

# 4.2 Sorting and Searching


## Whitelist Filter

Blacklist. A list of entities to be rejected for service.
Ex. Deny charges for overdrawn credit cards.

Whitelist. A list of entities to be accepted for service.
Ex. Accept messages only from listed friends.

Whitelist filter. Read a list of strings from a whitelist file,
then print out all strings from standard input that are in the whitelist.

whitelist

```
alice@home
bob@office
carl@beach
dave@boat
```

standard input

```
.
.
.
bob@office
carl@beach
marvin@spam
bob@office
bob@office
mallory@spam
dave@boat
eve@airport
alice@home
.
.
.
```

output

```
.
.
.
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
.
.
.
```

message contents
omitted for simplicity



## Search Client: Whitelist Filter

Whitelist filter. Read a list of strings from a whitelist file,
then print out all strings from standard input that are in the whitelist.

```
public class WhiteFilter
{
   public static void main(String[] args)
   {
   In in = new In(args[0]);
   String[] words;
   // Fill words[] with strings from In (stay tuned).
   while (!StdIn.isEmpty())
   {
      String key = StdIn.readString();
      if (search(key, words) != -1)
         StdOut.println(key);
      }
   }
}
```

```
% more whitelist.txt
alice@home
bob@office
carl@beach
dave@boat
```

```
% more test.txt
bob@office
carl@beach
marvin@spam
bob@office
bob@office
mallory@spam
dave@boat
eve@airport
alice@home
```

```
% java WhiteFilter list.txt < test.txt
bob@office
carl@beach
bob@office
bob@office
dave@boat
alice@home
```

## Sequential Search: Java Implementation

Scan through array, looking for key.

- search hit: return array index
- search miss: return -1

```
public static int search(String key, String[] a)
{
   for (int i = 0; i < a.length; i++)
      if ( a[i].compareTo(key) == 0 ) return i;
   return -1;
}
```



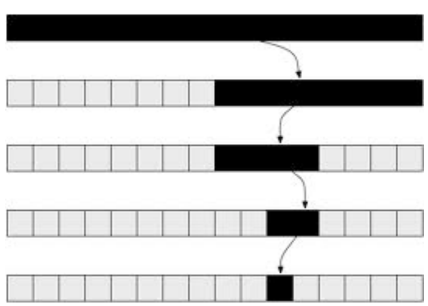
## TEQ on Searching 1

A credit card company needs to whitelist 10 million customer accounts, processing 1000 transactions per second.
Using sequential search, what kind of computer is needed?

A. Toaster
B. Cellphone
C. Your laptop
D. Supercomputer
E. Google server farm



# Binary Search

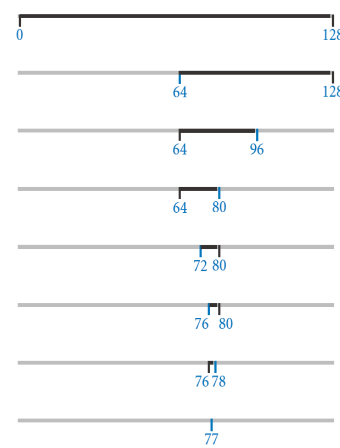




## Twenty Questions

Intuition. Find a hidden integer.

*interval*



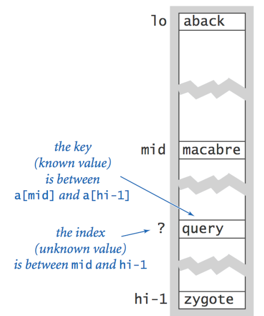

| *size* | *Q* | *A* |
|---|---|---|
| 128 | < 64 ? | *no* |
| 64 | < 96 ? | *yes* |
| 32 | < 80 ? | *yes* |
| 16 | < 72 ? | *no* |
| 8 | < 76 ? | *no* |
| 4 | < 78 ? | *yes* |
| 2 | < 77 ? | *no* |
| 1 | = 77 | |

## Binary Search

Idea:
- Sort the array (stay tuned)
- Play "20 questions" to determine the index associated with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, …

Binary search.
- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.




*Binary search in a sorted array (one step)*



## Binary Search: Java Implementation

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

```
public static int search(String key, String[] a)
{
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi)
{
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: `Arrays.binarySearch()`



## Binary Search: Mathematical Analysis

Analysis. To binary search in an array of size $N$: do one comparison, then binary search in an array of size $N / 2$.

$$N \to N/2 \to N/4 \to N/8 \to \dots \to 1$$

Q. How many times can you divide a number by 2 until you reach 1?
A. $\log_2 N$.

$$1$$
$$2 \to 1$$
$$4 \to 2 \to 1$$
$$8 \to 4 \to 2 \to 1$$
$$16 \to 8 \to 4 \to 2 \to 1$$
$$32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$1024 \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$



## TEQ on Searching 2

A credit card company needs to whitelist 10 million customer accounts, processing 1 thousand transactions per second.
Using binary search, what kind of computer is needed?

A. Toaster
B. Cellphone
C. Your laptop
D. Supercomputer
E. Google server farm

# Sorting

## TEQ on Sorting 0

Q. What's the fastest way to sort 1 million 32-bit integers?



## Sorting

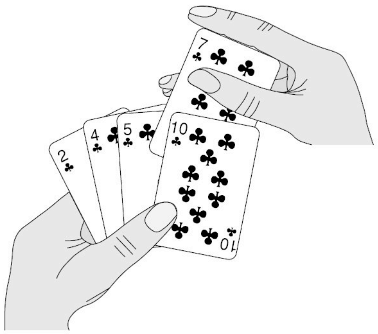
Sorting problem. Rearrange $N$ items in ascending order.

Applications. Binary search, statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list) ...

| | | |
|---|---|---|
| Hauser | → | Hanley |
| Hong | | Haskell |
| Hsu | | Hauser |
| Hayes | | Hayes |
| Haskell | | Hong |
| Hanley | | Hornet |
| Hornet | | Hsu |



# Insertion Sort



## Insertion Sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Insert each element into final position by exchanging it with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | and | had | him | his | was | you | the | but |
| 6 | 5 | and | had | him | his | was | the | you | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| | | and | had | him | his | the | was | you | but |

*Inserting* `a[6]` *into position by exchanging with larger entries to its left*



## Insertion Sort

**Insertion sort.**

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

| i | j | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|
| | | was | had | him | and | you | his | the | but |
| 1 | 0 | had | was | him | and | you | his | the | but |
| 2 | 1 | had | him | was | and | you | his | the | but |
| 3 | 0 | and | had | him | was | you | his | the | but |
| 4 | 4 | and | had | him | was | you | his | the | but |
| 5 | 3 | and | had | him | his | was | you | the | but |
| 6 | 4 | and | had | him | his | the | was | you | but |
| 7 | 1 | and | but | had | him | his | the | was | you |
| | | and | but | had | him | his | the | was | you |

*Inserting* `a[1]` *through* `a[N-1]` *into position (insertion sort)*



## Insertion Sort: Java Implementation

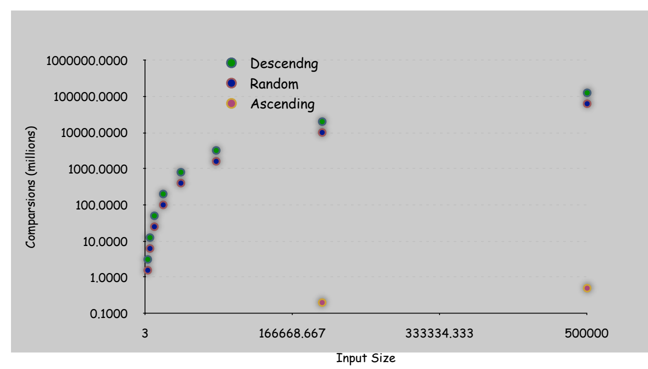
```java
public class Insertion
{

   public static void sort(String[] a)
   {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1].compareTo(a[j]) > 0)
               exch(a, j-1, j);
            else break;
   }


   private static void exch(String[] a, int i, int j)
   {
      String swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```



## Insertion Sort: Empirical Analysis

**Observation.** Number of comparisons depends on input family.

- Descending: $\sim N^2 / 2$.
- Random: $\sim N^2 / 4$.
- Ascending: $\sim N$.



## Insertion Sort: Mathematical Analysis

Worst case. [descending]

- Iteration $i$ requires $i$ comparisons.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) \sim N^2 / 2$ compares.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| E | F | G | H | I | J | D | C | B | A |
| | | | | | | $i$ | | | |

Average case. [random]

- Iteration $i$ requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) / 2 \sim N^2 / 4$ compares

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | C | D | F | H | J | E | B | I | G |
| | | | | | | $i$ | | | |



## Insertion Sort: Scientific Analysis

Hypothesis: Running time is ~ a N $^b$ seconds

Initial experiments:

| N | Comparisons | Time | Ratio |
|---|---|---|---|
| 5,000 | 6.2 million | 0.13 seconds | |
| 10,000 | 25 million | 0.43 seconds | 3.3 |
| 20,000 | 99 million | 1.5 seconds | 3.5 |
| 40,000 | 400 million | 5.6 seconds | 3.7 |
| 80,000 | 1600 million | 23 seconds | 4.1 |

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB
- Timing: Skagen wristwatch.

Doubling hypothesis:

- b = lg 4 = 2, so running time is ~ a N $^2$
- checks with math analysis
- a ≈ 23 / 80000$^2$ = 3.5 × $10^{-9}$

Refined hypothesis: Running time is ≈ 3.5 × $10^{-9}$ N $^2$ seconds



## Insertion Sort: Scientific Analysis (continued)

Refined hypothesis: Running time is ≈ 3.5 × $10^{-9}$ N $^2$ seconds

Prediction: Running time for N = 200,000
should be 3.5 × $10^{-9}$ × 4 × $10^{10}$ ≈ 140 seconds

Observation:

| N | Time |
|---|---|
| 200,000 | 145 seconds |

Observation matches prediction and validates refined hypothesis.



## TEQ on Sorting 1

A credit card company uses insertion sort to sort 10 million customer account numbers, for use in whitelisting with binary search. What kind of computer is needed?

A. Toaster
B. Cellphone
C. Your laptop
D. Supercomputer
E. Google server farm

## Insertion Sort: Lesson

Lesson. Supercomputer can't rescue a bad algorithm.

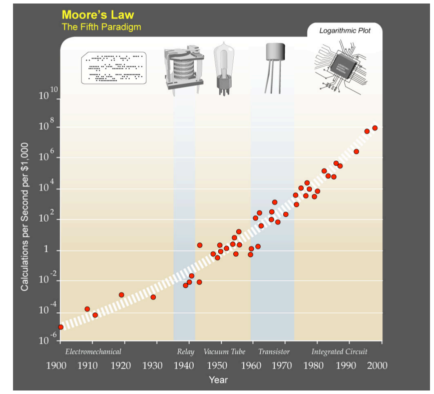
| Computer | Comparisons Per Second | Thousand | Million | Billion |
|---|---|---|---|---|
| laptop | $10^7$ | instant | 1 day | 3 centuries |
| super | $10^{12}$ | instant | 1 second | 2 weeks |



## Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.




http://en.wikipedia.org/wiki/Moore's_law

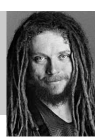


## Moore's Law and Algorithms

Quadratic algorithms do not scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

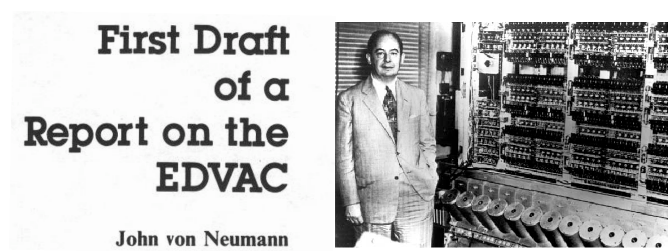
*"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding." – Jaron Lanier*



Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.



# Mergesort



## Mergesort

Mergesort.
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

*input*
`was had him and you his the but`

*sort left*
`and had him was you his the but`

*sort right*
`and had him was but his the you`

*merge*
`and but had him his the was you`



## Mergesort: Example

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| E | G | M | R | E | S | O | R | E | T | A | X | M | P | E | L |
| E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| E | G | M | R | E | O | R | S | E | T | A | X | M | P | E | L |
| E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| E | M | G | R | E | S | O | R | E | T | X | A | M | P | L | E |
| E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| E | G | M | R | E | O | R | S | A | E | T | X | M | P | E | L |
| E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| E | M | G | R | E | S | O | R | E | T | A | X | M | P | E | L |
| E | G | M | R | E | O | R | S | A | E | T | X | E | L | M | P |
| E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |



## Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

| i | j | k | aux[k] | a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | and | had | him | was | but | his | the | you |
| 0 | 4 | 0 | and | and | had | him | was | but | his | the | you |
| 1 | 4 | 1 | but | and | had | him | was | but | his | the | you |
| 1 | 5 | 2 | had | and | had | him | was | but | his | the | you |
| 2 | 5 | 3 | him | and | had | him | was | but | his | the | you |
| 3 | 5 | 4 | his | and | had | him | was | but | his | the | you |
| 3 | 6 | 5 | the | and | had | him | was | but | his | the | you |
| 3 | 6 | 6 | was | and | had | him | was | but | his | the | you |
| 4 | 7 | 7 | you | and | had | him | was | but | his | the | you |

*Trace of the merge of the sorted left half with the sorted right half*



## Merging

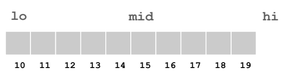
Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// Merge into auxiliary array.
int i = lo, j = mid;
for (int k = 0; k < N; k++)
{
   if      (i == mid) aux[k] = a[j++];
   else if (j == hi)  aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
   else                               aux[k] = a[i++];
}

// Copy back.
for (int k = 0; k < N; k++)
   a[lo + k] = aux[k];
```

## Mergesort: Java Implementation

```
public class Merge
{
   public static void sort(String[] a)
   {  sort(a, 0, a.length);  }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi)
   {
      int N = hi - lo;
      if (N <= 1) return;

      // Recursively sort left and right halves.
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // Merge sorted halves (see previous slide).
   }

}
```

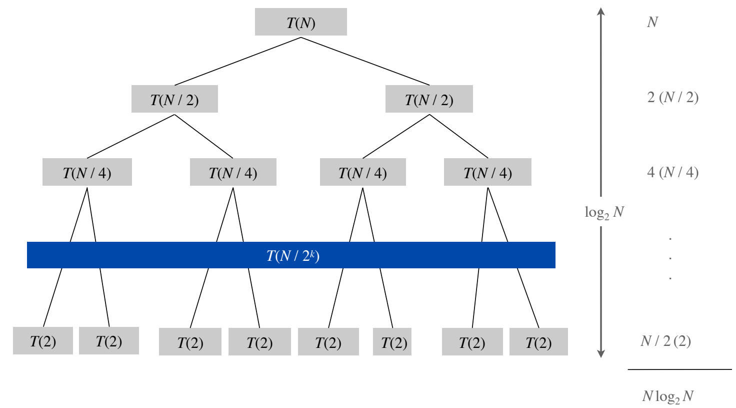





## Mergesort: Mathematical Analysis

Analysis. To mergesort array of size $N$, mergesort two subarrays of size $N/2$, and merge them together using $\leq N$ comparisons.

we assume $N$ is a power of 2






## Mergesort: Mathematical Analysis

Mathematical analysis.

| analysis | comparisons |
|---|---|
| worst | $N \log_2 N$ |
| average | $N \log_2 N$ |
| best | $1/2\, N \log_2 N$ |

Validation. Theory agrees with observations.

| N | actual | predicted |
|---|---|---|
| 10,000 | 120 thousand | 133 thousand |
| 20 million | 460 million | 485 million |
| 50 million | 1,216 million | 1,279 million |



## Mergesort: Scientific Analysis

Hypothesis. Running time is ~ c N lg N seconds

Initial experiments:

- c ≈ 3.2 / (4 × $10^6$ × 32) = 2.5 × $10^{-8}$

| N | Time |
|---|---|
| 4 million | 3.13 sec |
| 4 million | 3.25 sec |
| 4 million | 3.22 sec |

Refined hypothesis. Running time is 2.5 × $10^{-7}$ N lg N seconds.
Prediction: Running time for N = 20,000,000
should be about 2.5 × $10^{-8}$ × 2 × $10^7$ × 35 ≈ 17.5 seconds

Observation:

| N | Time |
|---|---|
| 20 million | 17.5 sec |

Observation matches prediction and validates refined hypothesis.

## TEQ on Sorting 2

A credit card company uses mergesort to sort 10 million customer account numbers, for use in whitelisting with binary search. What kind of computer is needed?

A. Toaster
B. Cellphone
C. Your laptop
D. Supercomputer
E. Google server farm



## Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

| Computer | Comparisons Per Second | Insertion | Mergesort |
|---|---|---|---|
| laptop | $10^7$ | 3 centuries | 3 hours |
| super | $10^{12}$ | 2 weeks | instant |

N = 1 billion



# Longest Repeated Substring



## Redundancy Detector

Longest repeated substring. Given a string, find the longest substring that appears at least twice.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |

Brute force.
- Try all indices $i$ and $j$ for start of possible match.
- Compute longest common prefix for each pair (quadratic+).

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| | $i$ | | | | | | | | $j$ | | | | | |

Applications. Bioinformatics, cryptography, …

## LRS application: patterns in music

Music is characterized by its repetitive structure

Mary Had a Little Lamb

Fur Elise

source: `http://www.bewitched.com/match/`



## LRS applications: patterns in sequences

Repeated sequences in real-world data are causal.

Ex 1. Digits of pi
- Q. are they "random"?
- A. No, but we can't tell the difference
- Ex. Length of LRS in first 10 million digits is 14

Ex 2. Cryptography
- Find LRS
- Check for "known" message header identifying place, date, person, etc.
- Break code

Ex 3. DNA
- Find LRS
- Look somewhere else for causal mechanisms
- Ex. Chromosome 11 has 7.1 million nucleotides



## Brute-force solution

Longest repeated substring. Given a string, find the longest substring that appears at least twice.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |

Brute force.
- Try all indices $i$ and $j$ for start of possible match.
- Compute longest common prefix (LCP) for each pair

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| | $i$ | | | | | | | | $j$ | | | | | |

Analysis.
- all pairs: 1 + 2 + ... + N ~ $N^2/2$ calls on LCP
- too slow for long strings



## Longest Repeated Substring: A Sorting Solution

*input string*

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
a a c a a g t t t a c a a g c
```

1. Form suffixes

*suffixes*

```
0  a a c a a g t t t a c a a g c
1  a c a a g t t t a c a a g c
2  c a a g t t t a c a a g c
3  a a g t t t a c a a g c
4  a g t t t a c a a g c
5  g t t t a c a a g c
6  t t t a c a a g c
7  t t a c a a g c
8  t a c a a g c
9  a c a a g c
10 c a a g c
11 a a g c
12 a g c
13 g c
14 c
```

2. Sort suffixes to bring repeated substrings together

*sorted suffixes*

```
0  a a c a a g t t t a c a a g c
11 a a g c
3  a a g t t t a c a a g c
9  a c a a g c
1  a c a a g t t t a c a a g c
12 a g c
4  a g t t t a c a a g c
14 c
10 c a a g c
2  c a a g t t t a c a a g c
13 g c
5  g t t t a c a a g c
8  t a c a a g c
7  t t a c a a g c
6  t t t a c a a g c
```

3. Compute longest prefix between adjacent suffixes

*longest repeated substring*

```
  1               9
a a c a a g t t t a c a a g c
```

## Longest Repeated Substring: Java Implementation

Suffix sorting implementation.

```
int N = s.length();
String[] suffixes = new String[N];
for (int i = 0; i < N; i++)
   suffixes[i] = s.substring(i, N);
Arrays.sort(suffixes);
```

Longest common prefix: `lcp(s, t)`.

- longest string that is a prefix of both s and t
- Ex: `lcp("acaagtttac", "acaagc") = "acaag"`.
- easy to implement (you could write this one).

Longest repeated substring. Search only adjacent suffixes.

```
String lrs = "";
for (int i = 0; i < N-1; i++)
{
   String x = lcp(suffixes[i], suffixes[i+1]);
   if (x.length() > lrs.length()) lrs = x;
}
```



## Java substring operation

Memory representation of strings.

`s = "aacaagtttacaagc";`

| D0 | D1 | D2 | D3 | D4 | D5 | D6 | D7 | D8 | D9 | DA | DB | DC | DD | DE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |

s

| A0 | A1 |
|---|---|
| D0 | 15 |

address length

- A `String` is an address and a length.
- Characters can be shared among strings.
- `substring()` computes address, length (instead of copying chars).

`t = s.substring(5, 15);`

t

| B0 | B1 |
|---|---|
| D5 | 10 |

Consequences.

- `substring()` is a constant-time operation (instead of linear).
- Creating suffixes takes linear space (instead of quadratic).
- Running time of LRS is dominated by the string sort.



## TEQ on Sorting 3

Q. Four researchers A, B, C and D are looking for long repeated subsequences in a genome with over 1 billion characters.

A. has a grad student do it.

B. uses brute force (check all pairs) solution.

C. uses sorting solution with insertion sort.

D. uses sorting solution with mergesort.

Which one is more likely to find a cancer cure?



## Longest Repeated Substring: Empirical Analysis

| Input File | Characters | Brute | Suffix Sort | Length |
|---|---|---|---|---|
| LRS.java | 2,162 | 0.6 sec | 0.14 sec | 73 |
| Amendments | 18,369 | 37 sec | 0.25 sec | 216 |
| Aesop's Fables | 191,945 | 3958 sec | 1.0 sec | 58 |
| Moby Dick | 1.2 million | 43 hours † | 7.6 sec | 79 |
| Bible | 4.0 million | 20 days † | 34 sec | 11 |
| Chromosome 11 | 7.1 million | 2 months † | 61 sec | 12,567 |
| Pi | 10 million | 4 months † | 84 sec | 14 |

† estimated

Lesson. Sorting to the rescue; enables new research.

Many, many, many other things enabled by fast sort and search!

## Summary

Binary search. Efficient algorithm to search a sorted array.

Mergesort. Efficient algorithm to sort an array.

Applications. Many, many, many things are enabled by fast sort and search.