COS 495 - Lecture 16
Autonomous Robot Navigation

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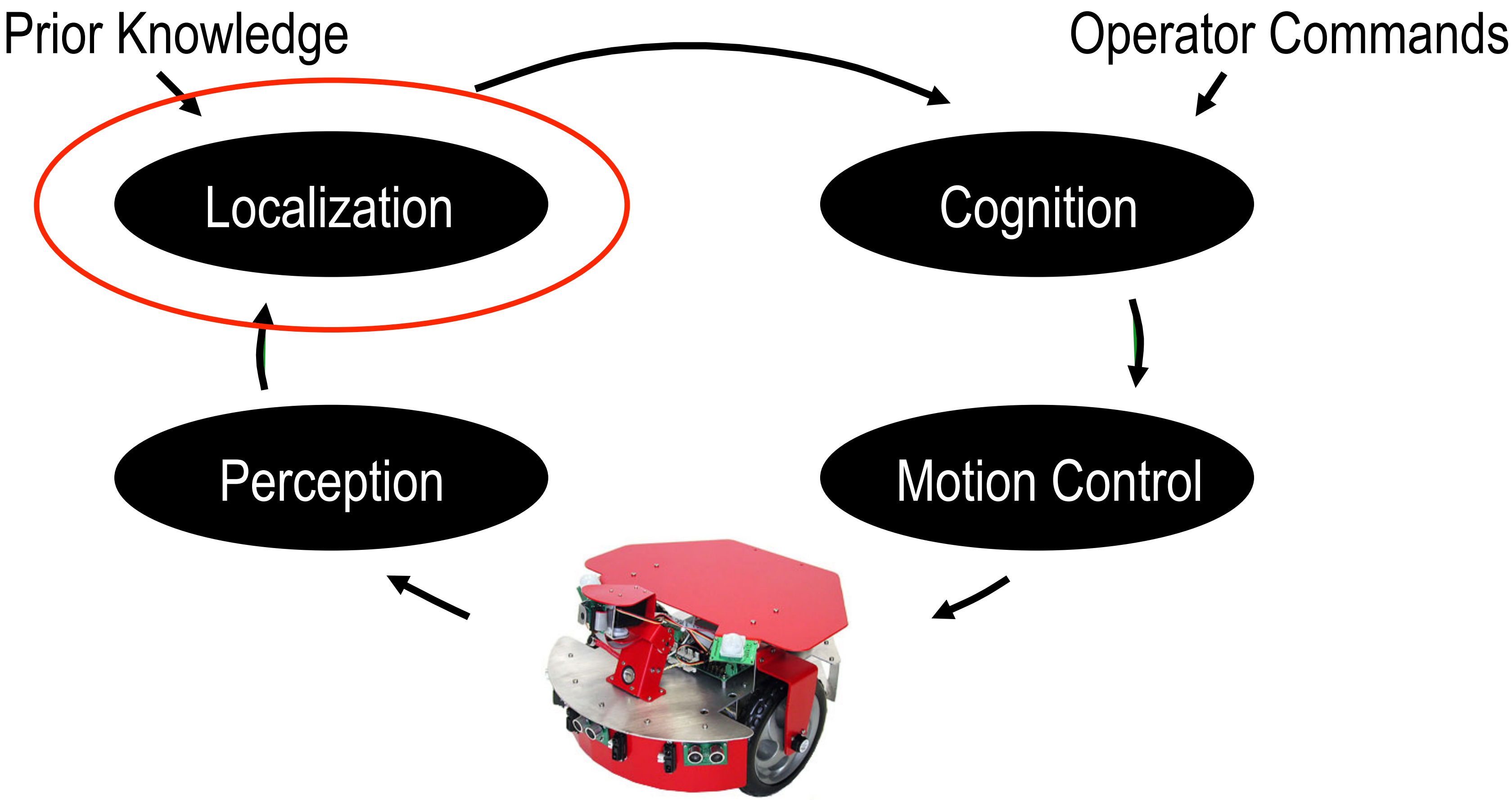
Figures courtesy of Siegwart & Nourbakhsh

Control Structure



Introduction to the Kalman Filter

1. KF Representations
2. Two Measurement Sensor Fusion
3. Single Variable Kalman Filtering
4. Multi-Variable KF Representations

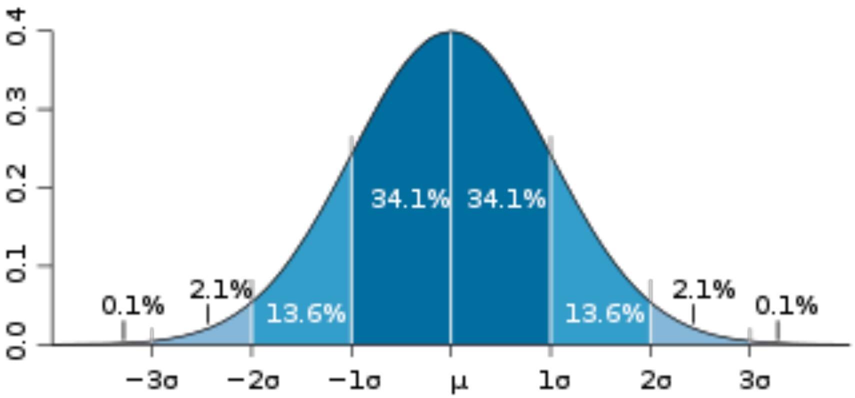
KF Representations

- What do Kalman Filters use to represent the states being estimated?

Gaussian Distributions!

KF Representations

- Single variable Gaussian Distribution
  - Symmetrical
  - Uni-modal
  - Characterized by
    - Mean μ
    - Variance σ^2
  - Properties
    - Propagation of errors
    - Product of Gaussians



KF Representations

- Single Var. Gaussian Characterization
  - Mean
    - Expected value of a random variable with a continuous Probability Density Function $p(x)$

$$\mu = \mathrm{E}[X] = \int x\, p(x)\, dx$$

    - For a discrete set of K samples

$$\mu = \sum_{k=1}^{K} x_k / K$$

KF Representations

- Single Var. Gaussian Characterization
  - Variance
    - Expected value of the difference from the mean squared

$$\sigma^2 = \mathrm{E}[(X-\mu)^2] = \int (x-\mu)^2 p(x)\, dx$$

    - For a discrete set of K samples

$$\sigma^2 = \sum_{k=1}^{K} (x_k - \mu)^2 / K$$

KF Representations

- Single variable Gaussian Properties
  - Propagation of Errors

$$\left.\begin{aligned} X &\sim N(\mu,\sigma^2) \\ Y &= aX+b \end{aligned}\right\} \Rightarrow \quad Y \sim N(a\mu+b, a^2\sigma^2)$$

  - Product of Gaussians

$$\left.\begin{aligned} X_1 &\sim N(\mu_1,\sigma_1^{\,2}) \\ X_2 &\sim N(\mu_2,\sigma_2^{\,2}) \end{aligned}\right\} \Rightarrow$$

$$p(X_1)\cdot p(X_2) \sim N\left(\frac{\sigma_2^{\,2}}{\sigma_1^{\,2}+\sigma_2^{\,2}}\mu_1 + \frac{\sigma_1^{\,2}}{\sigma_1^{\,2}+\sigma_2^{\,2}}\mu_2, \quad \frac{1}{\sigma_1^{\,-2}+\sigma_2^{\,-2}} \right)$$

KF Representations

- Single variable Gaussian Properties…
  - We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

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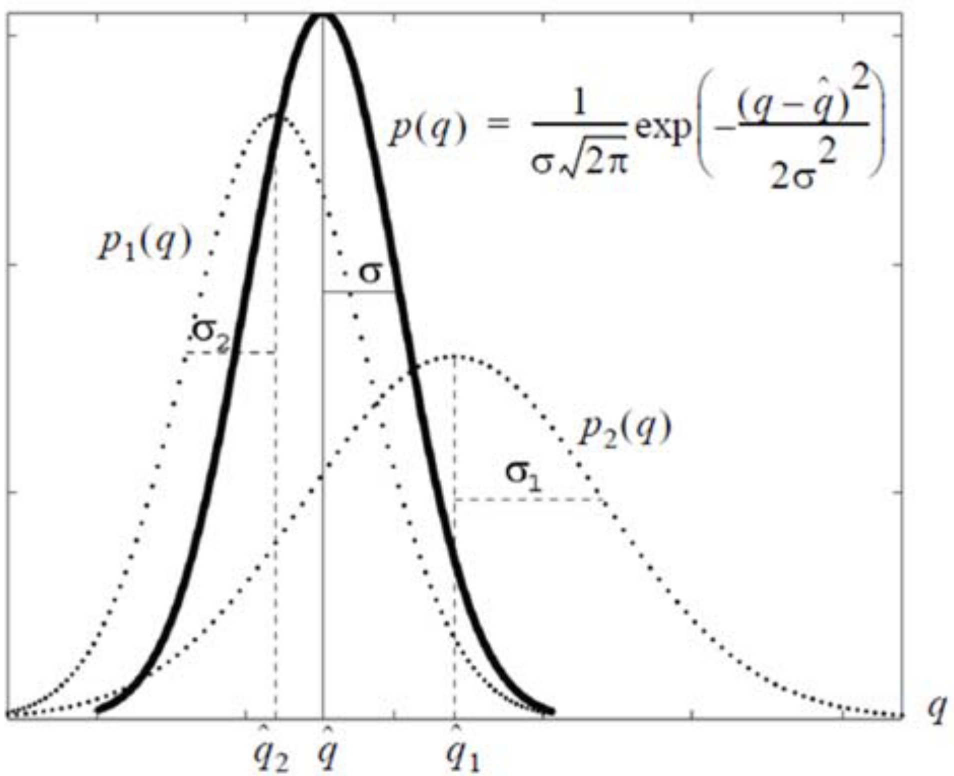
Fusing Two Measurements

- Example
  - Given two measurements q_1 and q_2, how do we fuse them to obtain an estimate $\hat{q}$?
  - Assume measurements are modeled as random variables that follow a Gaussian distribution with variance σ_1^2 and σ_2^2 respectively

Fusing Two Measurements

- Example (cont'):



Fusing Two Measurements

- Example (cont'):
  - Lets frame the problem as minimizing a weighted least squares cost function:

$$S = \sum_{i=1}^{n} w_i(\hat{q} - q_i)^2$$

$$\frac{\partial S}{\partial \hat{q}} = \frac{\partial}{\partial \hat{q}} \sum_{i=1}^{n} w_i(\hat{q} - q_i)^2 = 2\sum_{i=1}^{n} w_i(\hat{q} - q_i) = 0$$

Fusing Two Measurements

- Example (cont'):
  - If $n=2$ and $w_i = 1/\sigma_i^2$

$$\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(q_2 - q_1)$$

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Single Variable KF

- Example: Fusing two Measurements

$$\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(q_2 - q_1)$$

  - We can reformulate this in KF notation

$$\hat{x}_t = \hat{x}_{t-1} + K_t(z_t - \hat{x}_{t-1})$$
$$K_t = \frac{\sigma_{t-1}^2}{\sigma_{t-1}^2 + \sigma_t^2}$$

Single Variable KF

- KF for a Discrete Time System

$$\hat{x}_t = \hat{x}_{t-1} + K_t (z_t - \hat{x}_{t-1})$$

$$K_t = \frac{\sigma_{t-1}^2}{\sigma_{t-1}^2 + \sigma_t^2}$$

$$\sigma_t^2 = \sigma_{t-1}^2 - K_t \sigma_{t-1}^2$$

  - Where

$\hat{x}_t$ is the current state estimate

σ_t^2 is the associated variance

z_t^2 is the most recent measurement

K is the Kalman Gain

Kalman Filter Introduction

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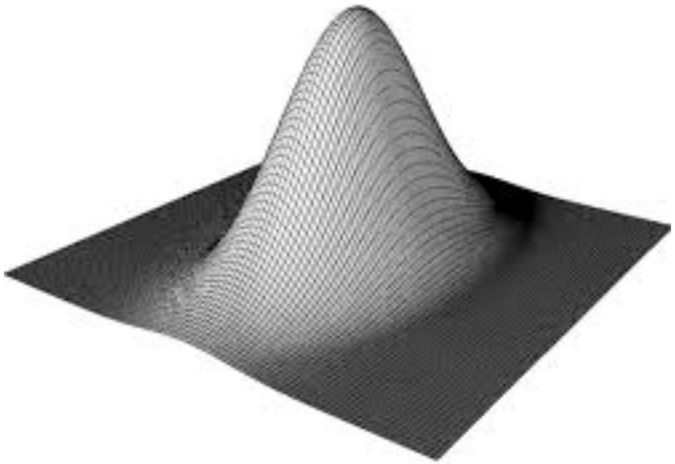
Representations in KF

- Multi-variable Gaussian Distribution
  - Symmetrical
  - Uni-modal
  - Characterized by
    - Mean Vector μ
    - Covariance Matrix Σ
  - Properties
    - Propagation of errors
    - Product of Gaussians



Representations in KF

- Multi-Var. Gaussian Characterization
  - Mean Vector
    - Vector of expected values of n random variables

$$\boldsymbol{\mu} = \mathrm{E}[\boldsymbol{X}] = [\ \mu_0\ \ \mu_1\ \ \mu_2\ \dots\ \mu_n\ ]^T$$

$$\mu_i = \int x_i\, p(x_i)\, dx_i$$

Representations in KF

- Multi-Var. Gaussian Characterization
  - Covariance
    - Expected value of the difference from the means squared

$$\sigma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathrm{E}[(X_i - \mu_i)(X_j - \mu_j)]$$

    - **Covariance** is a measure of how much two random variables change together.
    - **Positive** $\boldsymbol{\sigma_{ij}}$ – when variable i is **above** its expected value, then the other variable j tends to also be **above** its μ_j
    - **Negative** $\boldsymbol{\sigma_{ij}}$ – when variable i is **above** its expected value, then the other variable j tends to be **below** its μ_j

Representations in KF

- Multi-Var. Gaussian Characterization
  - Covariance
    - For continuous random variables

$$\sigma_{ij} = \iint (x_i - \mu_i)\,(x_j - \mu_j)\, p(x_i\,, x_j)\, dx_i\, dx_j$$

    - For discrete set of K samples

$$\sigma_{ij} = \sum_{k=1}^{K} (x_{i,k} - \mu_i)(x_{j,k} - \mu_j)/K$$

Representations in KF

- Multi-Var. Gaussian Characterization
  - Covariance Matrix
    - Covariance between each pair of random variables

$$\Sigma = \begin{bmatrix} \sigma_{00} \ \sigma_{01} & \dots \ \sigma_{0n} \\ \sigma_{10} \ \sigma_{11} & \dots \ \sigma_{1n} \\ \vdots & \\ \sigma_{n0} \ \sigma_{n1} & \dots \ \sigma_{nn} \end{bmatrix}$$

Note: $\sigma_{ii} = \sigma_i^2$

Representations in KF

- Multi variable Gaussian Properties
  - Propagation of Errors

$$\left.\begin{array}{r} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

  - Product of Gaussians

$$\left.\begin{array}{r} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\}$$

$$\Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2}\mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2}\mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}}\right)$$

Next…

Apply the Kalman Filter to multiple variables in the form of a KF.