Homework 3

Out: *Nov19* Due: *Dec 1*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince me. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 A *cycle cover* of a graph (directed or undirected) is a collection of cycles such that every vertex is in exactly one cycle. Describe a polynomial-time algorithm to decide whether or not a cycle cover exists.

§2 Suppose we wish to solve the following flow problem. There are n nodes in an undirected graph, and the edges should be thought of as pipes of a certain capacity. For each pair $\{i, j\}$ we wish to send 1 unit of fluid between them. All these flows must be routed through the pipes and should not violate any capacity. Let z be the minimum number such that if all edge pipes have capacity z then the flows can be routed in the network. Express the problem of finding z as a linear program, and argue that it can be solved in polynomial time.

§3 Use the multiplicative weights method to design a algorithm that solves the above linear program approximately. How long does your algorithm take to find z correctly up to an additive error $\epsilon > 0$?

§4 For a 5-regular undirected graph $G = (V, E)$ we define the *conductance* $\Phi(G)$ to be:

$$\min_{S \subseteq V} \frac{\left|E(S, \overline{S})\right|}{|S| \cdot \left|\overline{S}\right|}.$$

(a) Show that the diameter of the graph is $O(\log n / n\Phi(G))$. (Hint: Consider doing a breadth first search from any point. How many levels does it take to double the number of nodes that are in the BFS tree?)

(b) Show that if z_{OPT} is the optimum value of LP in the previous question then $z_{OPT} \geq 1/\Phi(G)$.

(c) Show that the dual of the LP in the previous question is equivalent to the following problem. We assign a nonnegative weights w_e to each edge e. This induces a distance function among the vertices, and we let d_{ij} be the distance between i, j. We have to minimize the sum of the w_e's while keeping $\sum_{i<j} d_{ij} = 1$.

(d) Show that $z_{OPT} \leq O(\log n / \Phi(G))$. (Hint: Consider the weighted graph defined by the dual weights w_e. Replace edge e with a sequence of unweighted edges and consider the conductance of this graph. You have to do other things too.)

§5 Let L be the normalized Laplacian of a graph, namely the matrix whose diagonal entries are 1 and the offdiagonal entry (i,j) is $-1/\sqrt{d_i d_j}$. Show that this matrix is positive semidefinite, has all its eigenvalues between 0 and 2, and the second eigenvalue is the minimum over all $x \in \Re^n$ satisfying $\sum_i d_i x_i = 0$ of the following expression:

$$\frac{\sum_{\{i,j\}\in E}(x_i - x_j)^2}{\sum_i d_i x_i^2}.$$

Show that if the graph is 5-regular then second eigenvalue is upper bounded by $O(n\Phi(G))$.

§6 In class we defined zero-sum two-person games, which are given by a single payoff matrix A. Now consider a game specified by two matrices A, B that are $n \times m$. If player 1 makes move $i \in \{1, \ldots, n\}$ and player 2 makes a move $j \in \{1, \ldots, m\}$ then player 1 wins A_{ij} and player 2 wins B_{ij}. (The zero-sum game is a special case where $A = -B$.) Show that there exist two mixed strategies p_1, p_2 for the two players such that each is a profit maximizing response to the other. (This pair of strategies is called a *Nash equilibrium*, the discovery for which John Nash won a Nobel prize in economics.)

You can use Brouwer's fixed theorem, which says that if S is a convex compact set in $\Re^n$ and f is a continuous function from S to S then f has a *fixed point*, namely, some $x \in S$ such that $f(x) = x$.

§7 In the *ultimatum game* Kid 1 is asked to share 4 cupcakes with Kid 2. Kid 1 has to suggest a nonzero and integral amount x he is are willing to give to the other. Kid 2 has to suggest a minimum amount y he is willing to accept. If $x \geq y$ then Kid 1 gives x cupcakes to Kid 2. If $x < y$ then neither gets anything.

Write matrices A, B for this game and compute a Nash Equilibrium. Is the equilibrium unique? If not, can you compute another?

§8 Write an SDP relaxation for the MAX-2SAT problem. Use a Goemans-Williamson type rounding algorithm to achieve an approximation ratio better than the 3/4 we achieved in class using an LP relaxation. Improve it as much as you can.