COS 597C: How to Solve it **Fall 2009**

Assignment #5

Due: Thursday, December 3 *Sanjeev Arora*

Please be succinct, and give a proof sketch rather than gory details.

1. Show that most boolean functions on n bits require circuits of size $\Omega(2^n/n)$. (Assume circuits are composed of AND, OR, and NOT gates, and maximum fanin of a gate is 2.)

2. A *sorting circuit* has inputs that are numbers, and each gate is a *comparator*. Given two numbers as input it produces two numbers as output. The input a, b produces output a, b if $a \leq b$ and otherwise produces output b, a. Describe a sorting circuit of size $O(n \log^2 n)$ that sorts n numbers.

3. (Disproof of direct sum conjecture for circuits) Use the previous questions to describe a boolean function f such that for sufficiently large k, solving k instances of f requires only circuits of size $O(\log^2 k)$ times the size required for a single instance.

4. For any matrix A with nonnegative entries we can define two games as follows. Game 1: Player I plays a distribution x on the rows, and Player II responds with a single column j, and then player I pays $\sum_i x_i A_{ij}$ to player II. Game 2: Player II plays a distribution x on the rows, and Player I responds with a single row i, and then player I pays $\sum_j y_j A_{ij}$ to player II. Show that if the players play optimally then the two games have the same payoff.

   Formalize the game "Rock Paper Scissor" using this formalism and report the optimum payoff.

5. For any $2^n \times 2^n$ matrix M with $+1, -1$ entries show that the discrepancy of a rectangle $A \times B$ is at most $\lambda_{max}(M)\sqrt{|A|\,|B|}/2^{2n}$. Compute the maximum discrepancy of any rectangle when M is the *inner product* function which maps $x, y \in GF(2)^n$ to the mod-2 sum $\sum_i x_i y_i$.