Laplacian Mesh Editing

COS 526 – Fall 2010

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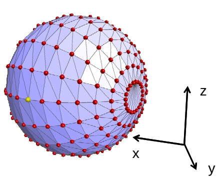
(most of the) Slides borrowed from: Olga Sorkine

Outline

- Differential surface representation
- Ideas and applications
  - Compact shape representation
  - Mesh editing and manipulation
  - Membrane and flattening
  - Generalizing Fourier basis for surfaces

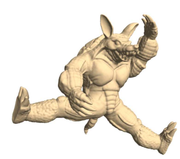
Motivation

- Meshes are great, but:
  - Geometry is represented in a *global* coordinate system
    - Single Cartesian coordinate of a vertex doesn't say much



Differential coordinates

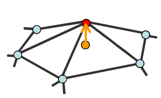
- Represent a point ***relative*** to it's neighbors.
- Represent ***local detail*** at each surface point
  - better describe the shape
- Linear transition from global to differential
- Useful for operations on surfaces where surface details are important



Differential coordinates

"Local control for mesh morphing", Alexa 01

- Detail = surface – *smooth*(surface)
- Smoothing = averaging

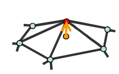


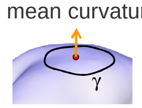
$$\boldsymbol{\delta}_i = \mathbf{v}_i - \frac{1}{d_i}\sum_{j\in N(i)} \mathbf{v}_j$$

$$\boldsymbol{\delta}_i = \sum_{j\in N(i)} \frac{1}{d_i}\left(\mathbf{v}_i - \mathbf{v}_j\right)$$

Connection to the smooth case

- The direction of $\boldsymbol{\delta}_i$ approximates the normal
- The size approximates the mean curvature





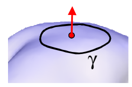
$$\boldsymbol{\delta}_\mathbf{i} = \frac{1}{d_i}\sum_{\mathbf{v}\in N(i)}\left(\mathbf{v}_\mathbf{i} - \mathbf{v}\right) \qquad \frac{1}{len(\gamma)}\int_{\mathbf{v}\in\gamma}\left(\mathbf{v}_\mathbf{i} - \mathbf{v}\right)ds$$

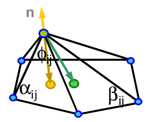
$$\lim_{len(\gamma)\to 0}\frac{1}{len(\gamma)}\int_{\mathbf{v}\in\gamma}\left(\mathbf{v}_\mathbf{i} - \mathbf{v}\right)ds = H(\mathbf{v}_\mathbf{i})\mathbf{n}_\mathbf{i}$$

Weighting schemes

$$\delta_i = \frac{\sum_{j\in N(i)} w_{ij}\left(\mathbf{v}_i - \mathbf{v}_j\right)}{\sum_{j\in N(i)} w_{ij}}$$

- Ignore geometry
  $\delta_{\text{umbrella}} : w_{ij} = 1$
- Integrate over circle around vertex
  $\delta_{\text{mean value}} : w_{ij} = \tan \phi_{ij}/2 + \tan \phi_{ij+1}/2$
- Integrate over Voronoi region of vertex
  $\delta_{\text{cotangent}} : w_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}$

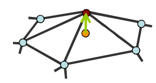




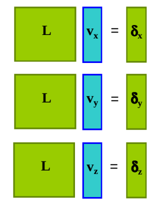
Laplacian matrix

[Taubin 95]

- The transition between the δ and xyz is linear:

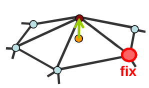


$$\delta_i = \sum_{j\in N(i)} w_{ij}\left(\mathbf{v}_i - \mathbf{v}_j\right)$$

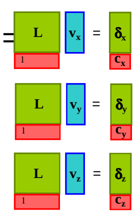


Basic properties

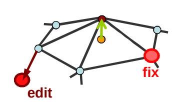
- $\mathrm{rank}(L) = n - c$ $(n - 1$ for connected meshes)
- We can reconstruct the xyz geometry from δ up to translation

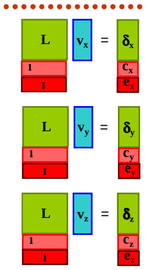


$L\mathbf{x} =$

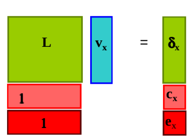


Reconstruction



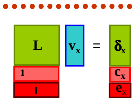


Reconstruction



$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \left(\left\| L\mathbf{x} - \boldsymbol{\delta}_x \right\|^2 + \sum_{s=1}^{k} \left| x_k - c_k \right|^2 \right)$$

Reconstruction



$$\mathbf{A}\ \mathbf{x} = \mathbf{b}$$

Normal Equations:

$$\mathbf{A}^T\mathbf{A}\ \mathbf{x} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{x} = \underbrace{(\mathbf{A}^T\mathbf{A})^{-1}}_{\text{compute once}}\ \mathbf{A}^T\mathbf{b}$$

What we have so far

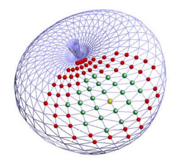
- Laplacian coordinates δ
  - Local representation
  - Translation-invariant
- Linear transition from δ to *xyz*
  - can constrain more that 1 vertex
  - least-squares solution

Editing using differential coordinates

- The editing process from the user's point of view:

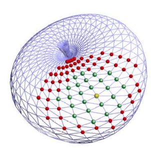
1) First, a ROI , anchors and a handle vertex should be set.

2) Then the edit is Performed By moving this vertex.

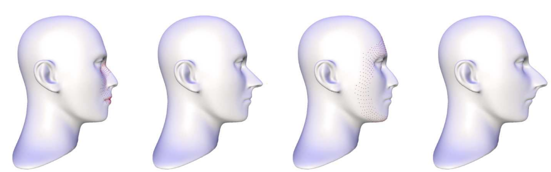


Editing using differential coordinates

- The user moves the handle and interactively the surface changes.
- The stationary anchors are responsible for smooth transition of the edited part to the rest of the mesh.
- This is done using increasing weight with geodesic distance in the soft spatial equations.

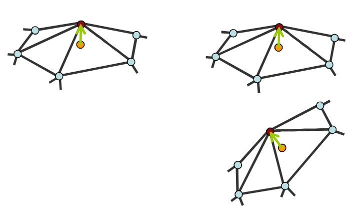


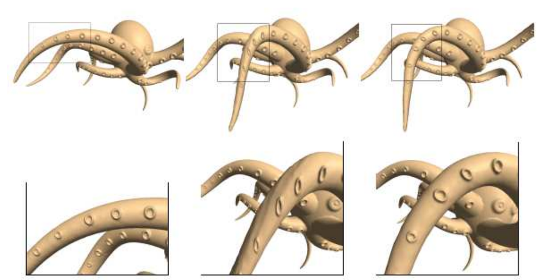
Example



Is that the state of the art?

- Laplacians are translation invariant but NOT rotation invariant.



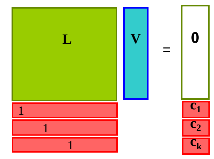


What else can we do with it?

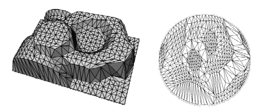
- Membrane 2D (flattening), 3D Shape interpolation.
- Compact shape representation
  - Quantization
  - New bases for geometry rep.
- “Good” function basis over the surface.

Membrane surfaces

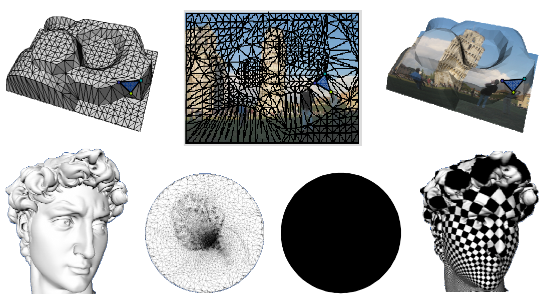
- Use zero Laplacians.

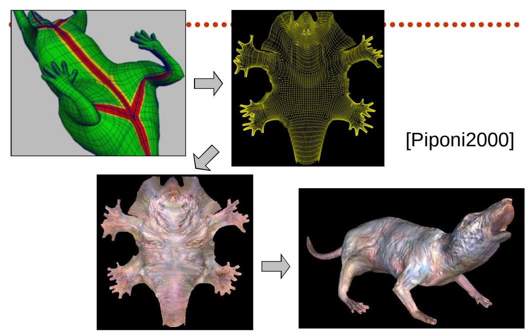


In 2D:



Texture Mapping

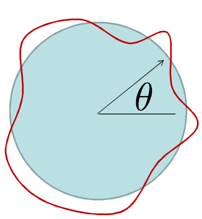




[Piponi2000]

Geometrical function basis

- Harmonics: eigenfunctions of the Laplacian.



$$\frac{d^2}{d\theta^2} f(\theta) = \lambda f(\theta)$$

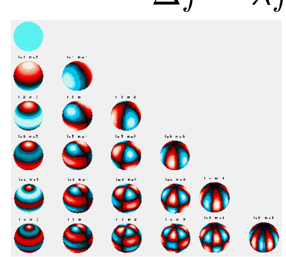
$$\begin{cases} f(\theta) = \sin(k\theta) \\ f(\theta) = \cos(k\theta) \end{cases}$$

Good basis for function over the circle:

$$g(\theta) = A_0 + \sum_{k=1}^{\infty} A_n \cos(k\theta) + \sum_{k=1}^{\infty} B_n \sin(k\theta)$$

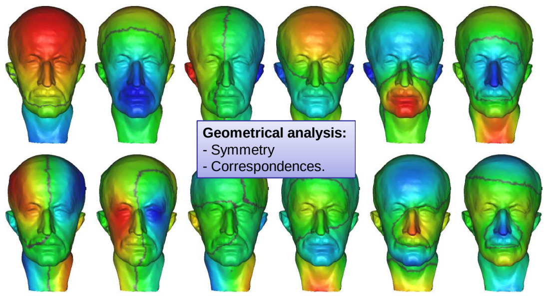
Geometrical function basis

- Spherical harmonics. $\Delta f = \lambda f$

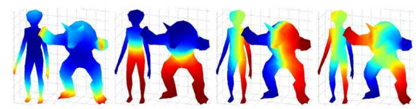


Geometrical function basis

$L f = \lambda f$



Correspondences between surfaces



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