COS 487: Theory of Computation	**Fall 2010**

Mid-Term Exam

Due: Monday, November 8 — *Sanjeev Arora*

Instructions: Give yourself 5 hours to do the test. Write answers legibly in the space provided. If you need extra space for an answer, you may attach extra sheets, *which the instructor will read at his discretion.* You can consult any notes/handouts from this class as well as the text. You cannot consult another person or source in any way. You can however Feel free to quote, without proof, any results proved in class or the text.
DO NOT READ THE TEST BEFORE YOU ARE READY TO WORK ON IT.

Your Name:

Write and sign the honor code pledge[1] here:

Prof. Arora will be available during this week by appointment. If you have questions during the exam you can try to reach him between 10am and 10pm at 609-356-9076 or Chris Beck at 203-984-8289. Both will also check email frequently and answer it as quickly as possible.

[1] The pledge is "I pledge my honor that I have not violated the honor code during this exam."

Question 1 (25 points) This question lists five languages over $\{0,1\}$. For each classify it in one of the following classes and give brief explanations: *Class 1:* Regular. *Class 2:* Context-free but not regular. *Class 3:* Decidable but not context-free. *Class 4:* Recognizable but not Decidable. *Class 5:* Not recognizable.

(i) (5 points) $L = \{a^i b^j c^k \mid i \cdot j = k\}$.

(ii) (5 points) Fix a number n. $L = \{a^i b^j c^k \mid i \cdot j = k \bmod n\}$.

(iii) (5 points) $L = \{x\#y \mid x, y \in \{0,1\}^*, x \neq y\}$.

(iv) (5 points) $L = \{\langle M \rangle \mid M$ is a pushdown automaton, and $L(M) \cap 1^*$ is not regular $\}$.

(v) (5 points) If M is a TM, say that a pair of states (q_1, q_2) is a potentially infinite pair if, on some input w, M transitions $q_1 \to q_2$ an infinite number of times.
$L = \{\langle M, q_1, q_2 \rangle \mid M \text{ is a turing machine with } (q_1, q_2) \text{ a potentially infinite pair}\}$.

Question 2 (25 points) In the first four parts of this question, answer *yes*, *no*, or *unknown / open*. Give a short justification of your answer. No credit will be given if the justification is wrong. For V a language over $\{0, 1, \#\}$, define

$$L_V = \{x \in \{0,1\}^* \mid \exists y \in \{0,1\}^*, x\#y \in V\} \ .$$

(i) (5 points) If V is regular does this imply L_V is regular?

(ii) (5 points) If V is context free does this imply L_V is context free?

(iii) (5 points) If V is decidable does this imply L_V is decidable?

(iv) (5 points) If $V \in \mathbf{P}$ does this imply $L_V \in \mathbf{P}$?

(v) (5 points) List two philosophical implications of the Church Turing Thesis that were either discussed in class or which occured to you later.

Question 3 (25 points)

As in homework 3, suppose that an advanced civilization has discovered a way to decide the halting problem.

Suppose that now, two-way communication is possible: any computer on earth may send a string $\langle M, w \rangle$ to the aliens and receive back (within 4 years) a correct answer to whether or not M halts on string w. Describe a language that is still undecidable by earth's computers (i.e., Turing Machines) with this extra capabillity, and prove your answer.

(Extra credit (5 points): Suppose the aliens then discover a way to decide this language as well. Can you describe yet another language that remains undecidable after that?)

Question 4 (25 points) Say that a real number α between 0 and 1 is *computable* if a Turing machine may output arbirarily good approximations of it – there exists a machine M_α which, on input 1^n, outputs the first n binary digits of α. (Note that e, π etc are computable according to this definition.)

For L a language over $\{0,1\}$, we define

$$\Phi_L = \sum_{x \in L} 2^{-(2|x|+1)}.$$

Note that since there are 2^n strings of length n, Φ_L is always at most 1. You can think of Φ_L as a probability that a random string is in L, when we choose strings according to the following process:

- First, choose a number $n \geq 0$ according to the distribution $\Pr[x] = 2^{-(1+x)}$.
- Then, choose a string of length n uniformly at random.

Show that if L is the halting problem, then Φ_L is not a computable number.

THAT'S ALL FOLKS. GOOD LUCK. NEXT QUESTION IS OPTIONAL.

Question 5 (Extra Credit) (25 points; you can work on this question after the alloted 5 hours) Let L be a regular language. Prove that Φ_L is rational.