

# Constrained Optimization



**COS 323**

# Last time

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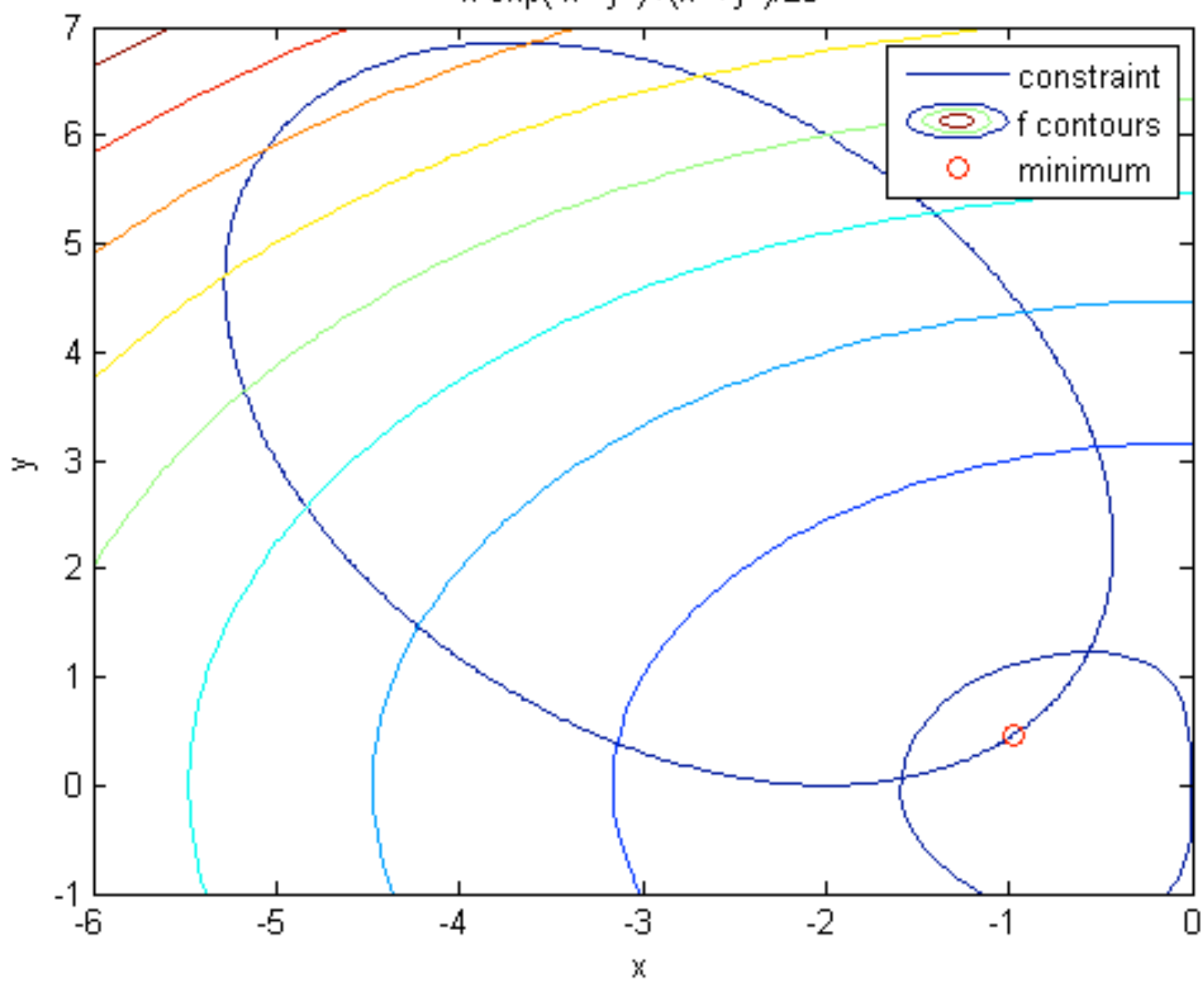
- Introduction to optimization
  - objective function, variables, [constraints]
- 1-dimensional methods
  - Golden section, discussion of error
  - Newton's method
- Multi-dimensional methods
  - Newton's method, steepest descent, conjugate gradient
- General strategies, value-only methods

# Today

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- Linear constrained optimization
  - Linear programming (LP)
  - Simplex method for LP
- General optimization
  - With equality constraints: Lagrange multipliers
  - With inequality: KKT conditions + Quadratic programming

$$x \exp(-x^2-y^2) + (x^2+y^2)/20$$



# Linear Objective, Linear Constraints

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Standard form: maximize objective

$$\zeta = c_1x_1 + c_2x_2 + \dots$$

with primary constraints

$$x_1 \geq 0, x_2 \geq 0, \dots$$

and additional constraints

$$a_{11}x_1 + a_{12}x_2 + \dots \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots \leq b_2$$

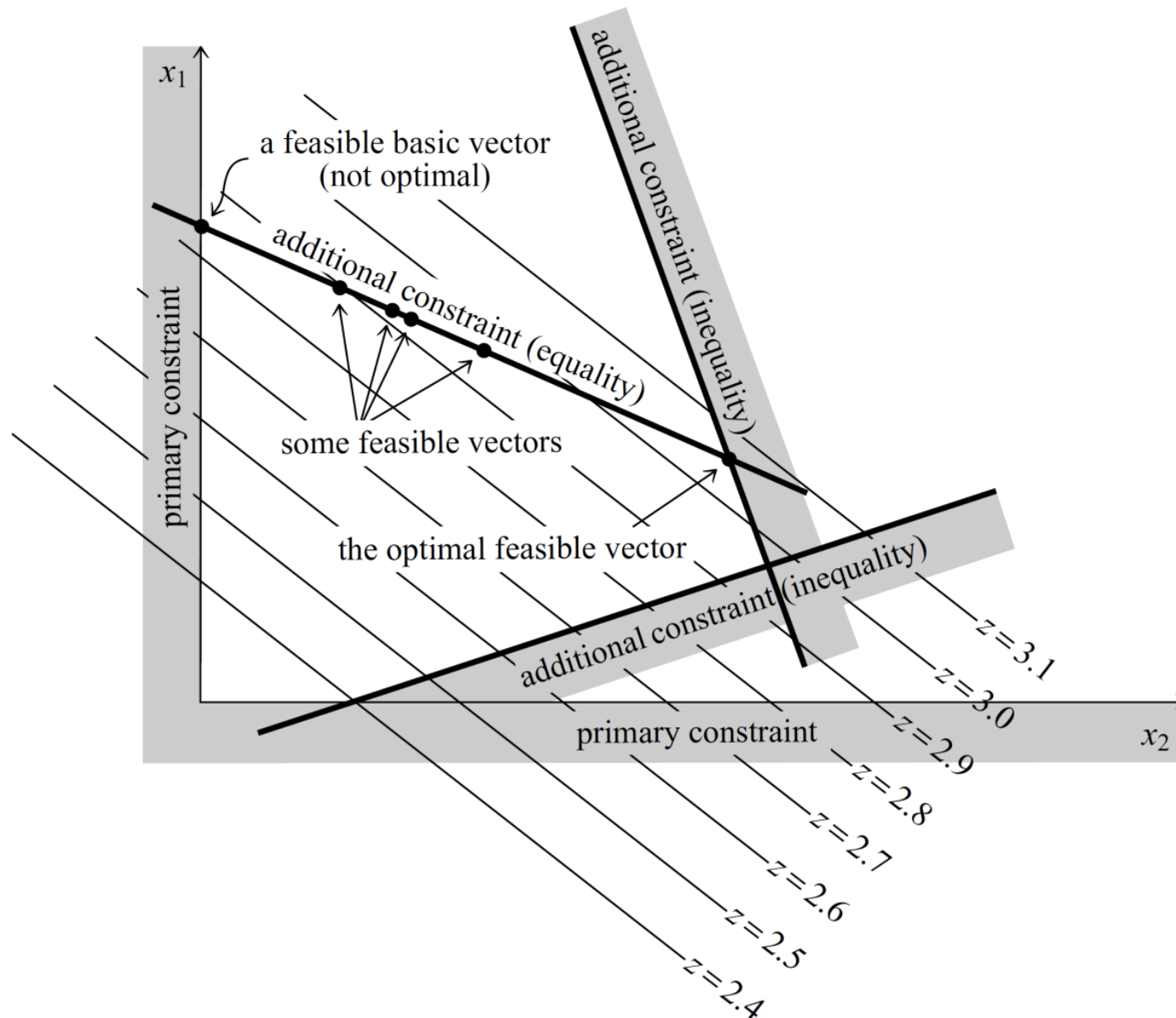
# Why Linear Programming?

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- What is the cheapest combination of foods that meets your nutritional needs?
- How do you minimize the risk in your investment portfolio, subject to achieving a given return?
- How should an airline assign crews to flights?
- Traveling salesman problem

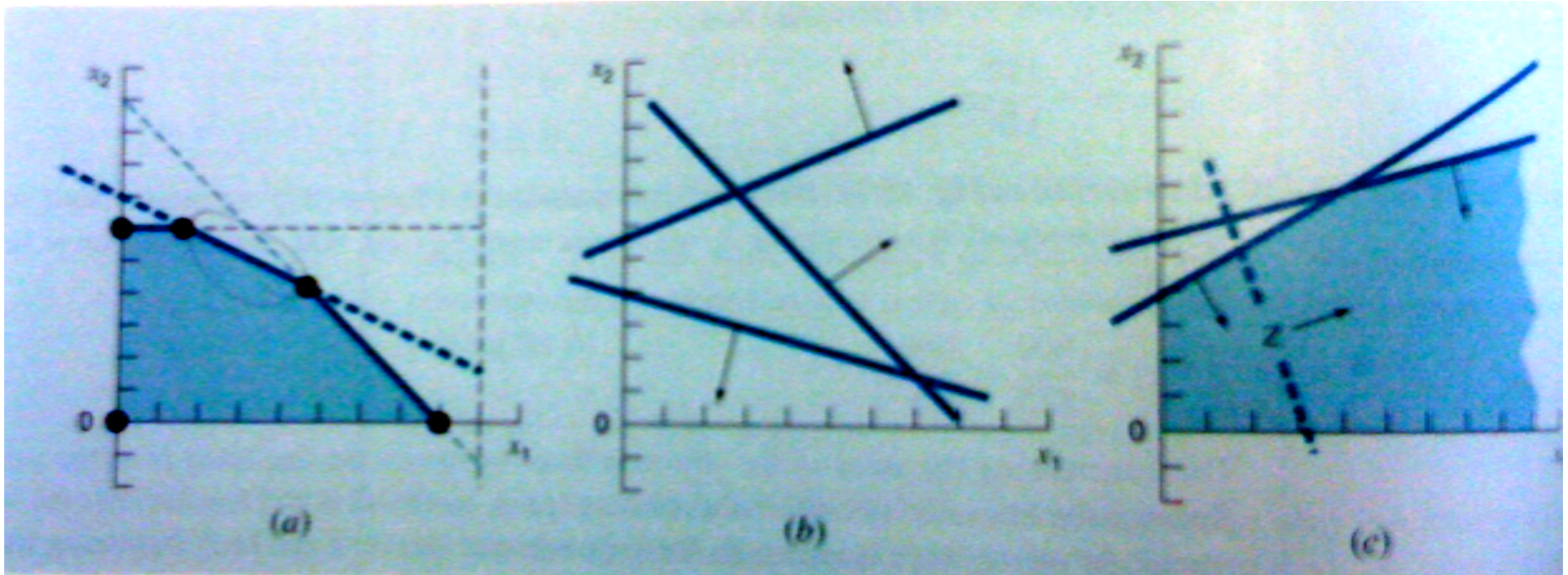
from John Mitchell @ RPI

# Linear objective + linear constraints



# Other possible outcomes

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# Simplex Method

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- [Dantzig's] Simplex Method
- Basic idea:
  - Phase I: Find a “basic feasible solution”: A vertex satisfying all constraints.
  - Phase II: Traverse vertices of the polytope along edges for which objective function is *strictly decreasing*
    - At any vertex, some  $n$  constraints are satisfied with exact equality

# Simplex Form

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- Transform each inequality constraint to an equality constraint using *slack variables*

$$x_1 + 2x_3 \leq 740$$

$$2x_2 - 7x_4 \leq 0$$

$$x_2 - x_3 + 2x_4 \geq \frac{1}{2}$$

$$x_1 + x_2 + x_3 + x_4 = 9$$



$$x_1 + 2x_3 + y_1 = 740$$

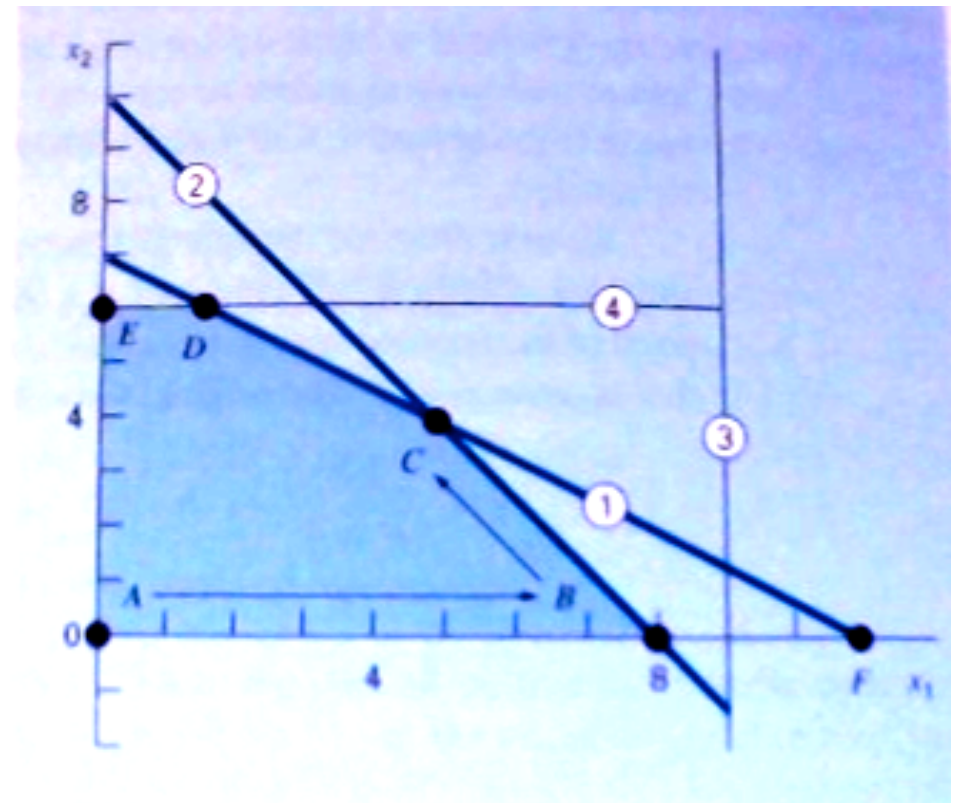
$$2x_2 - 7x_4 + y_2 = 0$$

$$x_2 - x_3 + 2x_4 - y_3 = \frac{1}{2}$$

$$x_1 + x_2 + x_3 + x_4 = 9$$

# An underdetermined system

- Normalized form:  
 $m$  equations with  $n$   
unknowns,  
with  $m < n$
- Solve by setting  $(n - m)$   
variables to 0 and  
solving for remaining  $m$   
unknowns



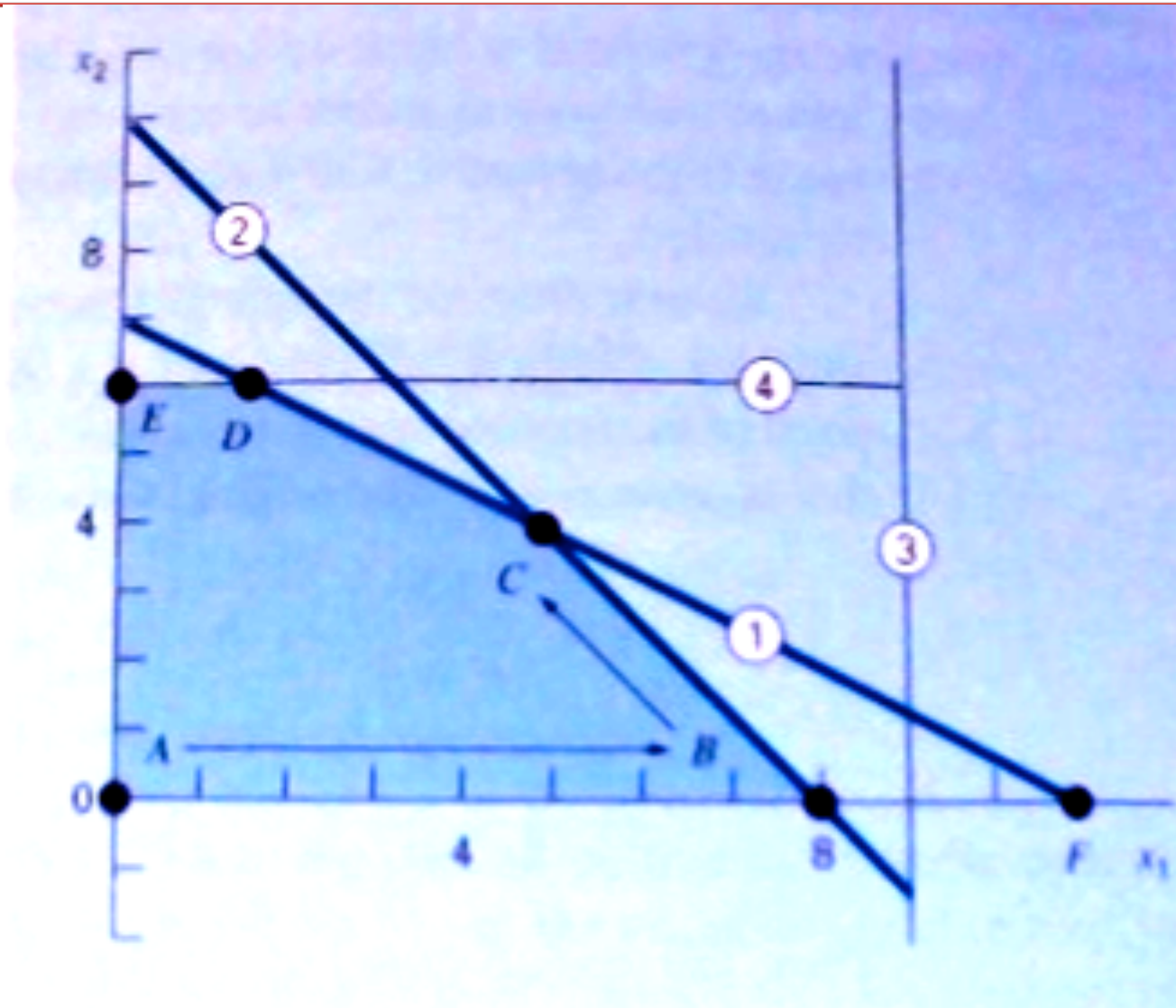
$$C_m^n = \frac{n!}{m!(n-m)!} \quad \text{systems of equations } \text{☹}$$

# Simplex method vertex traversal

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- Start with a *basic feasible solution*
  - Set  $(n-m)$  variables to 0: “non-basic variables”
  - Solve for remaining  $m$  “basic variables”: If they’re  $\geq 0$ , then it’s feasible
- Traverse an edge:
  - “Swap” a basic w/ a non-basic variable
  - Increase the value of the basic variable that has the biggest potential impact on the objective, and increase until another constraint is encountered (i.e., leaving basic variable becomes 0)

# Results of simplex method



## For more information

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- *Numerical Recipes in C* chapter on linear programming:

<http://www.nrbook.com/a/bookcpdf/c10-8.pdf>

- Interactive Java applet:

<http://campuscgi.princeton.edu/~rvdb/JAVA/pivot/simple.html>

# Comments on Simplex Method

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- In theory: can take very long – exponential in the input length
- In practice: efficient – # of iterations typically a few times # of constraints
  - Should take care to detect cycles
- There exist provably polynomial-time algorithms

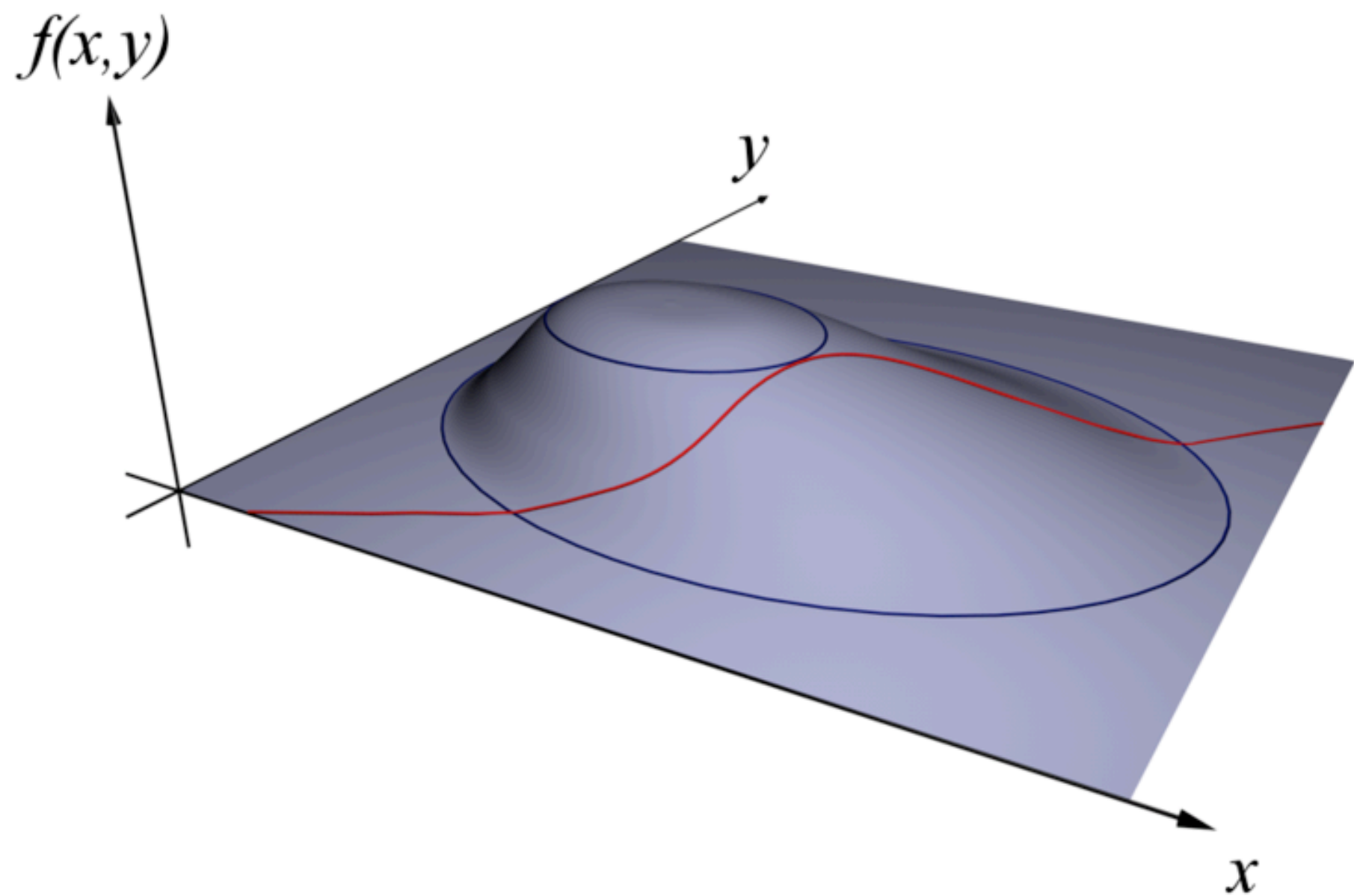
# Beyond linear optimization

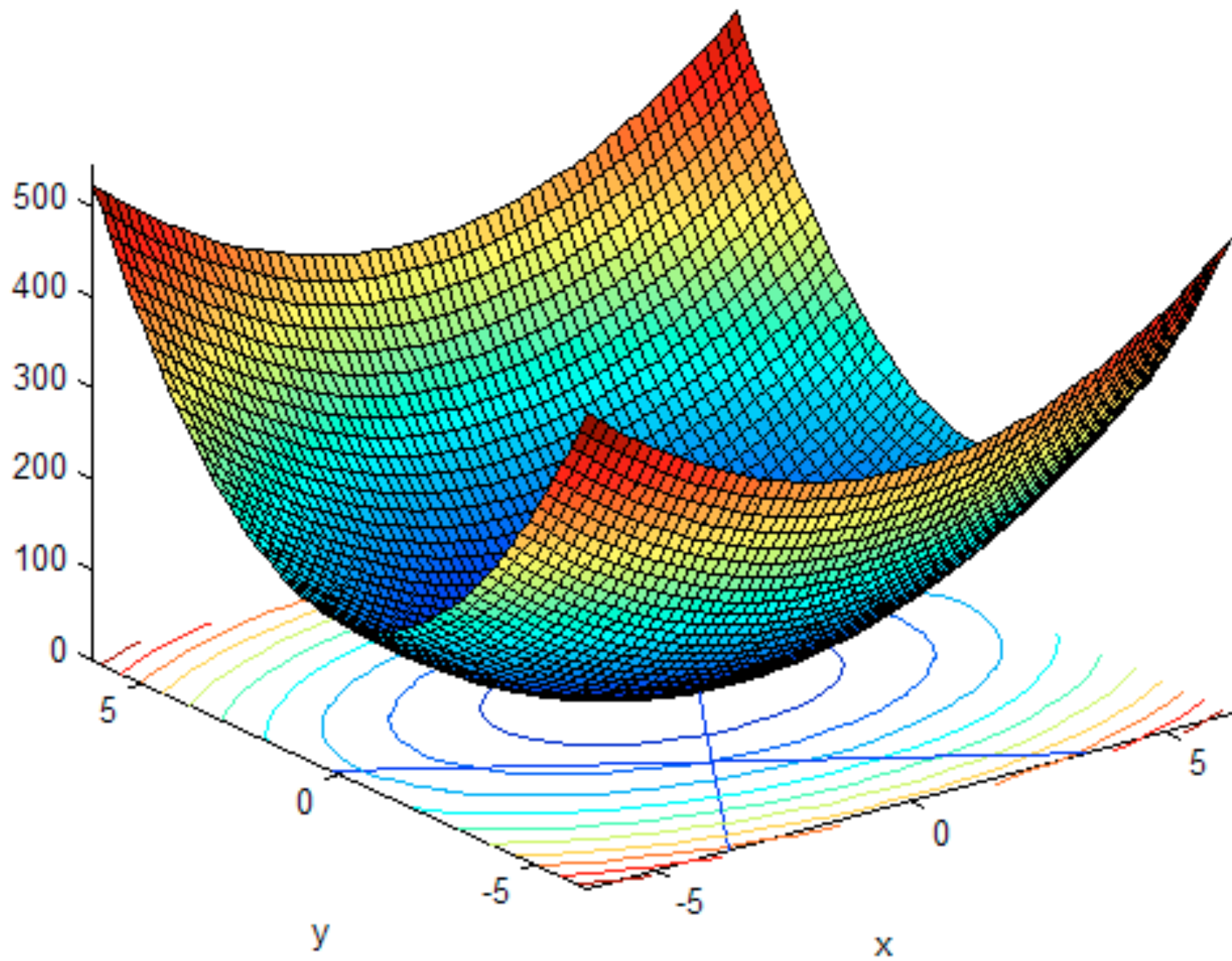
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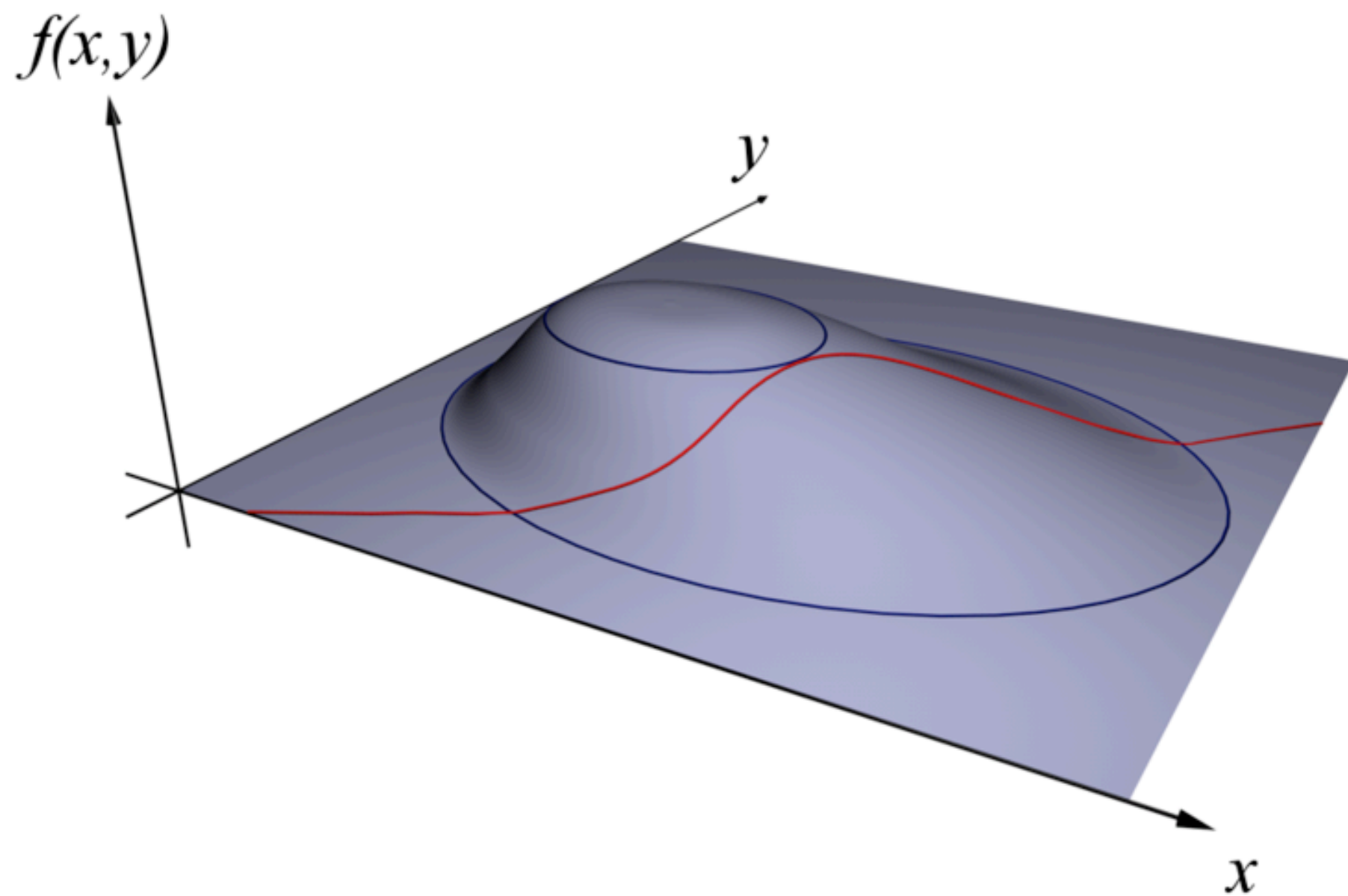
# General Optimization with Equality Constraints

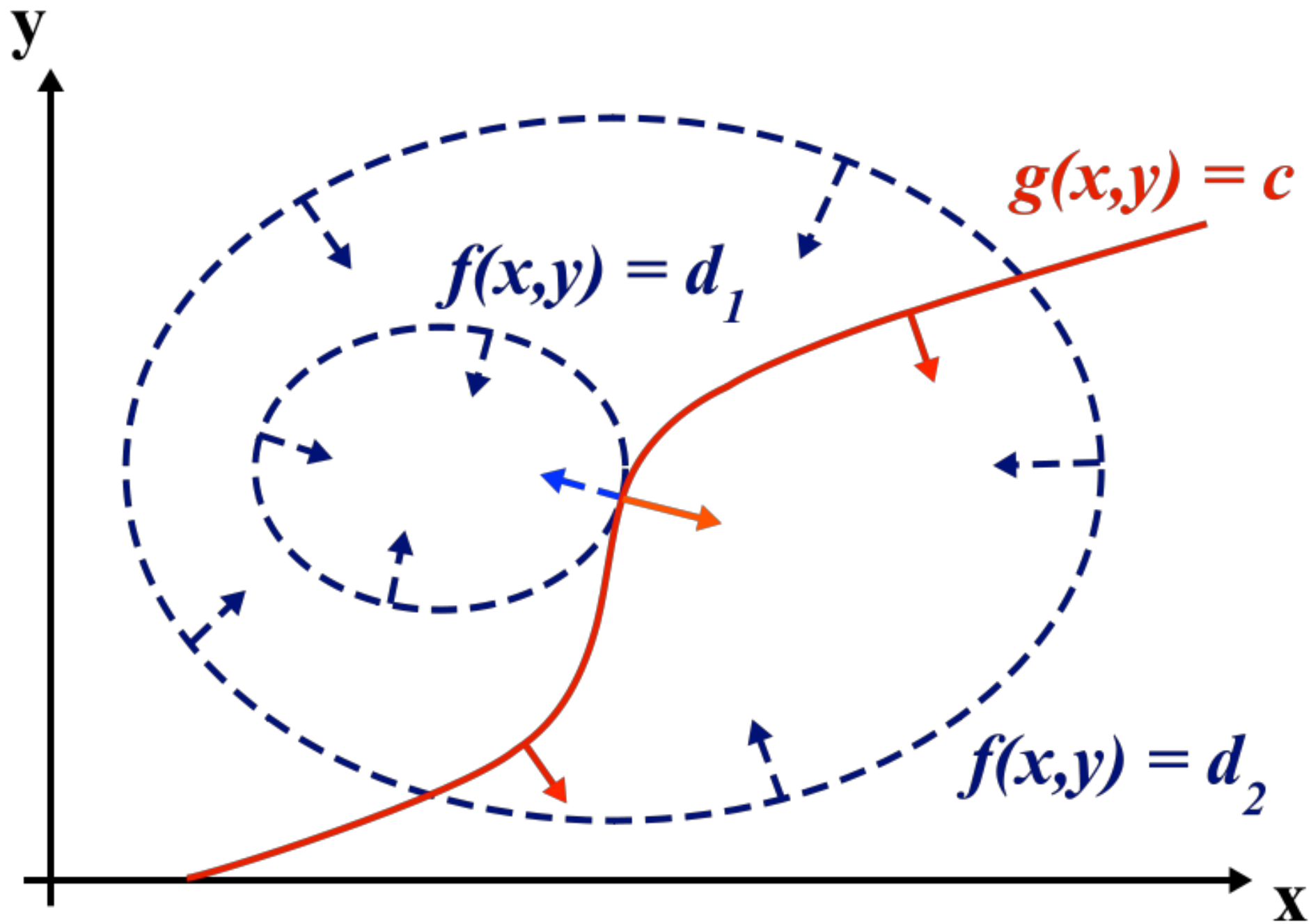
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- Minimize  $f(x)$  subject to  $g_i(x) = 0$









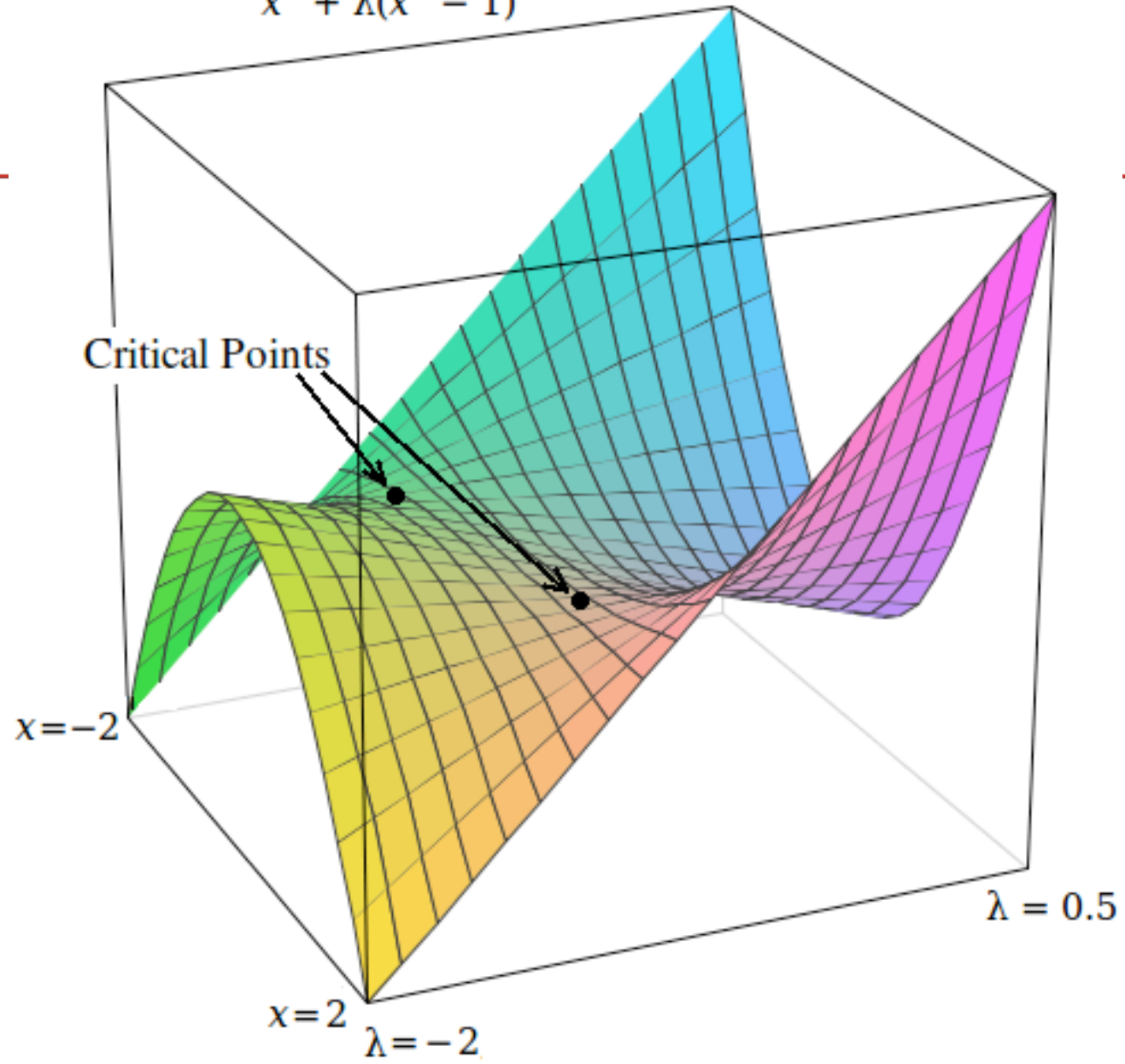
# General Optimization with Equality Constraints

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- Minimize  $f(x)$  subject to  $g_i(x) = 0$
- Method of Lagrange multipliers: convert to a higher-dimensional problem
- Find **critical points** of  $f(x) + \sum \lambda_i g_i(x)$

w.r.t.  $(x_1 \dots x_n; \lambda_1 \dots \lambda_k)$

$$x^2 + \lambda(x^2 - 1)$$



# Comments on Lagrange Multipliers

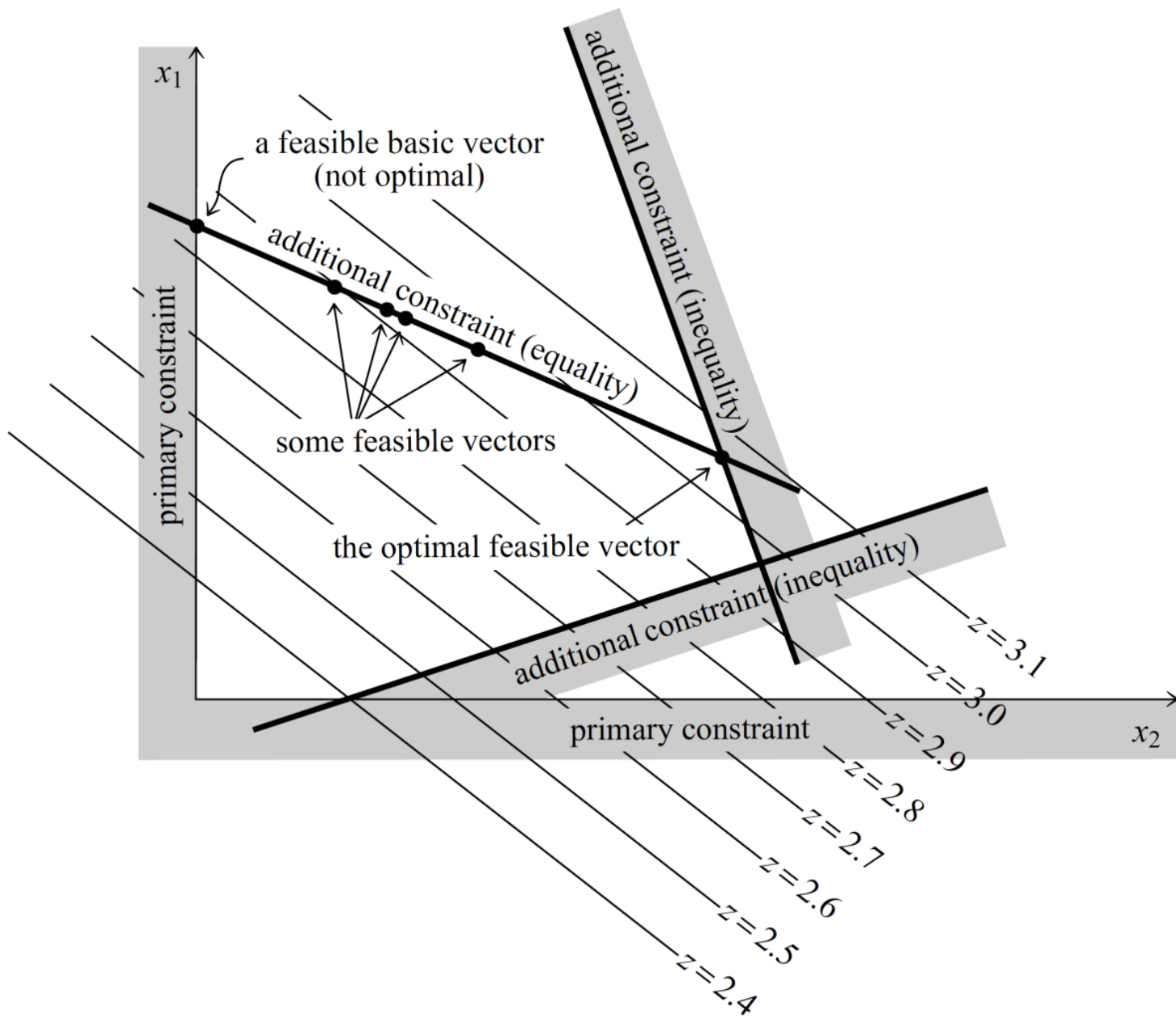
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- A way of re-defining an optimization problem in terms of a *necessary condition* for optimality (and feasibility)
  - *not* an algorithm for finding optimal points!
- Use other method to find critical points
- Sometimes lambdas are interesting in themselves
  - Lagrangian mechanics
  - “Shadow pricing” in economics: The “marginal cost” of a constraint

# General Optimization with Inequality Constraints

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- Minimize  $f(x)$  subject to  $h_i(x) \leq 0$
- More complicated: some inequality constraints might be irrelevant



# KKT Conditions for optimization with inequality constraints

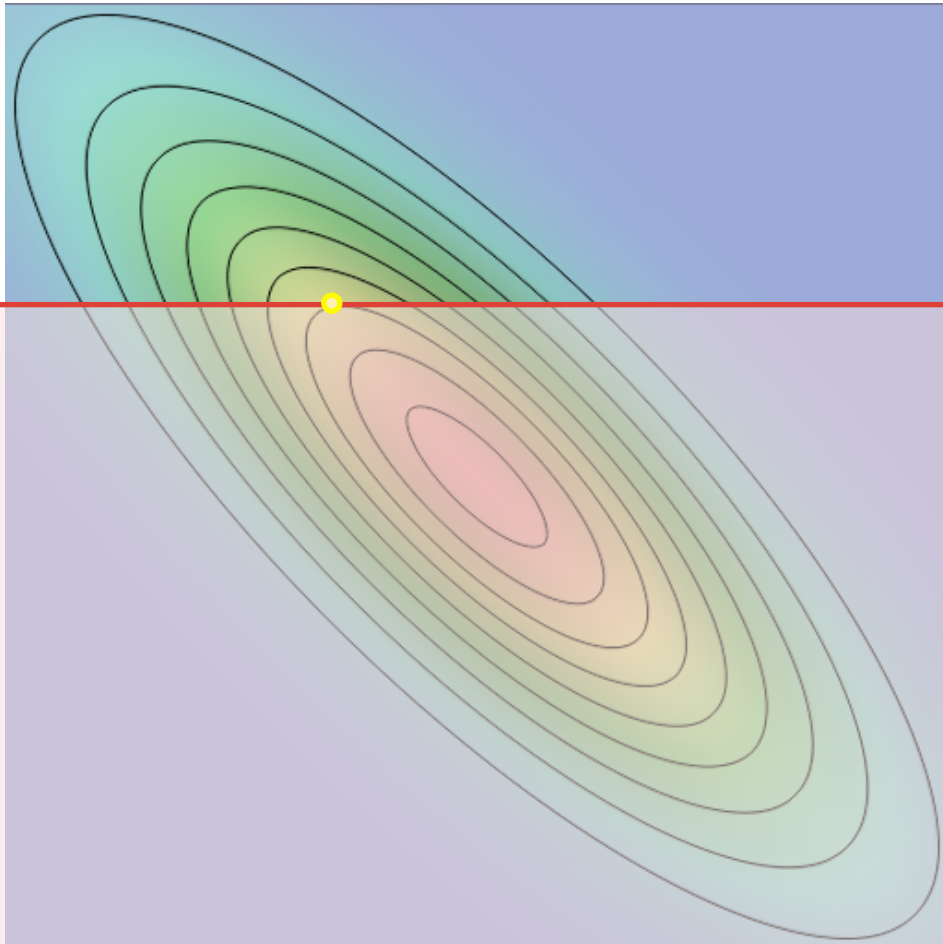
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- Karush-Kuhn-Tucker (KKT) conditions
  - Generalization of Lagrange multipliers
- Minimize  $f(x) + \sum \lambda_i h_i(x)$  w.r.t.  $(x_1 \dots x_n; \lambda_1 \dots \lambda_k)$
- Subject to

$$h_i(x) \leq 0, \quad \lambda_i(x) \geq 0, \quad \lambda_i(x) h_i(x) = 0$$

# KKT Conditions

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minimize  $f(x)$

with  $h(x) \leq 0$

1.  $\frac{\partial}{\partial x}(f(x)+\lambda h(x)) = 0$
2.  $\frac{\partial}{\partial \lambda}(f(x)+\lambda h(x)) = 0$
3.  $h(x) \leq 0$
4.  $\lambda \geq 0$
5.  $\lambda h(x) = 0$

# Quadratic Programming

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- The KKT conditions allow writing a system with **quadratic** objective and **linear** constraints as a linear program
  - Solve with simplex, etc.

# In practice

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- Matlab:
  - **linprog**
    - Option of using simplex method
  - **quadprog**
    - Option of getting Lagrange multiplier values out
  - See Matlab's official advice on choosing a solver
    - <http://www.mathworks.com/help/toolbox/optim/ug/brhkghv-18.html#bsbqd7i>
- Excel solver



# [Dantzig's] Simplex Method

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Maximize

$$z = a_{01}x_1 + a_{02}x_2 + \cdots + a_{0N}x_N$$

Subject to primary constraints

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \dots \quad x_N \geq 0$$

and to additional constraints of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{iN}x_N \leq b_i \quad (b_i \geq 0)$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jN}x_N \geq b_j \geq 0$$

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kN}x_N = b_k \geq 0$$