Developing the Rendering Equations

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As stated, physically based rendering simulates the movement of light throughout an environment. It is important that we understand the units involved in measuring light. As we will see, it is sometimes useful to use different units depending on the application. This also provides us with mathematical framework for describing the rendering process.

We will assume geometric optics in our measurements. This means that we will use the particle theory of light. We can get away with this because most visual phenomenon can be modeled with this assumption in place, diffraction and interference being the notable exceptions. We will also assume that the speed of light is infinite, which implies that any simulation is in a steady state. This is usually appropriate since the time it takes light to travel in common scenes is not perceivable.

The following sections touch briefly on several important concepts, which are handled in much detail by Glassner [3].

1 Solid Angles

Key concepts in the radiometric definitions are the ideas of solid angle and projection. When we think of a solid angle we usually think of some object projected onto a unit sphere. This projection is the solid angle of the object as view from the center of the sphere (Figure 1). The units for solid angles are steradians, sr, which are actually unitless but are usually left in for clarity.

The relationship between a differential area on a sphere and the corresponding differential solid angle can be described in the following way: A differential area, dA, on a unit sphere is equal to its solid angle, $d\hat{\omega}$. If dA is on a non-unit sphere, then the difference between the two is an r^2 term where r is the radius of a sphere. In Figure 2 describes this in detail. Here we see two hemispheres. The inside hemisphere has $r = 1$. Since dA has a horizontal side of length $r \;\sin\theta \; d\phi$ and a vertical side of length $r \; d\theta$ the differential area is:

$$dA = r^2 \sin\theta \; d\theta \; d\phi \tag{1}$$

and the differential solid angle is: $d\hat{\omega} = \sin\theta \; d\theta \; d\phi$

2 Projections

The relationship between the area of surface element dA and the projection of that surface onto a plane is:

$$\text{proj}_A = \cos\theta \; dA \;, \tag{2}$$

as shown in Figure 3.

Finally, we can consider a differential area dA' which does not lie on a great sphere. Projecting this onto a sphere is equivalent to projecting it onto a plane which is perpendicular to the ray running

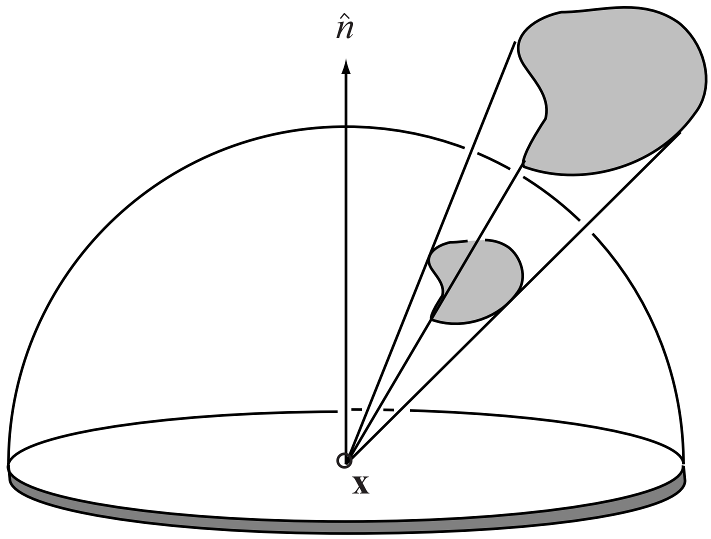

Figure 1: Solid Angle of an object viewed from $\mathbf{x}$

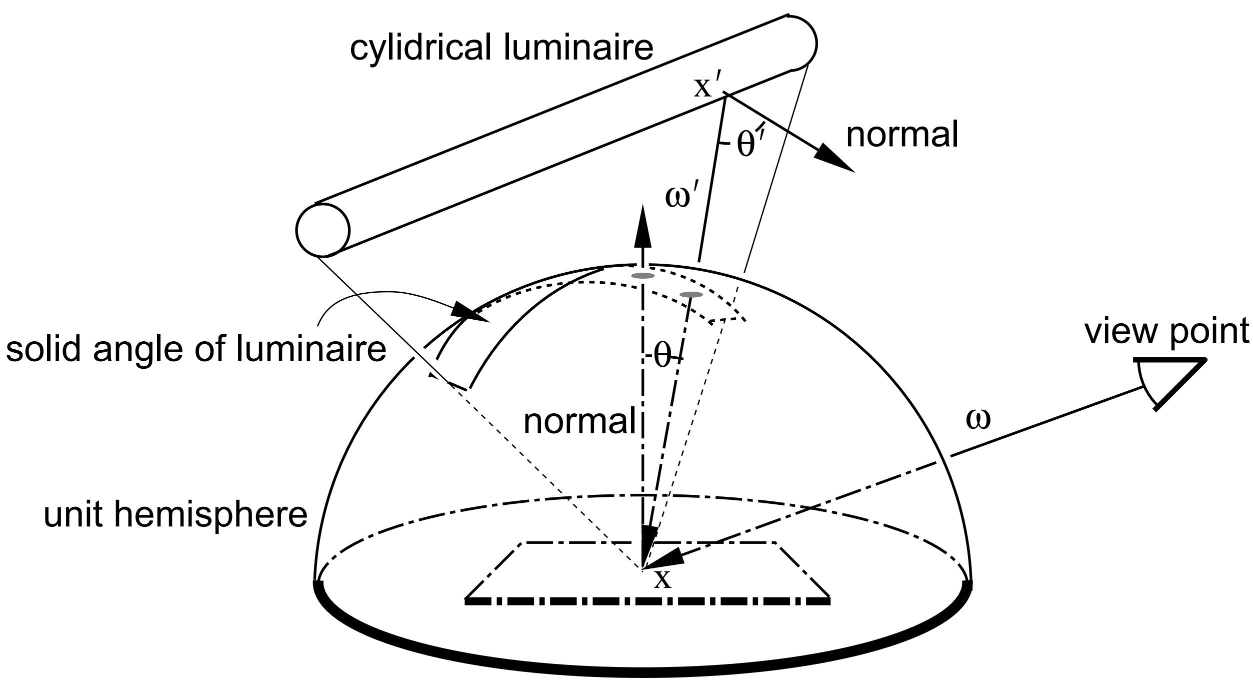

Figure 2: Relationship between area and solid angle on a sphere

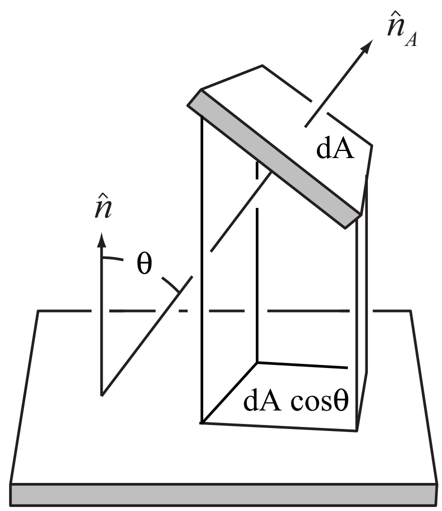

Figure 3: Projection of a surface element onto a plane

from the center of the sphere to the center of dA'. Thus from Equations 1, 3 and 2 we get the relationship between a differential solid angle $d\hat{\omega}'$ and an arbitrarily oriented differential area dA':

$$d\hat{\omega}' = \frac{dA' \cos\theta'}{||\mathbf{x}' - \mathbf{x}||^2} , \tag{3}$$

where $\mathbf{x}$ is the sphere center and $\mathbf{x}'$ is the center of dA'.

3 Radiometry

In general, physically based computer graphics algorithms do not chase light particles or photons around the environment. Usually the computational quantity of flow that is measured throughout an environment is *radiant flux* or *radiant power* which is generally denoted by the Greek letter Φ and measured in Watts. Radiant power has no meaning at a particular point in an environment, therefore we need different quantities to represent the interaction of radiant power and surfaces. The most important of these quantities is *radiance*.

4 Radiance

Radiance is a fundamental quantity usually associated with a light ray. The radiance leaving or arriving at a given point, $\mathbf{x}$, traveling in a given direction, $\hat{\omega}$, can be defined as the power per unit projected area perpendicular to the ray per unit solid angle in the direction of the ray. Following notation similar to the IES[1] standard we have:

$$L(\mathbf{x}, \hat{\omega}) = \frac{d^2\Phi(\mathbf{x}, \hat{\omega})}{dA \ \cos\theta \ d\hat{\omega}} , \tag{4}$$

[1]The Illumination Engineering Society or IES notation is the standard for illumination engineering. Notation and definitions can be found in the ANSI/IES report [5].

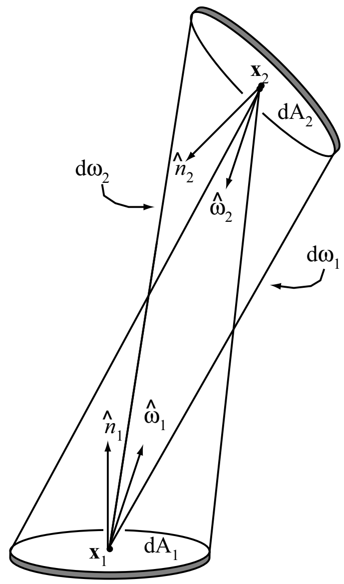

Figure 4: Radiance between differential surfaces.

where Φ is power, dA is the differential area surrounding $\mathbf{x}$, θ is the angle between the ray and the surface normal at $\mathbf{x}$, and $d\hat{\omega}$ is the differential solid angle in the direction of the ray.[2]

Radiance is a convenient quantity to associate with a light ray because it remains constant as it propagates along a direction (assuming a vacuum). To see that this is true we need to look closely at the definitions. We can reorganize the above definition in terms of radiant flux:

$$d\Phi(\mathbf{x}, \hat{\omega}) = L(\mathbf{x}, \hat{\omega}) \cos\theta d\hat{\omega} dA \ . \tag{5}$$

Using the geometry of Figure 4 and assuming a vacuum, the law of conservation of energy says that the flux leaving surface one in the direction of surface two, must arrive at surface two, more concisely:

$$d\Phi(\mathbf{x}_1, \hat{\omega}_1) = d\Phi(\mathbf{x}_2, \hat{\omega}_2) \ .$$

Thus

$$L(\mathbf{x}_1, \hat{\omega}_1) \cos\theta_1 d\hat{\omega}_1 dA_1 = L(\mathbf{x}_2, \hat{\omega}_2) \cos\theta_2 d\hat{\omega}_2 dA_2 \ . \tag{6}$$

From the previous definitions we see that $d\hat{\omega}_1 = (dA_2 \cos\theta_2)/r^2$ and $d\hat{\omega}_2 = (dA_1 \cos\theta_1)/r^2$ where $r^2 = \mathbf{x}_1 - \mathbf{x}_2^2$, $\theta_1 = (\hat{n}_1 \cdot \hat{\omega}_1)$ and $\theta_2 = (\hat{n}_2 \cdot \hat{\omega}_2)$. Dividing each side of Equation 6 by $dA_1(\cos\hat{\omega}_1 \ dA_2 \cos\hat{\omega}_2)/r^2$ we see that $L(\mathbf{x}_1, \hat{\omega}_1) = L(\mathbf{x}_2, \hat{\omega}_2)$. Notice that the definition of radiance lends itself to some confusion about the direction of flow. For this reason Arvo [1] uses the term *surface radiance*, $L_s(\mathbf{x}, \hat{\omega})$, to refer to light leaving $\mathbf{x}$ in direction $\hat{\omega}$ and *field radiance*, $L_f(\mathbf{x}, \hat{\omega})$, to refer to light arriving at $\mathbf{x}$ from direction $\hat{\omega}$.

Radiance is considered a fundamental quantity not only because it is convenient but because all other radiometric and photometric quantities can be derived from it as can be seen in the appendix.

[2]Note that Equation 4 should be written as a second order partial derivative in the form $\frac{\partial^2 \Phi}{\partial A \ \cos\theta \ \partial\hat{\omega}}$, but we will stick with convention.

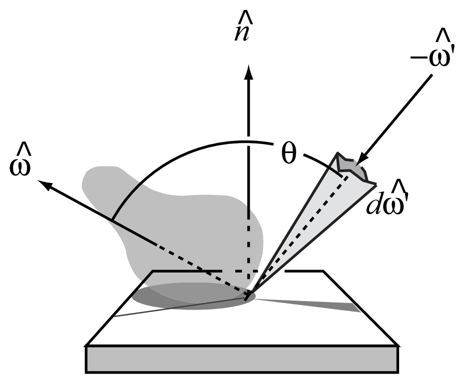

Figure 5: Geometry for BRDF.

5 BRDF and BTDF

Now that we have radiance to characterize the flow of light traveling between two surfaces a function is needed to describe the reflection of light off a surface. We would expect that the reflection of light off a surface is proportional to the light arriving at the surface. The function that describes this proportionality is the *bidirectional reflectance distribution function* or BRDF, Figure 5

$$f_r(\mathbf{x}, \hat{\omega}', \hat{\omega}) = \frac{dL_r(\mathbf{x}, \hat{\omega})}{L_f(\mathbf{x}, \hat{\omega}') \cos\theta d\hat{\omega}'} \; , \tag{7}$$

where L_f is the field radiance and L_r is the reflected radiance. Note that L_r is used instead of the surface radiance L_s. The reason for this distinction will become clear in the next section. Note also that the denominator of Equation 7 is irradiance as described in the appendix. A *physically plausible* BRDF maintains two important properties:

1. The BRDF must follow the *Helmholtz reciprocity principle*. This states that the BRDF will be the same if the incident and reflected light is reversed. Stated,

$$f_r(\mathbf{x}, \hat{\omega}', \hat{\omega}) = f_r(\mathbf{x}, \hat{\omega}, \hat{\omega}') \tag{8}$$

2. The BRDF must uphold the law of conservation of energy. Therefore the outgoing radiance must be less than or equal to the incoming radiance. If the BRDF is integrated over the hemisphere of reflected directions we will get the total reflectance for an incoming direction $\hat{\omega}'$. This value must be less than or equal to one:

$$R(\mathbf{x}, \hat{\omega}') = \int_\Omega f_r(\mathbf{x}, \hat{\omega}', \hat{\omega}) \cos\theta d\hat{\omega}' \leq 1.0 \; . \tag{9}$$

Several models for BRDF are described in Glassner [3] including the most commonly used models of Lambert and Phong, as well as more complicated models employing Fresnel equations and the empirical models of Ward [11]. An additional model which is not covered by Glassner but deserves mention is the modified Phong model of Lafortune and Willems [7]. Lafortune and Willems modify the Phong model so that it obeys the Helmholtz reciprocity principle. As pointed out by Shirley [10] it is difficult to tell whether or not it is necessary to have a physically plausible BRDF in order to produce realistic images.

For some surfaces that transmit light, the BRDF must be combined with the *bidirectional transmission distribution function*, BTDF. This allows us to render images of glass, lamp shades and ultra-thin metals.

6 The Rendering Equation

Previously, radiance was defined as means of expressing the light traveling between two surface. In the previous section, the BRDF was defined as the interaction of light with a surface. These two ideas can be combined to form an equation that describes the flow of light throughout an environment. Notice that by rewriting Equation 7 we get the following:

$$dL_r(\mathbf{x}, \hat{\omega}) = f_r(\mathbf{x}, \hat{\omega}', \hat{\omega}) L_f(\mathbf{x}, \hat{\omega}') \cos\theta d\hat{\omega}'$$

This is the reflected radiance in terms of the incoming radiance from one ray and the BRDF. The total reflected radiance at a point, $\mathbf{x}$, in direction, $\hat{\omega}$, combine with any emitted radiance, L_e, to form surface radiance, L_s:

$$L_s(\mathbf{x}, \hat{\omega}) = L_e(\mathbf{x}, \hat{\omega}) + \int_{\Omega_i} f_r(\mathbf{x}, \hat{\omega}', \hat{\omega}) L_f(\mathbf{x}, \hat{\omega}') \cos\theta d\hat{\omega}' \; , \tag{10}$$

where $\cos\theta = (\hat{n} \cdot -\hat{\omega}')$. This is the *rendering equation* in terms of directions as first introduced by Immel et al.[4]. Sometimes it is more convenient to express Equation 10 in terms of surfaces. We can do this by using the definition from Equation 3 to get:

$$L_s(\mathbf{x}, \hat{\omega}) = L_e(\mathbf{x}, \hat{\omega}) + \int_A g(\mathbf{x}, \mathbf{x}') f_r(\mathbf{x}, \hat{\omega}, \hat{\omega}') L_f(\mathbf{x}, \hat{\omega}') \frac{\cos\theta \cos\theta' dA}{||\mathbf{x}' - \mathbf{x}||^2} \; , \tag{11}$$

where $||\mathbf{x}' - \mathbf{x}||$ is the distance from $\mathbf{x}$ to $\mathbf{x}'$, $\cos\theta' = (\hat{n}' \cdot \hat{\omega}')$, and

$$g(\mathbf{x}, \mathbf{x}') = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is visible to } \mathbf{x}' \\ 0 & \text{otherwise .} \end{cases}$$

This geometry term is necessary since some surfaces might be blocked. Equation 11 is the form similar to that of Kajiya's landmark paper[6]. The geometry for the rendering equation can be seen in Figure 6.

We must keep in mind that $L_f(\mathbf{x}, \hat{\omega}') = L_s(\mathbf{x}', \hat{\omega}')$ in Equations 11 and 10 . By replacing L_f with L_s we see that Equations 11 and 10 are integral equations.

A Appendix: Radiometry and Photometry

This appendix was written in an attempt to clarify the relationship between radiometry and photometry. This clarification was necessary because our ray tracer associates a value of radiance with each ray traced. However, the illumination engineering community specifies luminaires with photometric values.

In order to use the value associated with a luminaire sample, we had to transform it into spectral radiance. It should be noted that in the literature the term *radiance* usually implies *spectral* radiance, averaged over a band of wavelengths (such as the red, green, or blue portions of the visible spectrum).

The first step was to understand the radiometric and photometric terminology according to ANSI/IES (1986)[5].

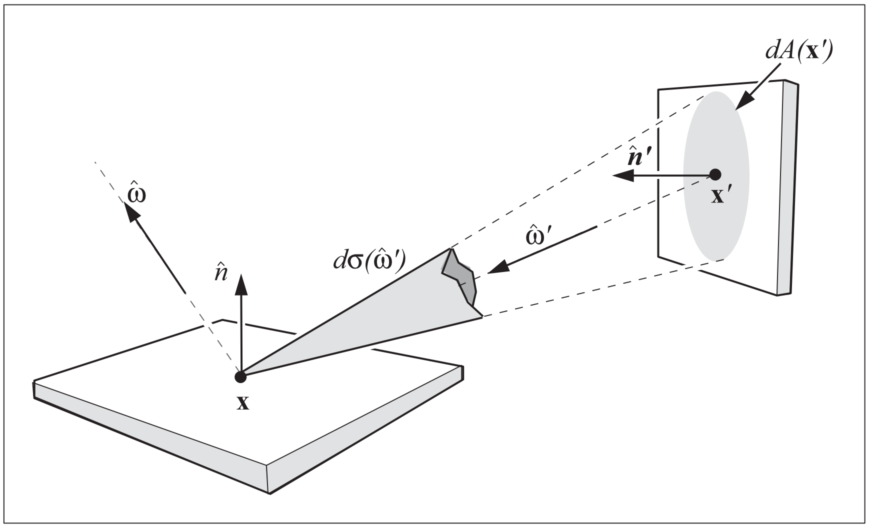

Figure 6: Geometry for the rendering equation

A.1 Important Radiometric Terms

1. **Radiant energy, Q.** Energy traveling in electro-magnetic waves, measured in joules.

   (a) **Spectral radiant energy, $Q_\lambda = dQ/d\lambda$,** measured in joules per nanometer, $joules/nm$.

2. **Radiant Flux (radiant power), $\Phi = dQ/dt$.** The time rate of flow of radiant energy, measured in joules per second or watts $= W$.

   (a) **Spectral Radiant Flux, $\Phi_\lambda = d\Phi/d\lambda$,** measured in W/nm.

3. **Radiant flux density, $d\Phi/dA$.** The quotient of the radiant flux incident on or emitted by a differential surface element dA at a point, divided by the area of the element. The preferred term for radiant flux density leaving a surface is exitance, M. The preferred term for radiant flux density incident on a surface is irradiance, E. Measured in watts per square meter, W/m^2.

   (a) **Spectral radiant flux density, $d\Phi_\lambda/(dA\ d\lambda)$.** In terms of exitance it is $M_\lambda/d\lambda$. In terms of irradiance it is $E_\lambda/d\lambda$. Measured in $W/(m^2\ nm)$.

4. **Radiant intensity, $I = d\Phi/d\omega$.** The radiant flux proceeding from a source per unit solid angle in a given direction. Measured in watts per steradian, W/sr.

   (a) **Spectral radiant intensity, $I_\lambda = dI/d\lambda$.** Measured in $W/(sr\ nm)$.

5. **Radiance, $L = d^2\Phi/[d\omega(dA\cos\theta)]$.** Power per unit projected area perpendicular to the ray per unit solid angle in the direction of the ray. Measured in $W/(m^2\ sr)$.

   (a) **Spectral radiance, L_λ.**
   $L_\lambda = d^3\Phi/[d\omega(dA\cos\theta)d\lambda]$. Measured in $W/(m^2\ sr\ nm)$.

A.2 Important Photometric Terms

Note that the symbols for radiometric and the corresponding photometric terms are the same. In cases where the terms might be confused radiometric terms will be identified by the subscript e and photometric terms will be identified by the subscript v.

1. **Luminous flux Φ.** Radiant flux evaluated in terms of a standardized visual response. Measured in lumens, lm.

$$\Phi_v = K_m \int_\Lambda \Phi_{e,\lambda} V(\lambda)\, d\lambda$$

   where

Φ_v	=	lumens
$\Phi_{e,\lambda}$	=	watts per nanometer
λ	=	nanometers
$V(\lambda)$	=	the spectral luminous efficiency
K_m	=	the spectral luminous efficacy in lumens per watt (lm/W)

   The above definition of luminous flux is for photopic vision and K_m has the value $683\, lm/W$. For scotopic vision $V(\lambda)$ is replaced by $V'(\lambda)$ and K_m is replaced by $K_{m'} = 1754\, lm/W$.

2. **Luminous flux density,** $d\Phi/dA$ This item is usually referred to as illuminance, E, if luminous flux density is incident on a surface element, and luminous exitance, M, if luminous flux density is leaving a surface element. Measured in lm/m^2

3. **Luminous intensity,** $I = d\Phi/d\omega$**.** The luminous flux per unit solid angle in a certain direction. Measured in lm/sr or candelas.

4. **Luminance,** $L = d^2\Phi/[d\omega(dA\cos\theta)]$**.** The definition is the same as radiance. The units are $lm/(m^2\ sr)$.

A.3 Deriving Everything from Radiance

All of the above definitions can be derived from spectral radiance. This is an important exercise which will help clarify the relationship between radiance and the other radiometric and photometric terms. In the following list, spectral radiance will be referred to as the function $L_e(x,\omega,\lambda)$.[3]

1. **Spectral Radiometry**

   - **Spectral radiant energy**

$$Q_{e,\lambda} = \int_T \int_\Omega \int_{x\in A} L_e(x,\omega,\lambda)\cos\theta\; dA\; d\omega\; dt$$

   - **Spectral radiant flux**

$$\Phi_{e,\lambda} = \int_\Omega \int_{x\in A} L_e(x,\omega,\lambda)\cos\theta\; dA\; d\omega$$

[3]We define only spectral radiometry since the corresponding radiometric terms can be found by integrating the spectral radiometric terms over the appropriate range of the light spectrum

- **Spectral radiant flux density** (in terms of irradiance)

$$E_{e,\lambda} = \int_{\Omega} L_e(x, \omega, \lambda) \cos\theta \; d\omega$$

- **Spectral radiant intensity**

$$I_{e,\lambda} = \int_{x \in A} L_e(x, \omega, \lambda) \; dA$$

2. **Photometry**

- **Luminous flux**

$$\Phi_v = K_m \int_{\Lambda} \int_{\Omega} \int_{x \in A} L_e(x, \omega, \lambda) V(\lambda) \cos\theta \; dA \; d\omega \; d\lambda$$

- **Luminous flux density**(in terms of illuminance)

$$E_v = K_m \int_{\Lambda} \int_{\Omega} L_e(x, \omega, \lambda) V(\lambda) \cos\theta \; d\omega \; d\lambda$$

- **Luminous intensity**

$$I_v = K_m \int_{\Lambda} \int_{x \in A} L_e(x, \omega, \lambda) V(\lambda) \; dA \; d\lambda$$

- **Luminance**

$$L_v = K_m \int_{\Lambda} L_e(x, \omega, \lambda) V(\lambda) \; d\lambda$$

A.4 IES Luminaires and Spectral Radiance

The IES photometric data file format [8] defines the three-dimensional distribution of light emitted by a luminaire. The distribution is defined for a point light source even though most luminaires are clearly not point sources. The file format specifies luminous intensities I_v for a set of vertical and horizontal directions, thus allowing for non-uniform distributions. To compute spectral radiance from this information we must make two assumptions: the distance from the luminaire to a point on the illuminated surface satisfies the "five-times" rule, and the spectral output of the luminaire is known. The five-times rule states that the luminaire can be modeled as a point source if distance from the luminaire to the point on the illuminated surface is greater than five times the maximum projected width of the luminaire as seen from the point. (In other words, the luminaire must not exceed a subtended angle of 0.2 radians as seen from the point.) If this rule is satisfied, the error for the predicted illuminance will be less than ± 1 percent [2].

The five-times rule allows us to model the luminaire as a photometrically homogeneous luminous aperture. That is, any point on the luminous surface of the luminaire will exhibit the same three-dimensional photometric distribution of luminous intensity as does the point source being used to represent the luminaire in the IES photometric data file.

Usually the type of lamp used in the luminaire will be defined in the IES file (although different lamps may be often be used when luminaire is installed). By maintaining a database of spectra that correspond to particular lamp types, we can satisfy the second assumption. Spectra from a number of generic lamp types are presented in the IES Lighting Handbook [9], while spectra for specific lamps are often available from the lamp manufacturers. These spectra are given in terms of watts per nanometer, or spectral radiant flux ($\Phi_{e,\lambda}$). This allows us to derive the spectral radiant exitance $L_{e,\lambda}$ as follows:

The known quantities are luminous intensity $I_v = d\Phi_v/d\omega$, spectral radiant flux $\Phi_{e,\lambda}$, the maximum spectral luminous efficacy $K_m = 683$, and the photopic luminous efficiency curve $V(\lambda)$. The goal is spectral radiance $L_{e,\lambda}$.

Based on our assumption that the luminous surface of the luminaire is photometrically homogeneous, we have:

$$L_{e,\lambda} = \frac{d\,I_{e,\lambda}}{d\,A\cos\theta} = \frac{I_{e,\lambda}}{A\cos\theta} \tag{12}$$

where A is the luminous surface area of the luminaire as seen from the point on the illuminated surface and θ is the mean angle between the luminous surface normal and the direction of the point. (Remember that we are modeling the luminaire as a point source.) Therefore, we will have a solution for $L_{e,\lambda}$ if we can solve for the spectral radiant intensity $I_{e,\lambda}$.

We also have:

$$L_v = \frac{dI_v}{dA\cos\theta} = \frac{I_v}{A\cos\theta} \tag{13}$$

Now it is evident that the luminance L_v at the point on the surface is directly proportional to the amount of luminous flux Φ_v received at that point. The same argument must therefore hold for spectral radiance: $L_{e,\lambda}$ is directly proportional to the spectral radiant flux $\Phi_{e,\lambda}$. This gives us:

$$\frac{L_{e,\lambda}}{L_v} = \frac{\Phi_{e,\lambda}}{\Phi_v} \tag{14}$$

Rearranging terms gives us:

$$L_{e,\lambda} = \frac{L_v\Phi_{e,\lambda}}{\Phi_v} = \frac{I_v\Phi_{e,\lambda}}{(A\cos\theta)\Phi_v} \tag{15}$$

However:

$$\Phi_v = K_m \int_\lambda \Phi_{e,\lambda} V(\lambda)\ d\lambda \tag{16}$$

and so spectral radiance can be defined as:

$$L_{e,\lambda} = \frac{I_v\Phi_{e,\lambda}}{(A\cos\theta)K_m \int_\lambda \Phi_{e,\lambda} V(\lambda)\ d\lambda} \tag{17}$$

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