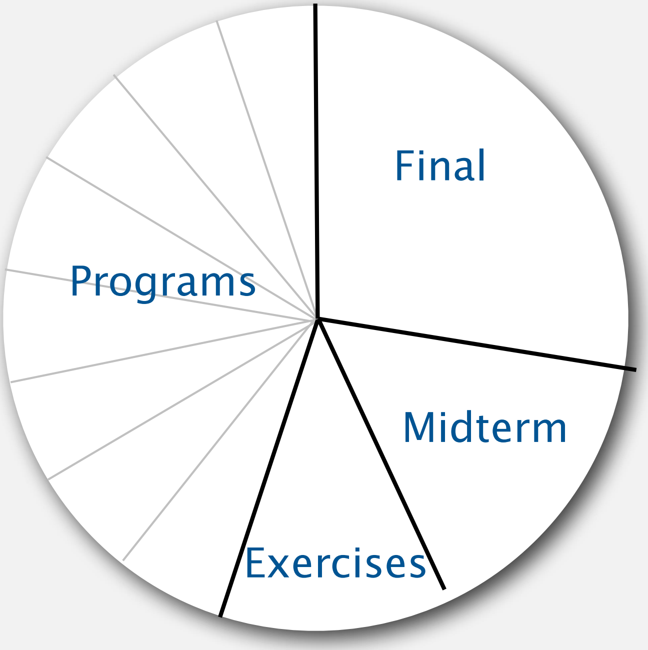
experimental feature (subject to change)

Exams. 15% + 30%

- Midterm (in class on Tuesday, October 23).
- Final (to be scheduled by Registrar).

Staff discretion. To adjust borderline cases.

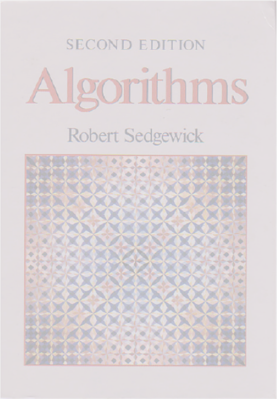
- Report errata.
- Contribute to Piazza discussions.
- Attend and participate in precept/lecture.



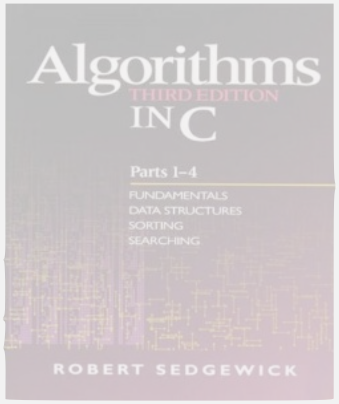
Resources (textbook)

Required reading. Algorithms 4th edition by R. Sedgewick and K. Wayne, Addison-Wesley Professional, 2011, ISBN 0-321-57351-X.

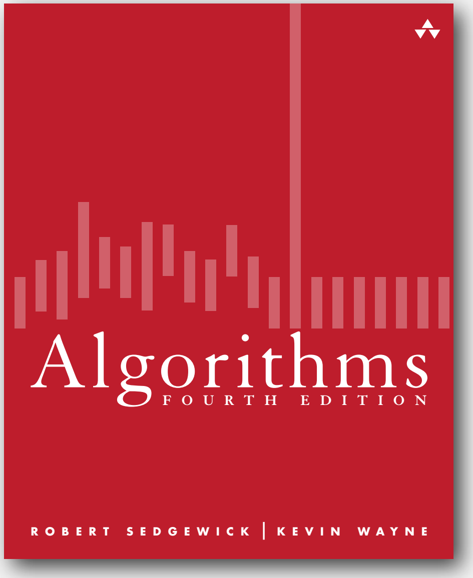
1st edition (1982)



2nd edition (1988)



3rd edition (1997)



Available in hardcover and Kindle.

- Online: Amazon ($60 to buy), Chegg ($40 to rent), ...
- Brick-and-mortar: Labyrinth Books (122 Nassau St). ← 30% discount with PU student ID
- On reserve: Engineering library.

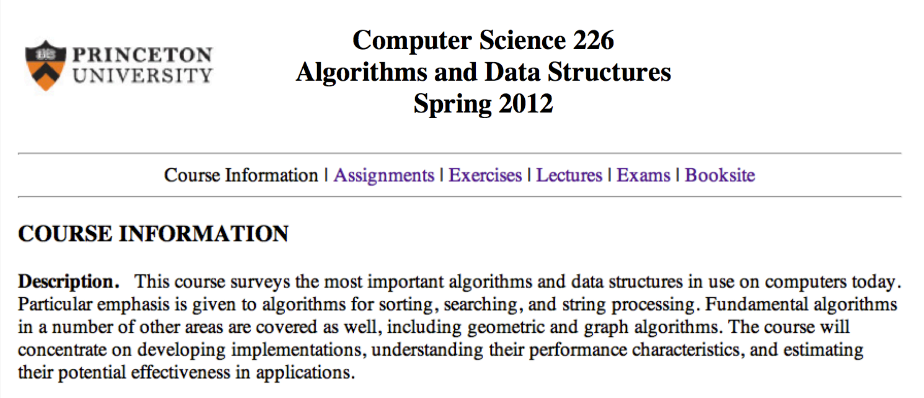
Resources (web)

Course content.

- Course info.
- Programming assignments.
- Exercises.
- Lecture slides.
- Exam archive.
- Submit assignments.

Booksites.

- Brief summary of content.
- Download code from book.



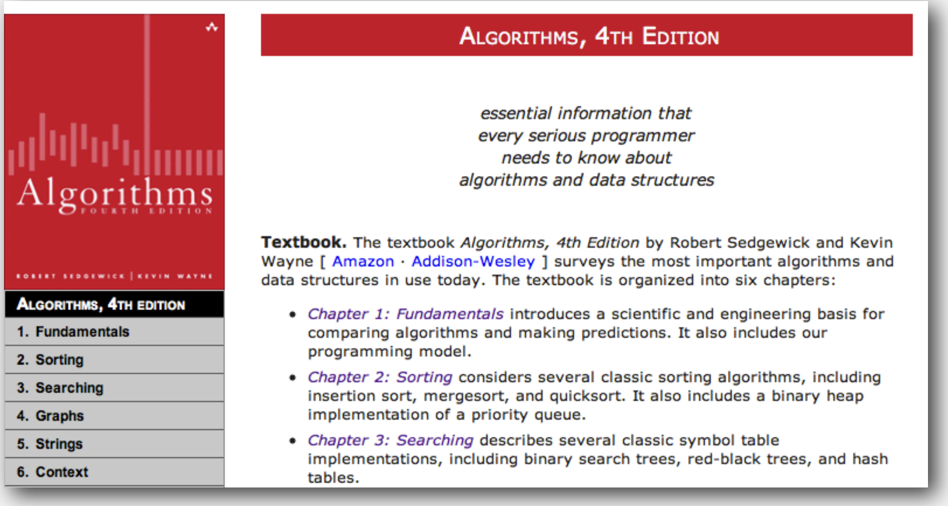
Computer Science 226
Algorithms and Data Structures
Spring 2012

Course Information | Assignments | Exercises | Lectures | Exams | Booksite

COURSE INFORMATION

Description. This course surveys the most important algorithms and data structures in use on computers today. Particular emphasis is given to algorithms for sorting, searching, and string processing. Fundamental algorithms in a number of other areas are covered as well, including geometric and graph algorithms. The course will concentrate on developing implementations, understanding their performance characteristics, and estimating their potential effectiveness in applications.

http://www.princeton.edu/~cos226



ALGORITHMS, 4TH EDITION

essential information that
every serious programmer
needs to know about
algorithms and data structures

Textbook. The textbook *Algorithms, 4th Edition* by Robert Sedgewick and Kevin Wayne [Amazon · Addison-Wesley] surveys the most important algorithms and data structures in use today. The textbook is organized into six chapters:

- *Chapter 1: Fundamentals* introduces a scientific and engineering basis for comparing algorithms and making predictions. It also includes our programming model.
- *Chapter 2: Sorting* considers several classic sorting algorithms, including insertion sort, mergesort, and quicksort. It also includes a binary heap implementation of a priority queue.
- *Chapter 3: Searching* describes several classic symbol table implementations, including binary search trees, red-black trees, and hash tables.

http://www.algs4.princeton.edu

What's ahead?

Lecture 1. [today] Union find.

Lecture 2. [Tuesday] Analysis of algorithms.

Precept 1. [Thursday/Friday] Meets this week.

Exercise 1. Due via Bb submission at 11pm on Monday.

Assignment 1. Due via electronic submission at 11pm on Wednesday.

Right course? See me.

Placed out of COS 126? Review Sections 1.1–1.2 of Algorithms, 4th edition (includes command-line interface and our I/O libraries).

Not registered? Go to any precept this week.

Change precept? Use SCORE. ← see Colleen Kenny-McGinley in CS 210 if the only precept you can attend is closed

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

Algorithms

FOURTH EDITION

ROBERT SEDGEWICK | KEVIN WAYNE

http://algs4.cs.princeton.edu

1.5 UNION-FIND

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

1.5 Union-Find

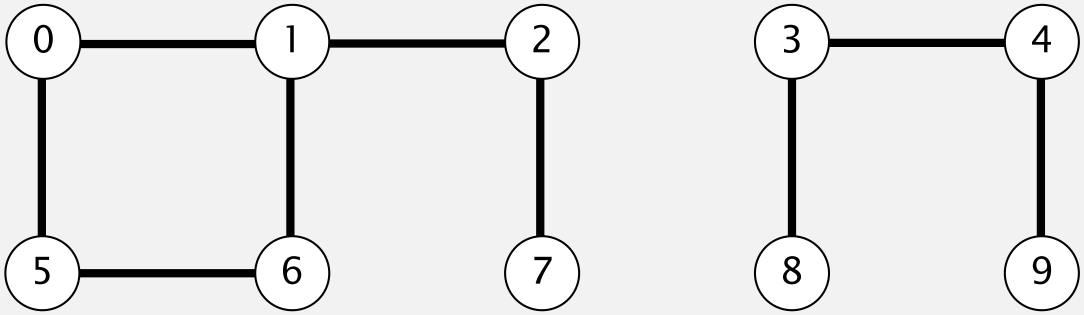
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Dynamic connectivity

Given a set of N objects.

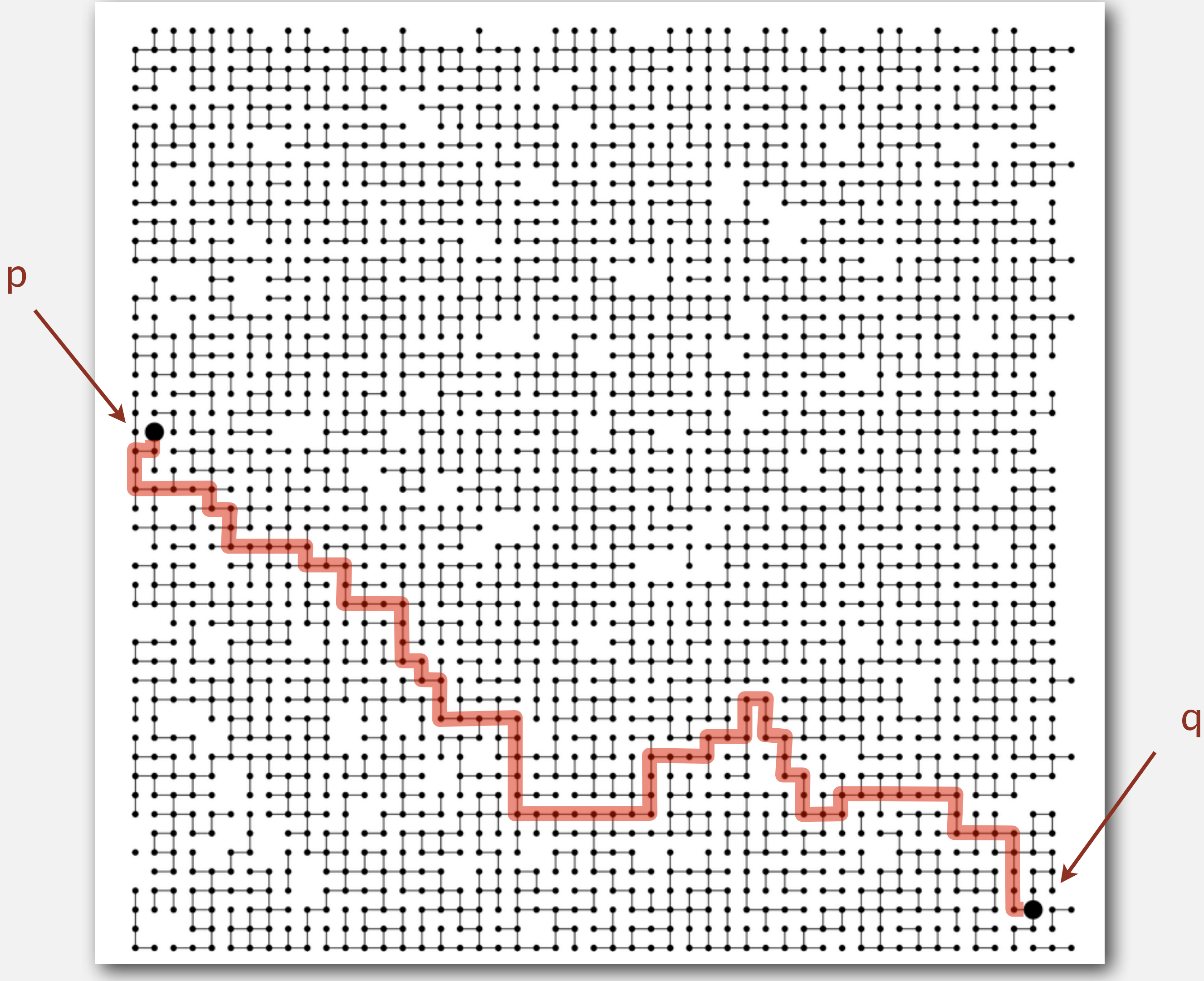
- Union command: connect two objects.
- Find/connected query: is there a path connecting the two objects?

```
union(4, 3)
union(3, 8)
union(6, 5)
union(9, 4)
union(2, 1)
connected(0, 7)    ×
connected(8, 9)    ✔
union(5, 0)
union(7, 2)
union(6, 1)
union(1, 0)
connected(0, 7)    ✔
```



Connectivity example

Q. Is there a path connecting p and q ?



A. Yes.

Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to N –1.

- Use integers as array index.
- Suppress details not relevant to union-find.

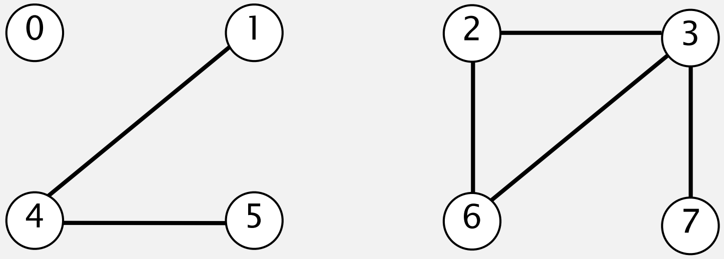
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.

Connected components. Maximal set of objects that are mutually connected.



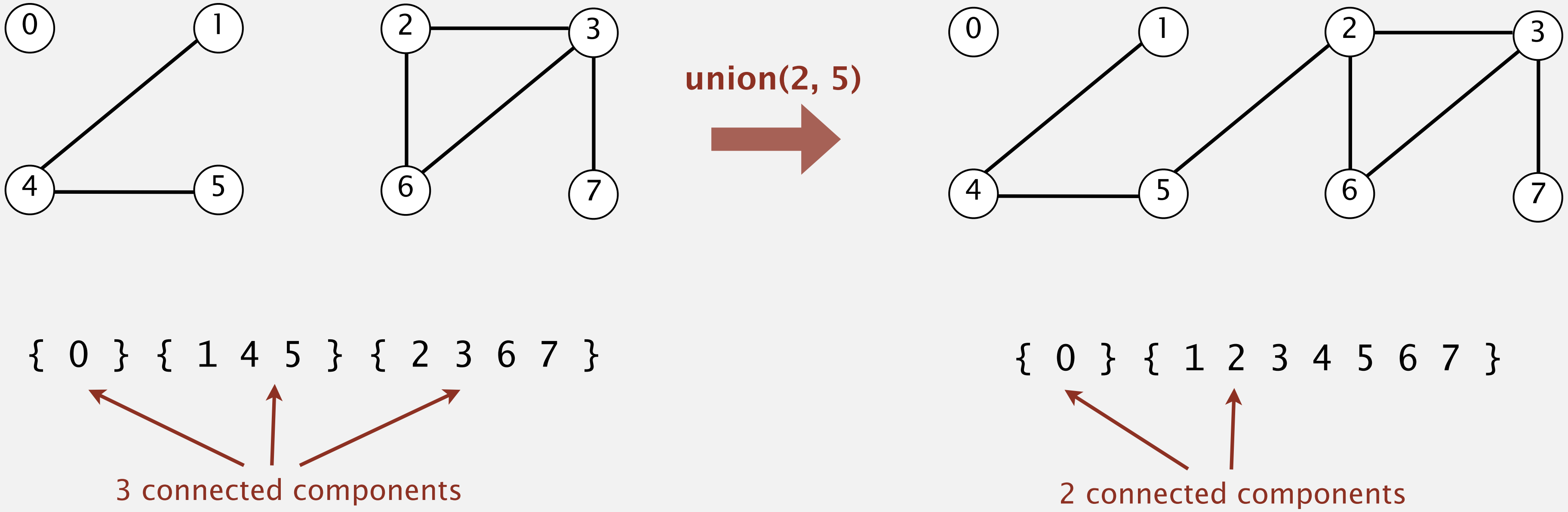
{ 0 } { 1 4 5 } { 2 3 6 7 }

3 connected components

Implementing the operations

Find query. Check if two objects are in the same component.

Union command. Replace components containing two objects with their union.



union(2, 5)

{ 0 } { 1 4 5 } { 2 3 6 7 }

3 connected components

{ 0 } { 1 2 3 4 5 6 7 }

2 connected components

Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Find queries and union commands may be intermixed.

public class UF		
	`UF(int N)`	*initialize union-find data structure with N objects (0 to $N-1$)*
`void`	`union(int p, int q)`	*add connection between p and q*
`boolean`	`connected(int p, int q)`	*are p and q in the same component?*
`int`	`find(int p)`	*component identifier for p (0 to $N-1$)*
`int`	`count()`	*number of components*

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (!uf.connected(p, q))
      {
         uf.union(p, q);
         StdOut.println(p + " " + q);
      }
   }
}
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

http://algs4.cs.princeton.edu

1.5 UNION-FIND

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Quick-find [eager approach]

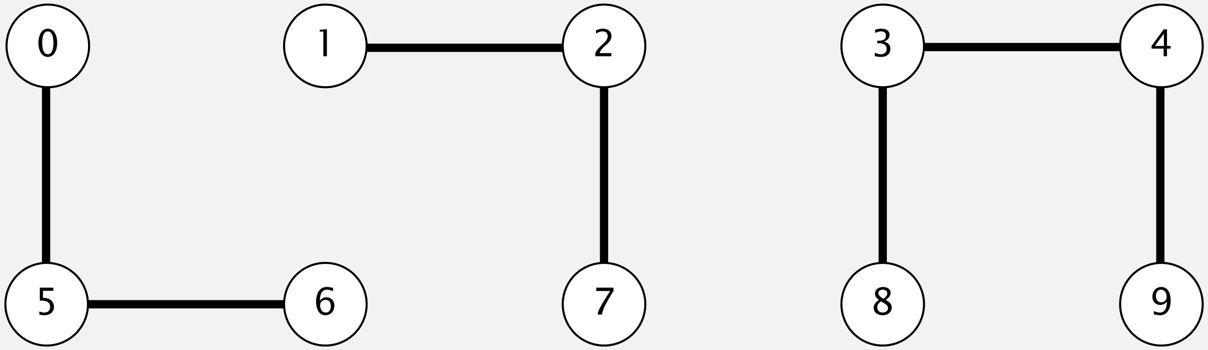
Data structure.

if and only if

- Integer array `id[]` of size `N`.
- Interpretation: `p` and `q` are connected iff they have the same id.

	0	1	2	3	4	5	6	7	8	9
`id[]`	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Quick-find [eager approach]

Data structure.

- Integer array id[] of size N.
- Interpretation: p and q are connected iff they have the same id.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Find. Check if p and q have the same id.

id[6] = 0; id[1] = 1
6 and 1 are not connected

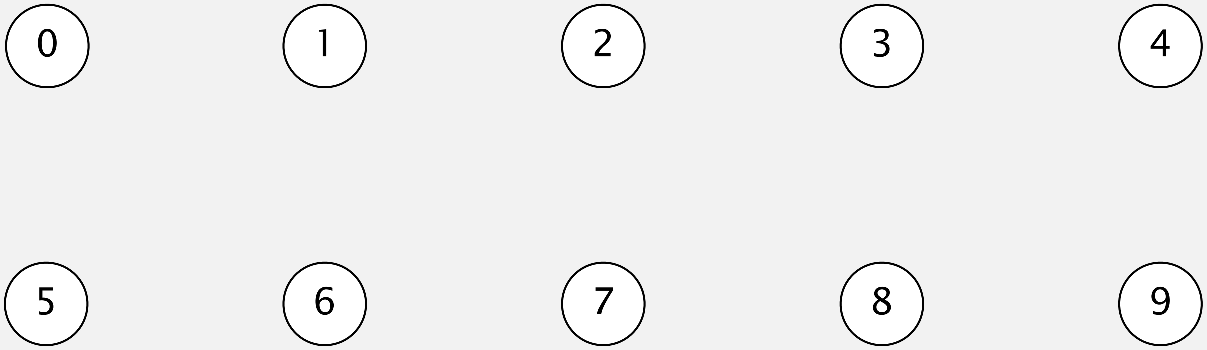
Union. To merge components containing p and q, change all entries whose id equals id[p] to id[q].

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

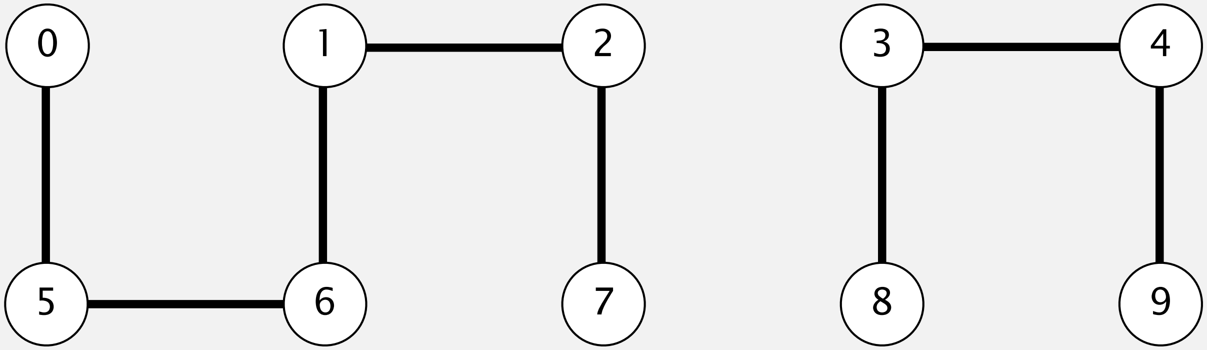
problem: many values can change

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean connected(int p, int q)
   {  return id[p] == id[q];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

check whether p and q
are in the same component
(2 array accesses)

change all entries with id[p] to id[q]
(at most 2N + 2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

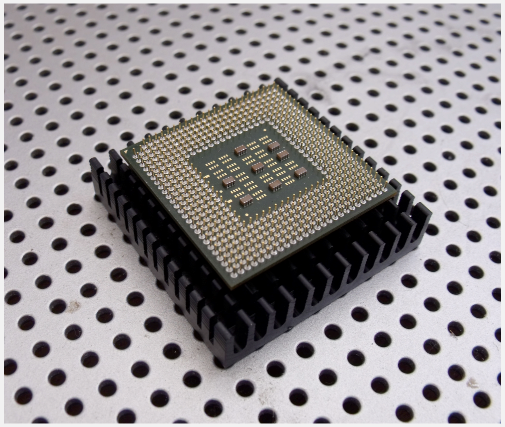
algorithm	initialize	union	find
quick-find	N	N	1

order of growth of number of array accesses

quadratic

Union is too expensive. It takes N^2 array accesses to process a sequence of N union commands on N objects.

Quadratic algorithms do not scale



Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

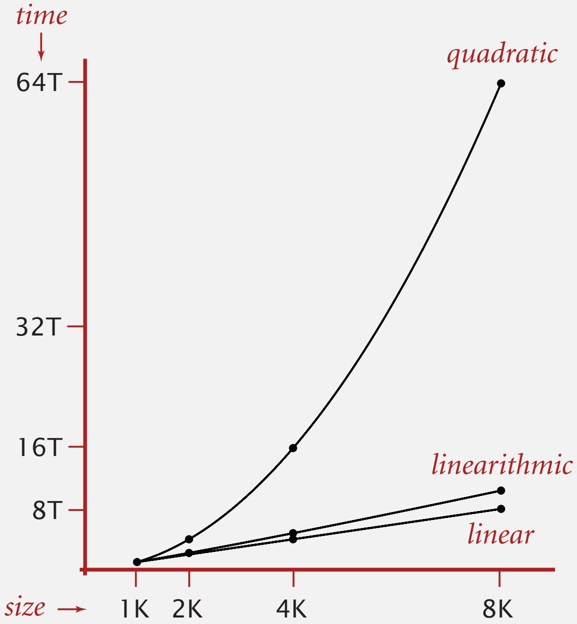
a truism (roughly) since 1950!

Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory ⇒ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!



Algorithms

Robert Sedgewick | Kevin Wayne

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1.5 Union-Find

- *dynamic connectivity*
- *quick find*
- ***quick union***
- *improvements*
- *applications*

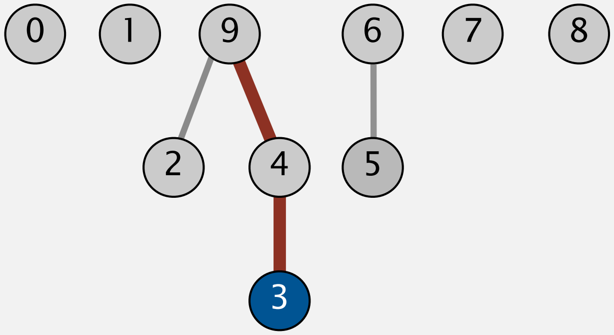
Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



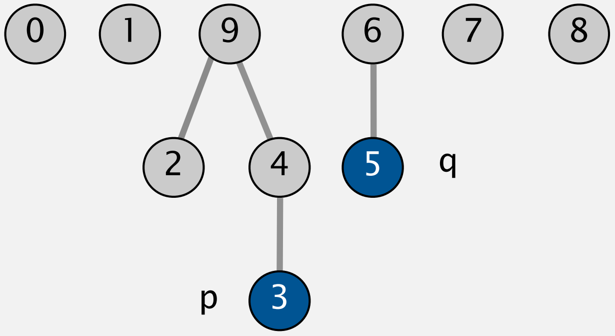
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



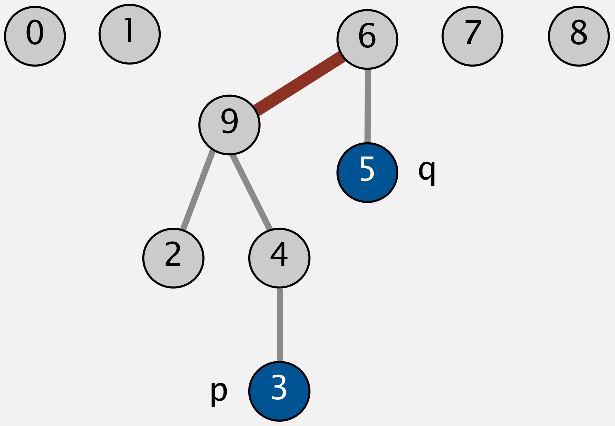
root of 3 is 9
root of 5 is 6
3 and 5 are not connected

Find. Check if `p` and `q` have the same root.

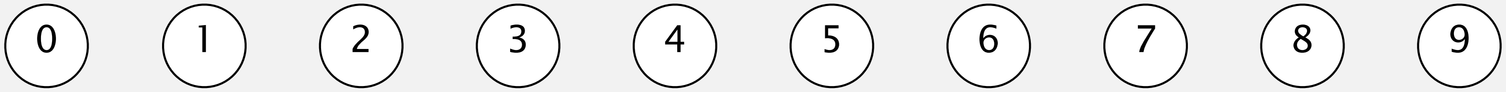
Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes

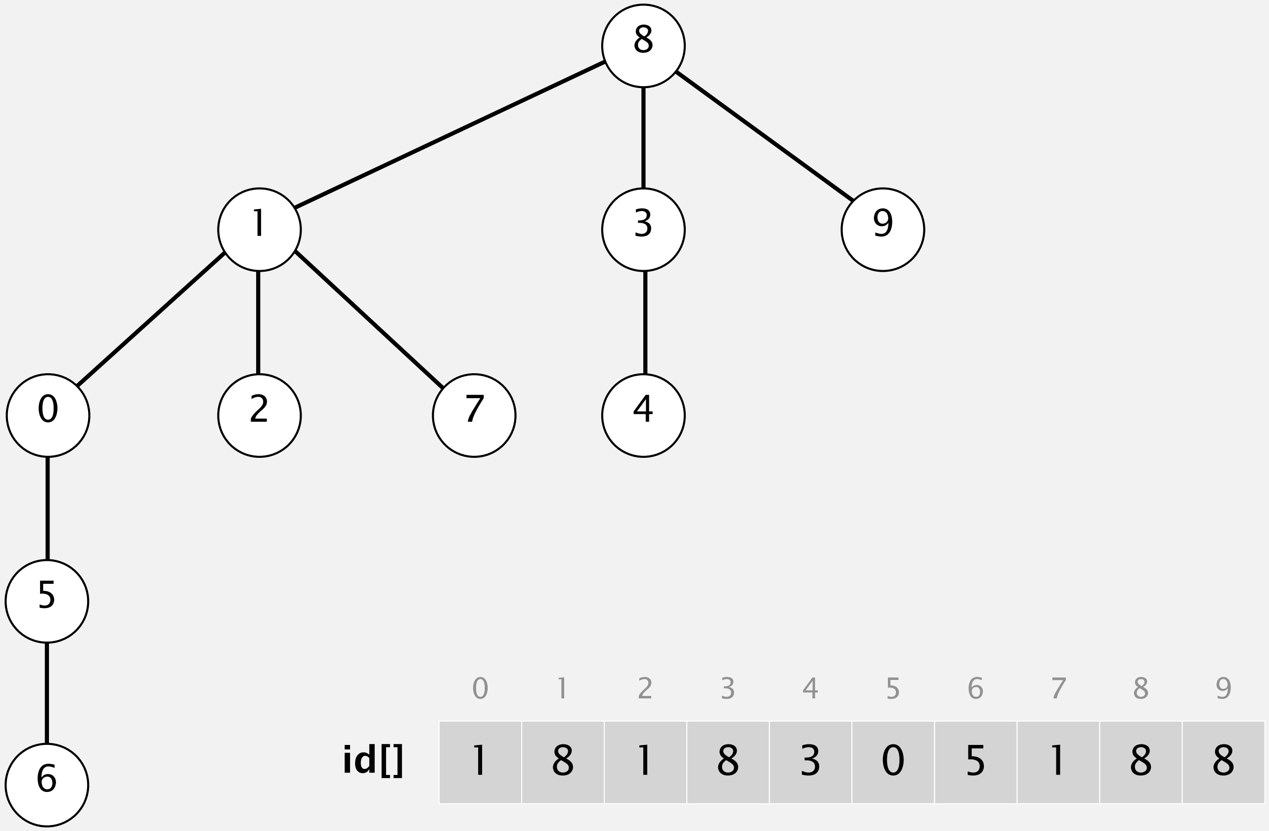


Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   private int root(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }

   public boolean connected(int p, int q)
   {
      return root(p) == root(q);
   }

   public void union(int p, int q)
   {
      int i = root(p);
      int j = root(q);
      id[i] = j;
   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

check if p and q have same root
(depth of p and q array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	N	N	1
quick-union	N	N †	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find too expensive (could be N array accesses).

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1.5 Union-Find

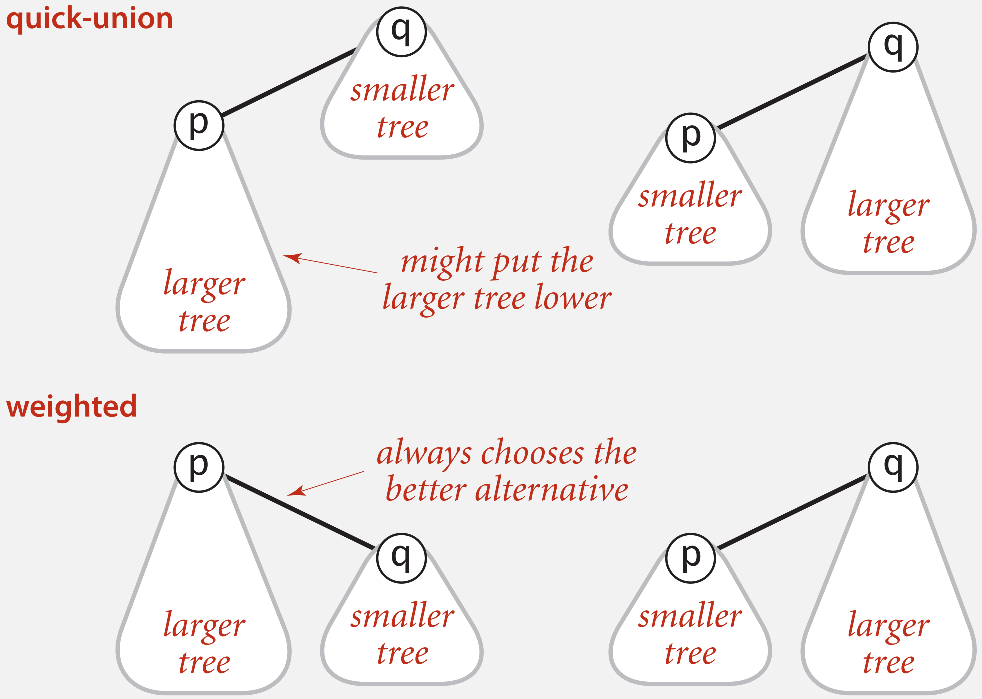
- *dynamic connectivity*
- *quick find*
- *quick union*
- ***improvements***
- *applications*

Improvement 1: weighting

Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives: union by height or "rank"

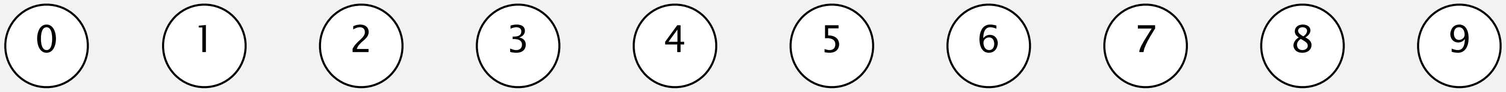

quick-union

weighted

might put the larger tree lower

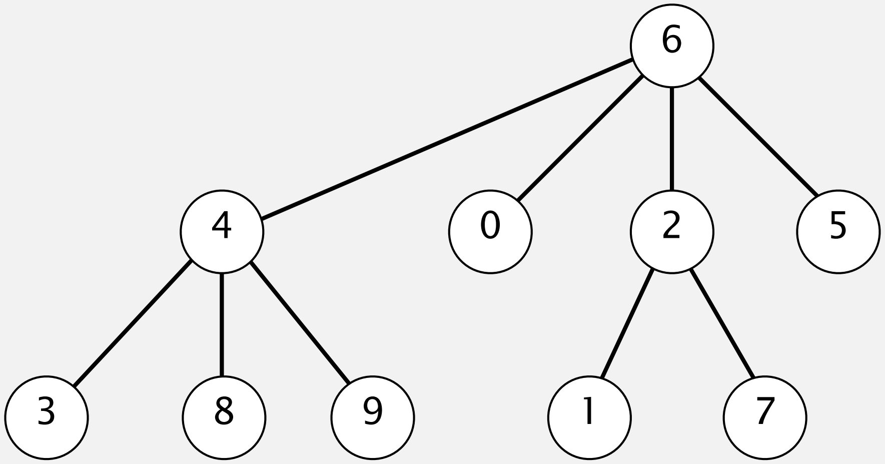
always chooses the better alternative

Weighted quick-union demo



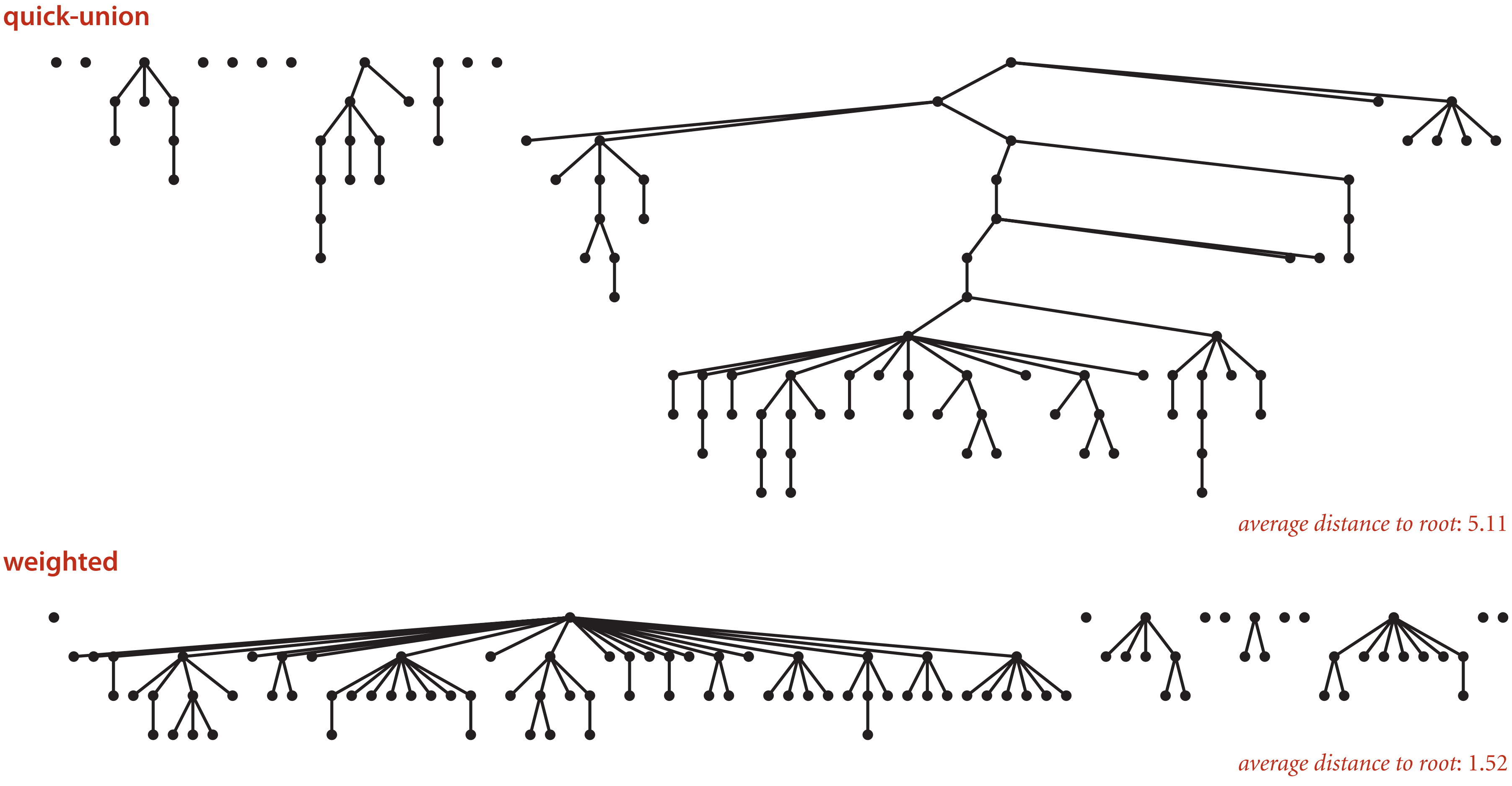
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Quick-union and weighted quick-union example


quick-union
average distance to root: 5.11
weighted
average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find. Identical to quick-union.

```
return root(p) == root(q);
```

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

```
int i = root(p);
int j = root(q);
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

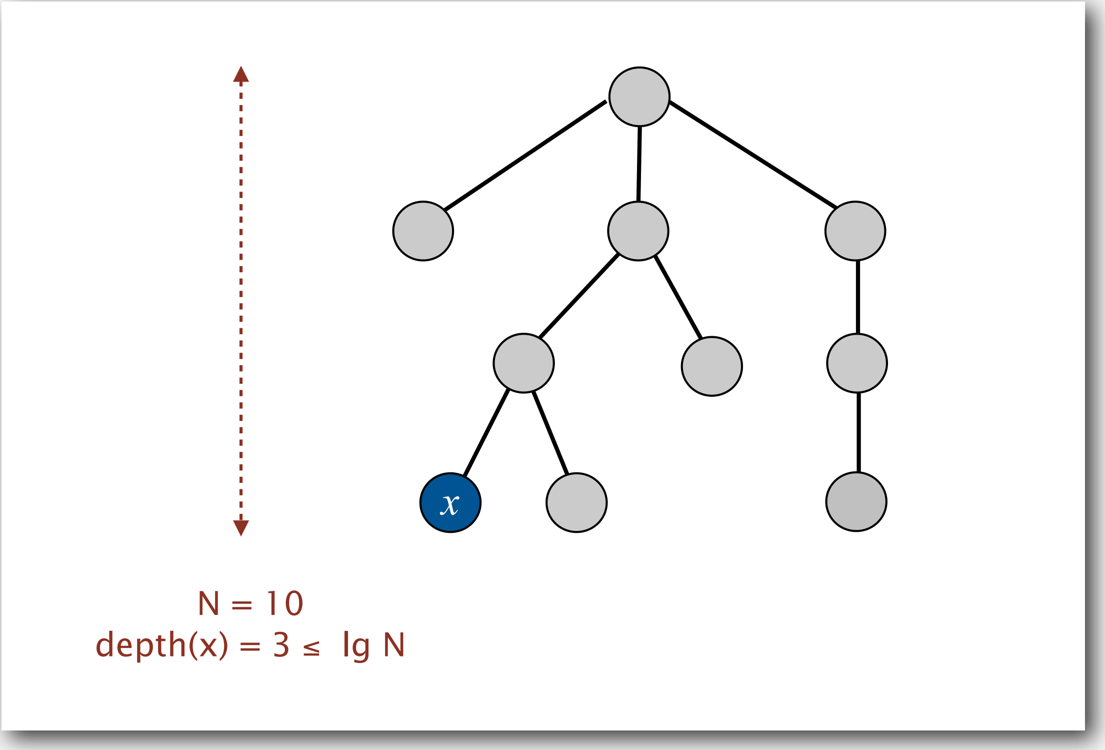
Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p and q.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



Weighted quick-union analysis

Running time.

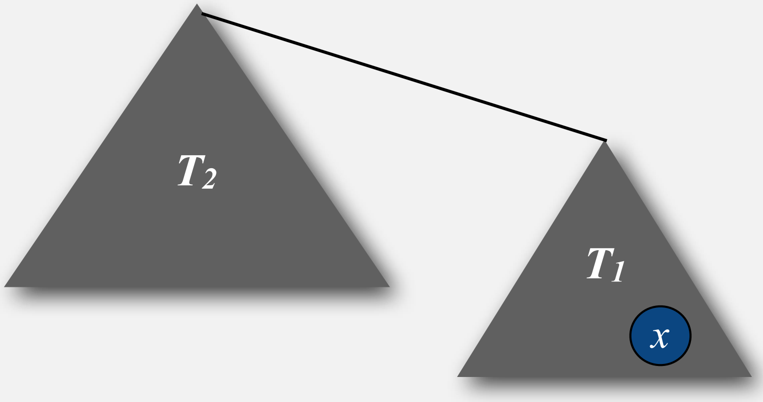
- Find: takes time proportional to depth of p and q.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

Pf. When does depth of x increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg N$ times. Why?



Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p and q.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	connected
quick-find	N	N	1
quick-union	N	N †	N
weighted QU	N	lg N †	lg N

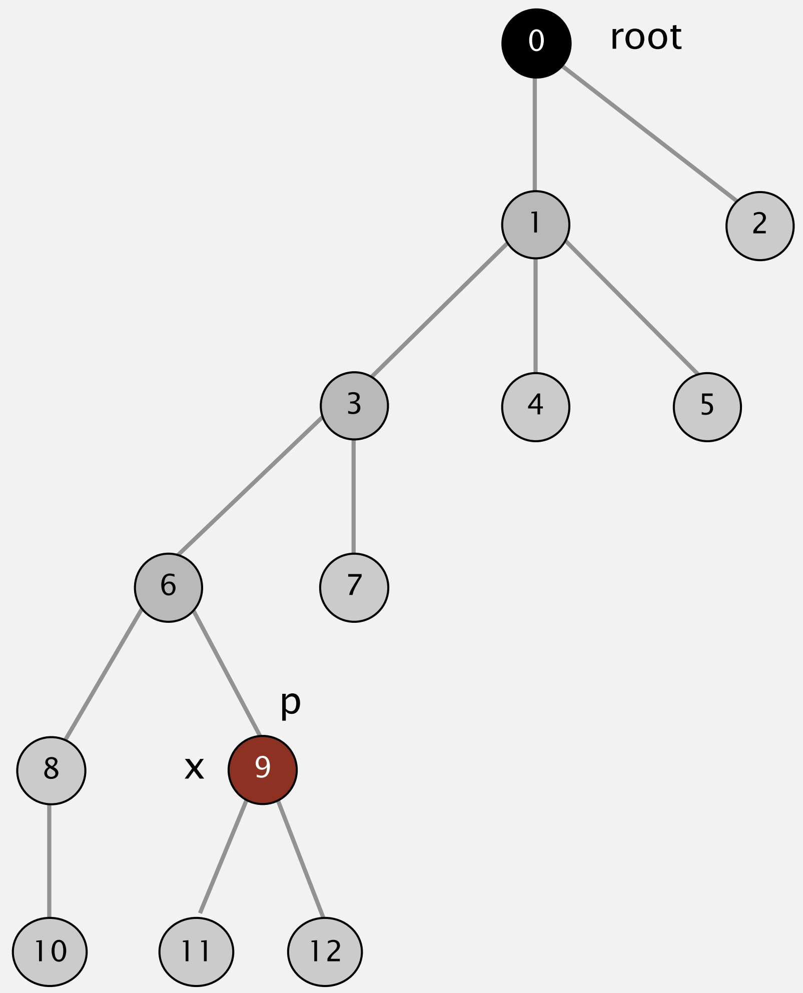
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

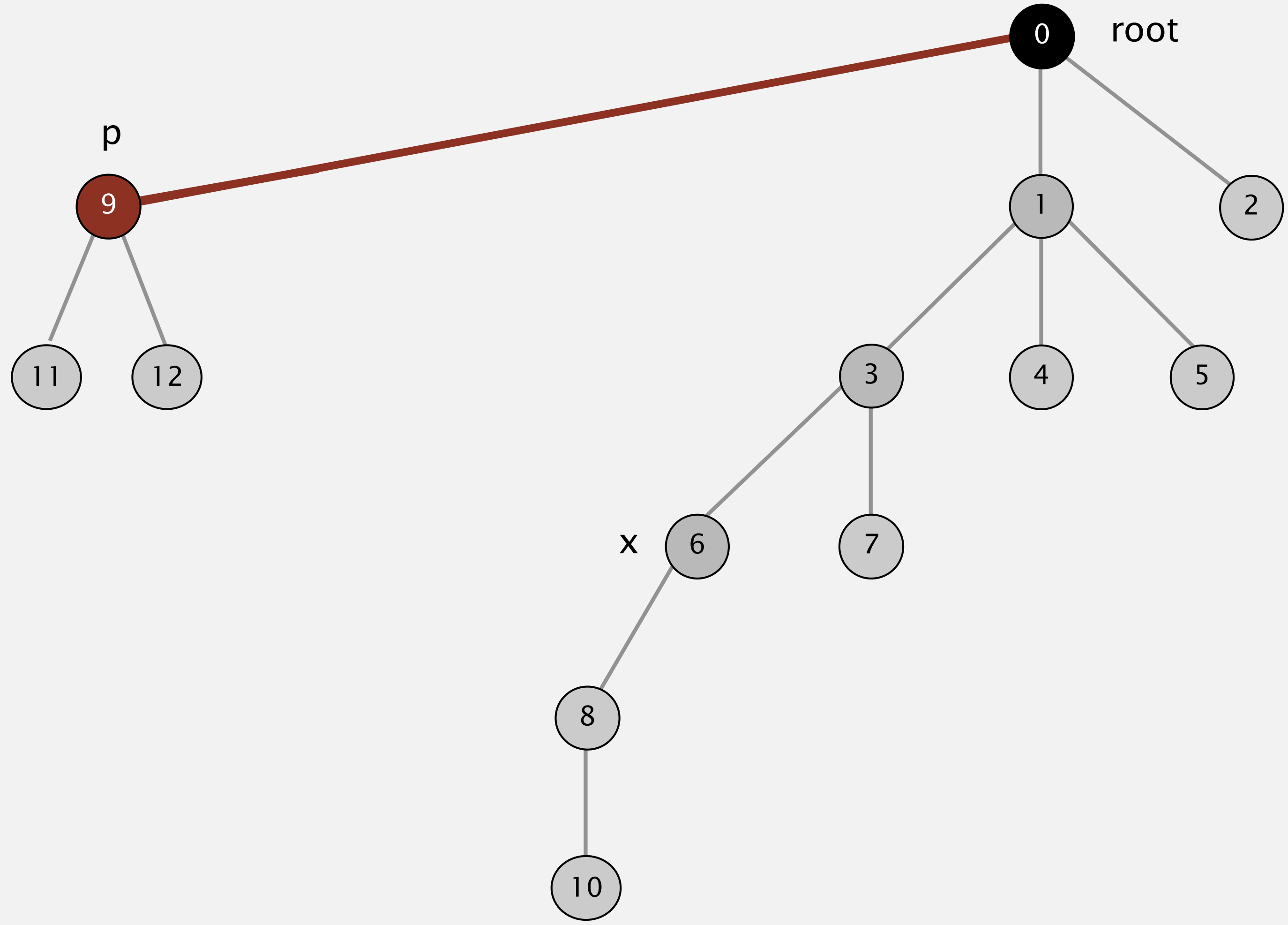
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



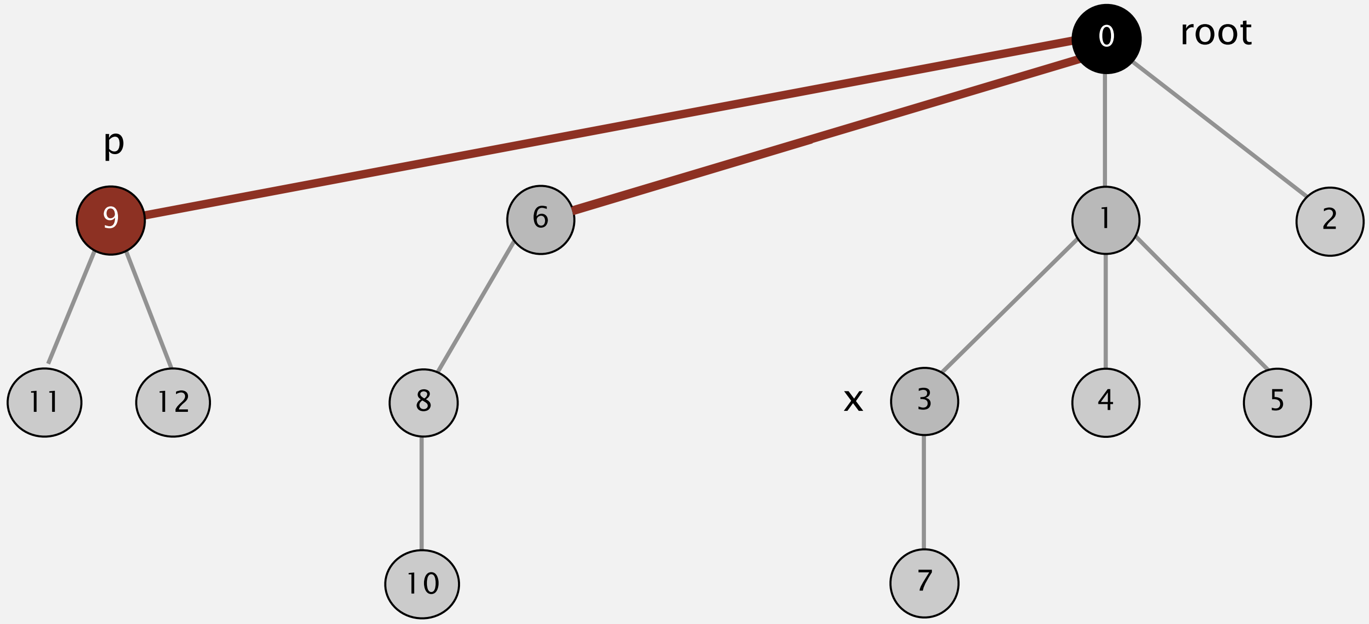
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



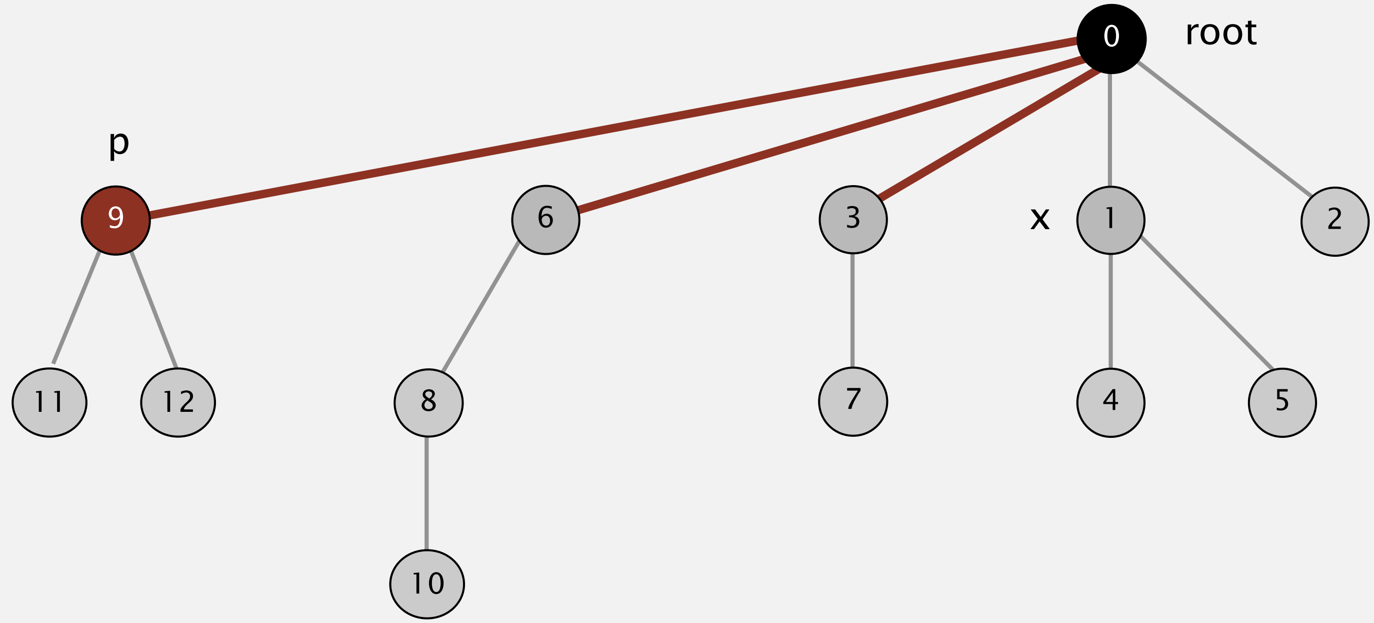
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



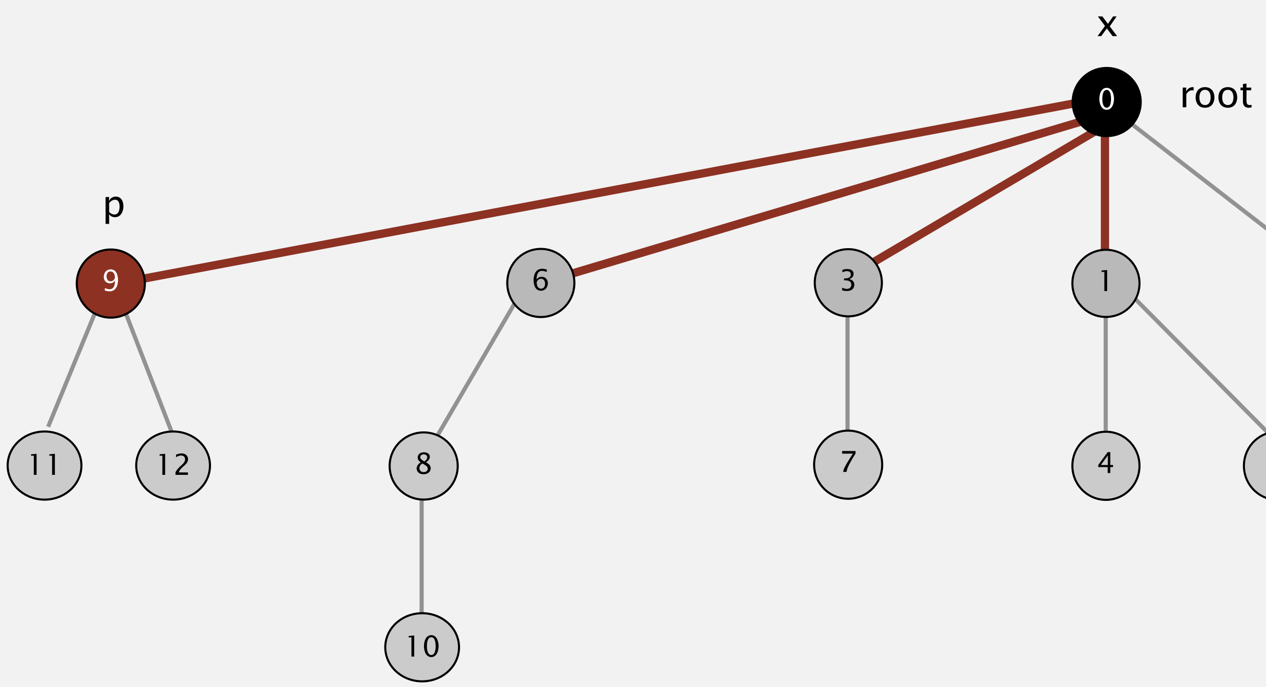
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id of each examined node to point to that root.



Path compression: Java implementation

Two-pass implementation: add second loop to `root()` to set the `id[]` of each examined node to the root.

Simpler one-pass variant: Make every other node in path point to its grandparent (thereby halving path length).

```
private int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression: amortized analysis

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of M union–find ops on N objects makes ≤ $c\,(N + M \lg^* N)$ array accesses.

- Analysis can be improved to $N + M\,\alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterate log function

Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

Bottom line. Weighted quick union (with path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

Algorithms

Robert Sedgewick | Kevin Wayne

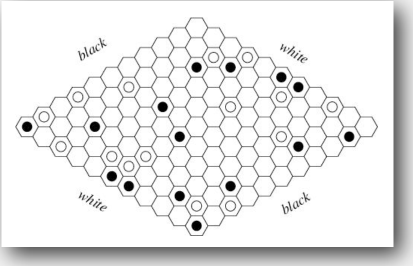
http://algs4.cs.princeton.edu

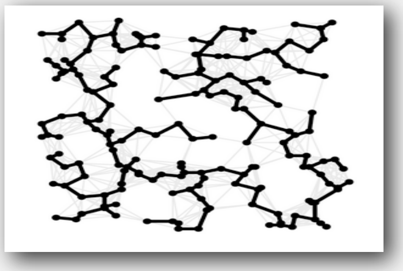
1.5 Union-Find

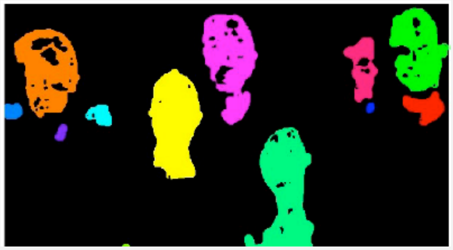
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Union-find applications

- Percolation.
- Games (Go, Hex).
- ✓ Dynamic connectivity.
- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's `bwlabel()` function in image processing.



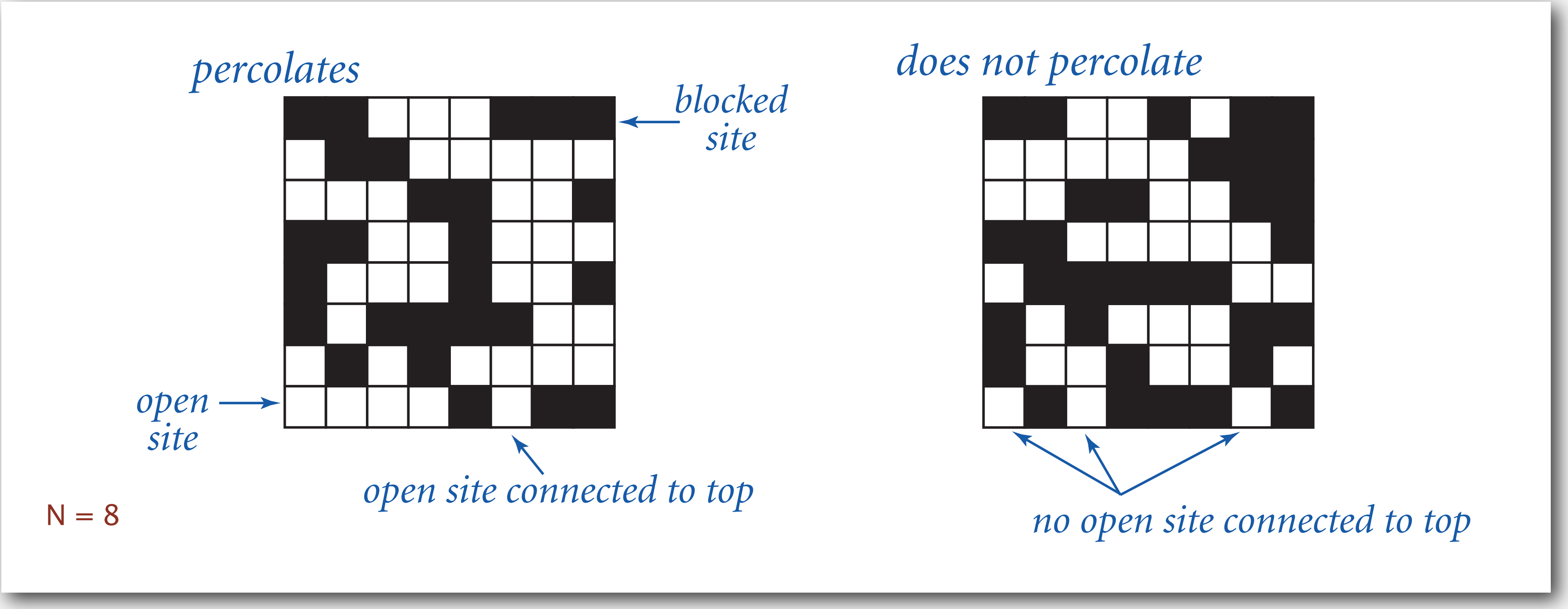




Percolation

A model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (or blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.



Percolation

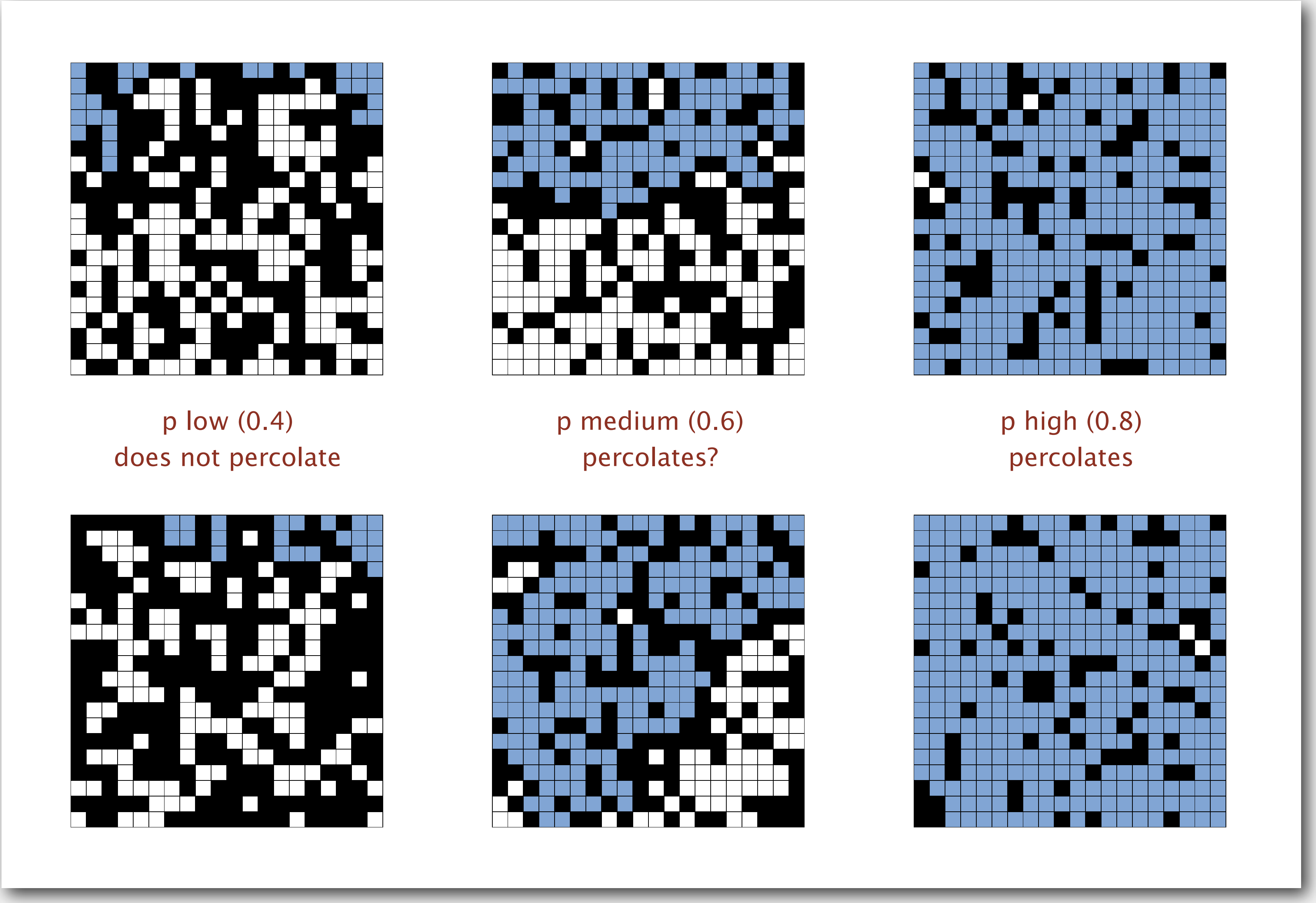
A model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (or blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

Depends on site vacancy probability p.



p low (0.4)
does not percolate

p medium (0.6)
percolates?

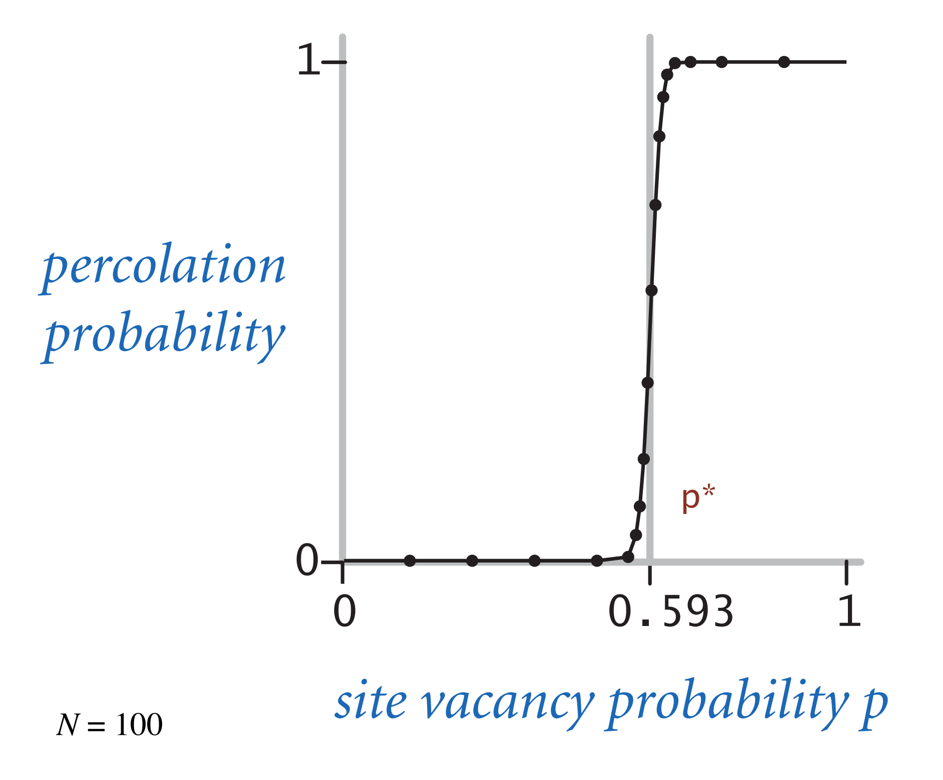
p high (0.8)
percolates

Percolation phase transition

When N is large, theory guarantees a sharp threshold p^*.

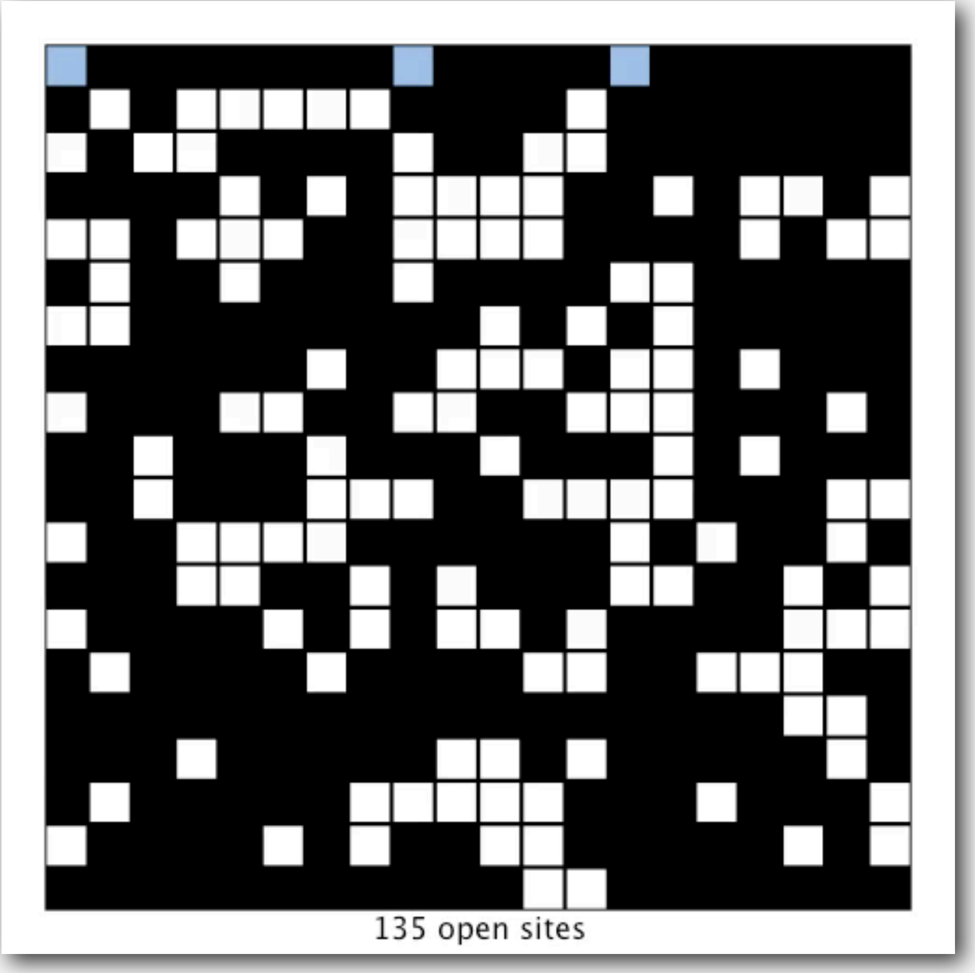
- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

Q. What is the value of p^* ?



Monte Carlo simulation

- Initialize N-by-N whole grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.



🟦 full open site (connected to top)

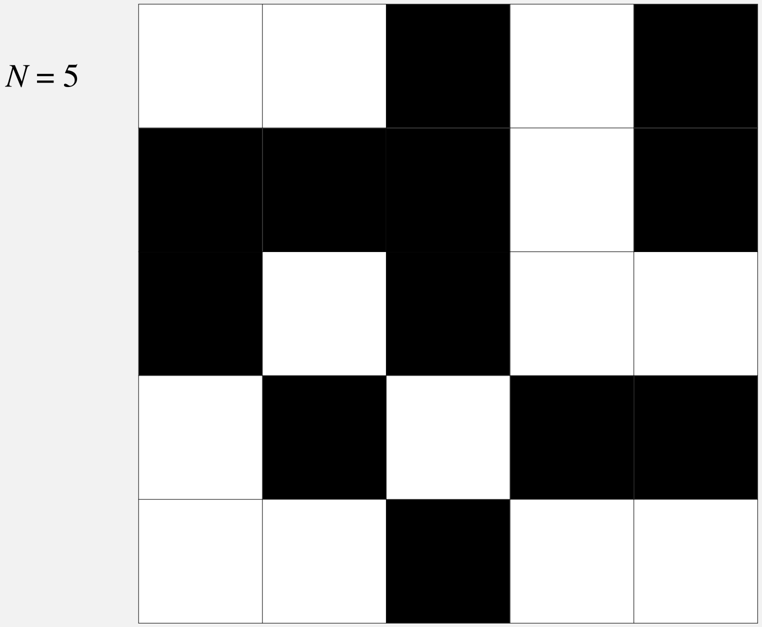
□ empty open site (not connected to top)

■ blocked site

$N = 20$

Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?



□ open site

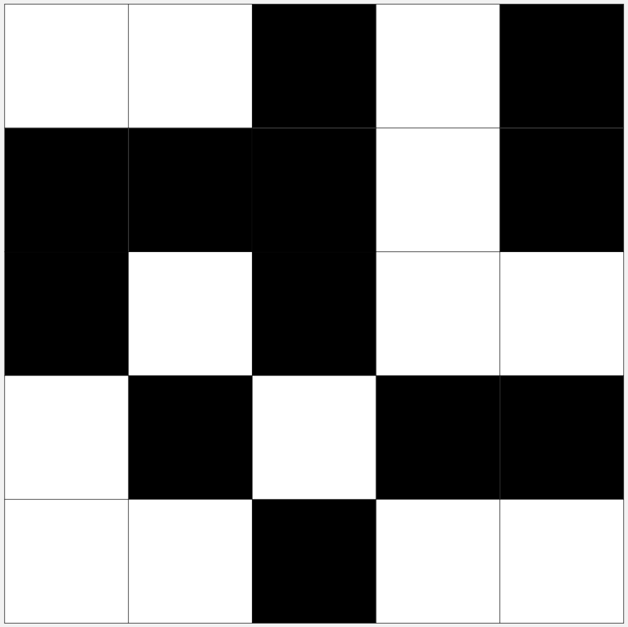
■ blocked site

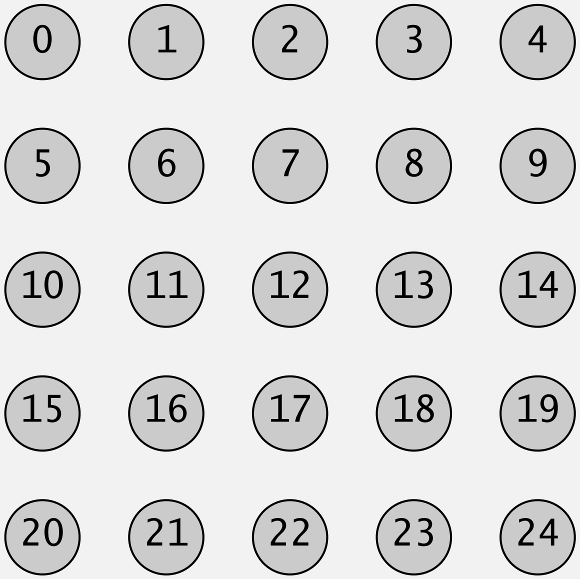
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.

$N = 5$





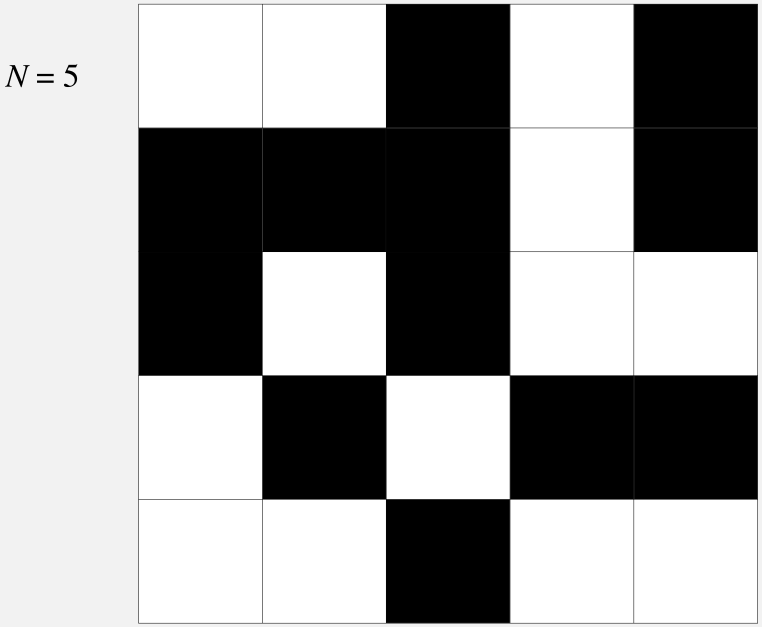
□ open site

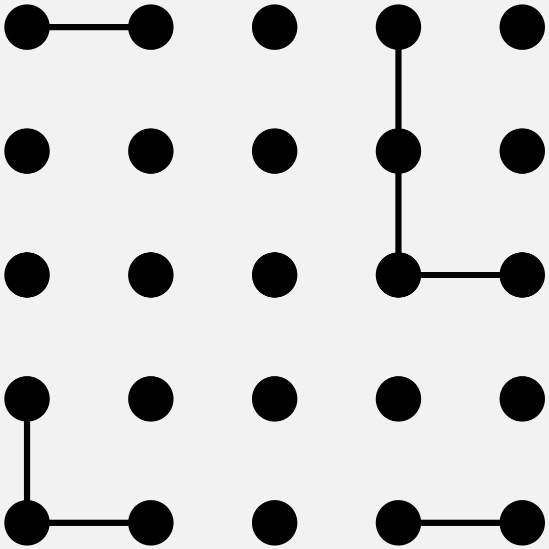
■ blocked site

Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component if connected by open sites.





□ open site

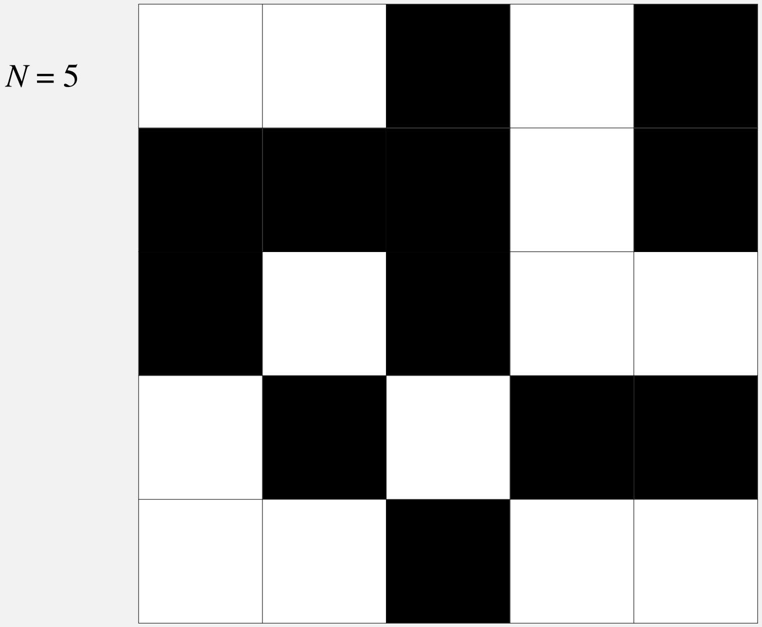
■ blocked site

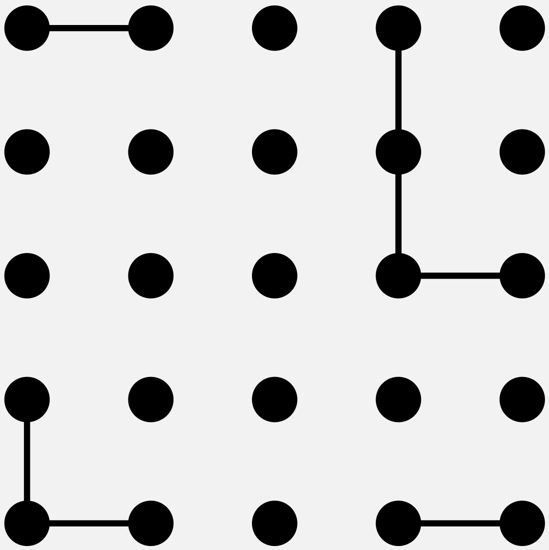
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component if connected by open sites.
- Percolates iff any site on bottom row is connected to site on top row.

brute-force algorithm: N^2 calls to connected()





top row

bottom row

□ open site

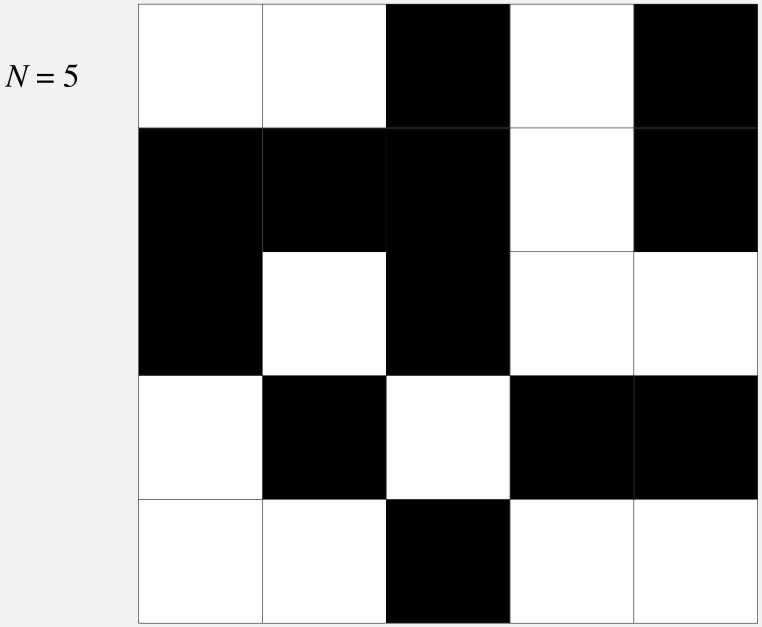
■ blocked site

Dynamic connectivity solution to estimate percolation threshold

Clever trick. Introduce 2 virtual sites (and connections to top and bottom).

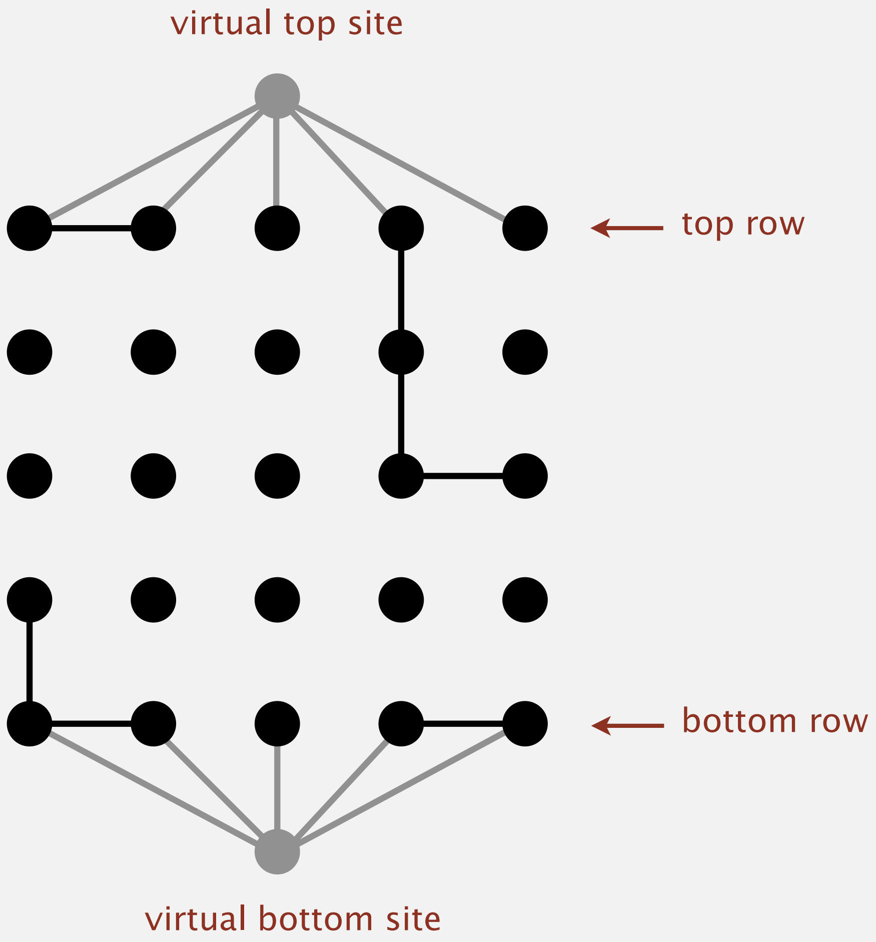
- Percolates iff virtual top site is connected to virtual bottom site.

efficient algorithm: only 1 call to connected()



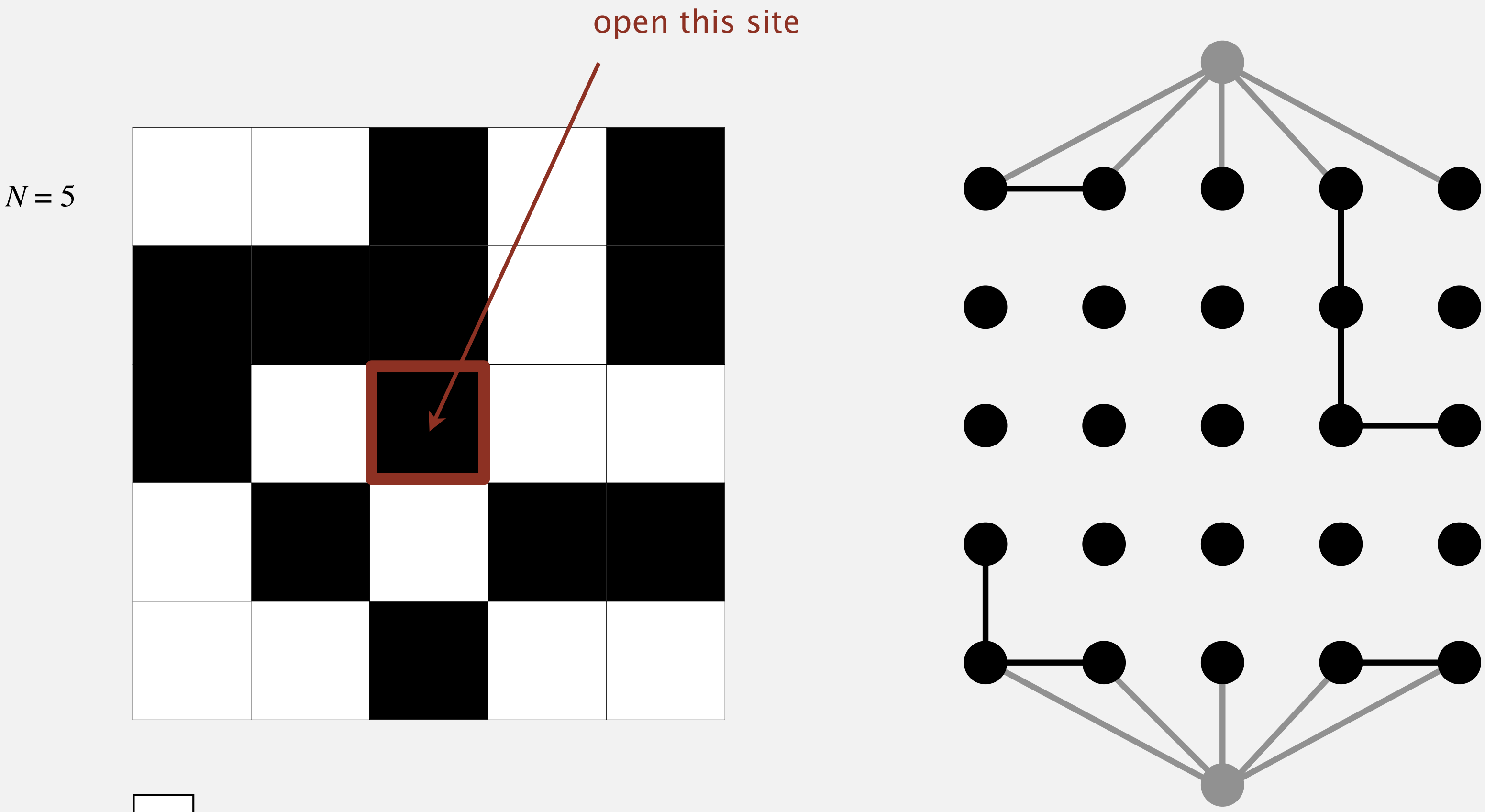
□ open site

■ blocked site



Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?



□ open site

■ blocked site

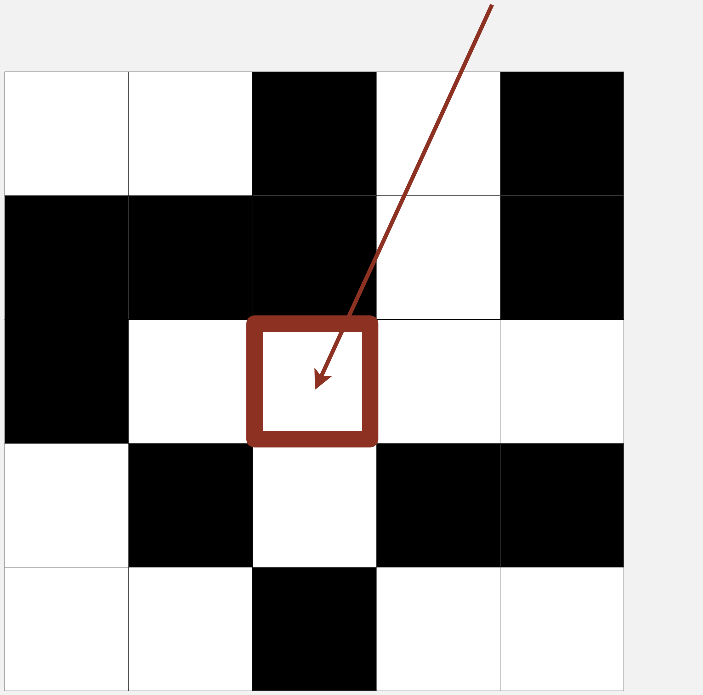
Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

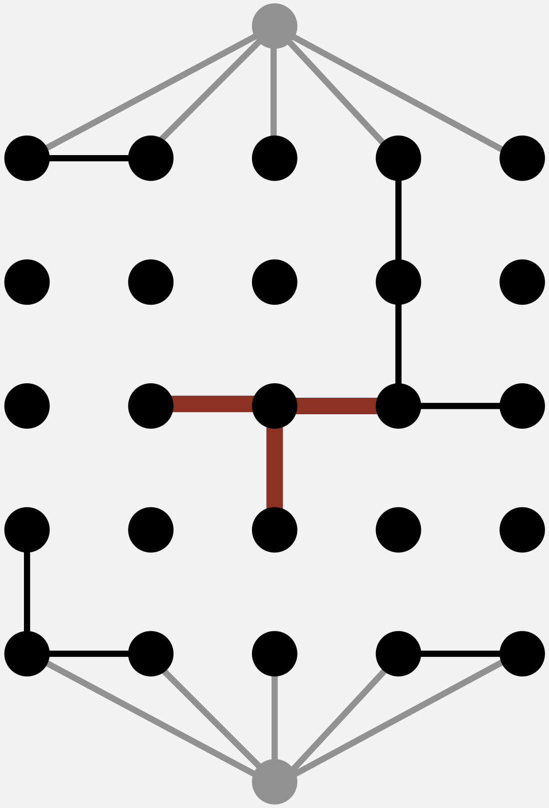
A. Mark new site as open; connect it to all of its adjacent open sites.

up to 4 calls to union()

$N = 5$



open this site



□ open site

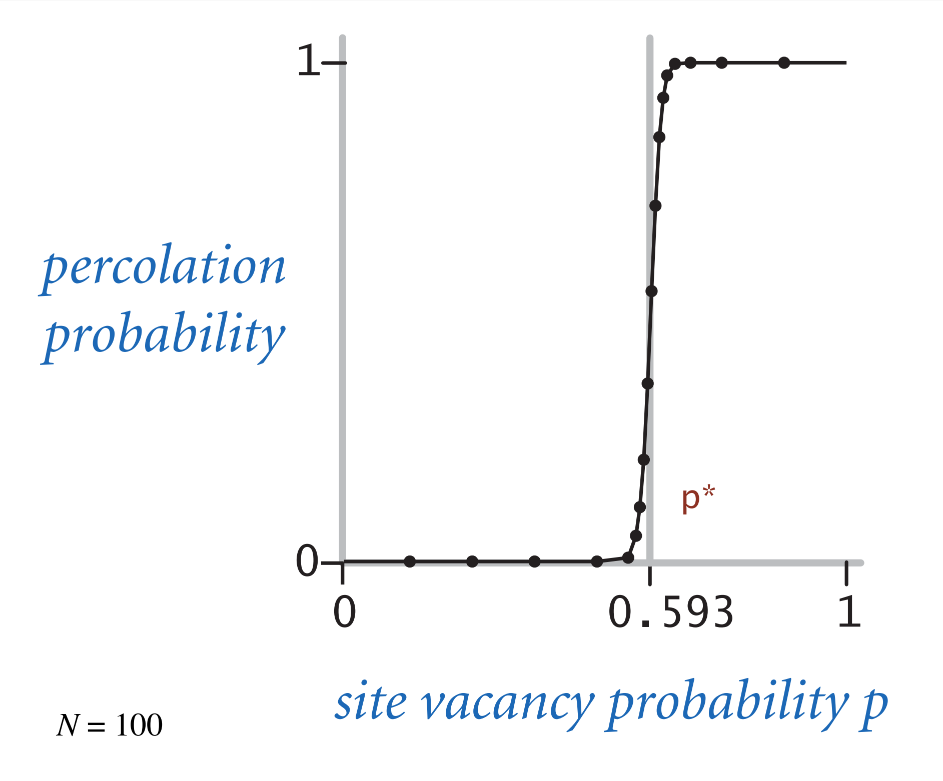
■ blocked site

Percolation threshold

Q. What is percolation threshold p^* ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.