Geometric Search

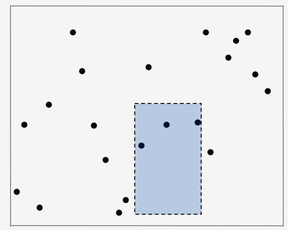
- range search
- space partitioning trees
- intersection search

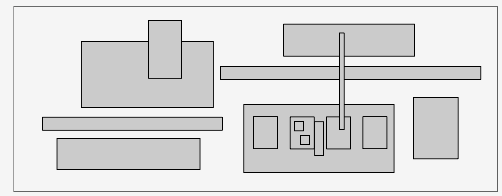
References:
Algorithms in C (2nd edition), Chapters 26-27
http://www.cs.princeton.edu/algs4/73range
http://www.cs.princeton.edu/algs4/74intersection

Overview

Geometric objects. Points, lines, intervals, circles, rectangles, polygons, ...
This lecture. Intersection among N objects.

Example problems.
- 1D range search.
- 2D range search.
- Find all intersections among h-v line segments.
- Find all intersections among h-v rectangles.





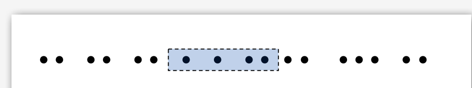
- range search
- space partitioning trees
- intersection search

1D range search

Extension of ordered symbol table.
- Insert key-value pair.
- Search for key k.
- Rank: how many keys less than k?
- Range count: how many keys between k_1 and k_2?
- Range search: find all keys between k_1 and k_2.

Application. Database queries.

Geometric interpretation.
- Keys are point on a line.
- How many points in a given interval?



insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
count G to K	2
search G to K	H I

1D range search: implementations

Ordered array. Slow insert, binary search for `lo` and `hi` to find range.
Hash table. No reasonable algorithm (key order lost in hash).

data structure	insert	rank	range count	range search
ordered array	N	log N	log N	R + log N
hash table	1	N	N	N
BST	log N	log N	log N	R + log N

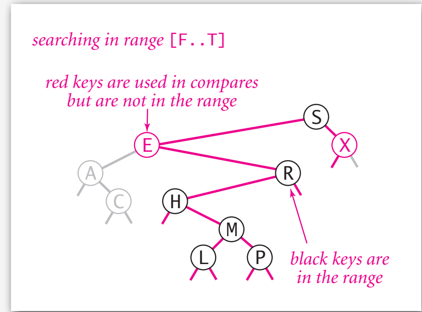
N = # keys
R = # keys that match

BST. All operations fast.

1D range search: BST implementation

Range search. Find all keys between `lo` and `hi`?

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).

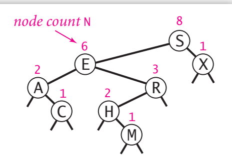


Worst-case running time. R + log N (assuming BST is balanced).

1D range count: BST implementation

Rank. How many keys < k ?

```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) return rank(key, x.left);
   else if (cmp > 0) return 1 + size(x.left) + rank(key, x.right);
   else              return size(x.left);
}
```



Range count. How many keys between `lo` and `hi`?

```
public int rangeCount(Key lo, Key hi)
{
   if (contains(hi)) return rank(hi) - rank(lo) + 1;
   else              return rank(hi) - rank(lo);
}
```


2D orthogonal range search

Extension of ordered symbol-table to 2D keys.

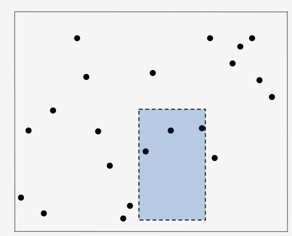
- Insert a 2D key.
- Search for a 2D key.
- Range count: how many keys lie in a 2D range?
- Range search: find all keys that lie in a 2D range?

Applications. Networking, circuit design, databases.

Geometric interpretation.

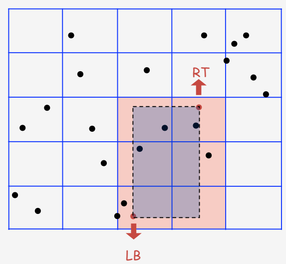
- Keys are point in the plane.
- How many points in a given h-v rectangle.

rectangle is axis-aligned



2D orthogonal range search: grid implementation

Grid implementation. [Sedgewick 3.18]
- Divide space into M-by-M grid of squares.
- Create list of points contained in each square.
- Use 2D array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only those squares that intersect 2D range query.

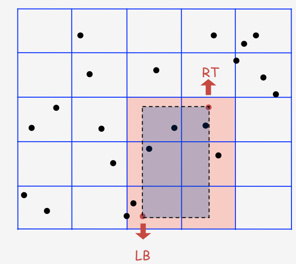


2D orthogonal range search: grid implementation costs

Space-time tradeoff.
- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

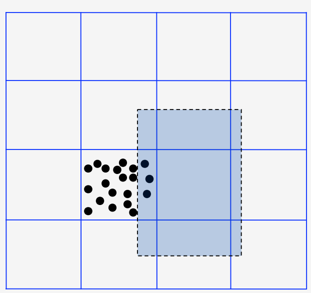
Choose grid square size to tune performance.
- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]
- Initialize: O(N).
- Insert: O(1). $M \approx \sqrt{N}$
- Range: O(1) per point in range.



Clustering

Grid implementation. Fast, simple solution for well-distributed points.
Problem. Clustering a well-known phenomenon in geometric data.



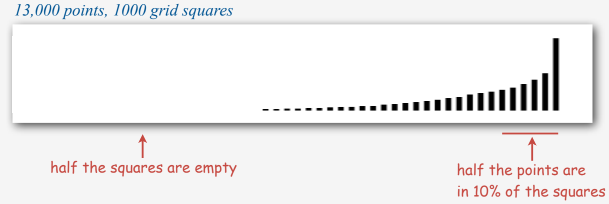
Lists are too long, even though average length is short.
Need data structure that gracefully adapts to data.

Clustering

Grid implementation. Fast, simple solution for well-distributed points.
Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.

13,000 points, 1000 grid squares

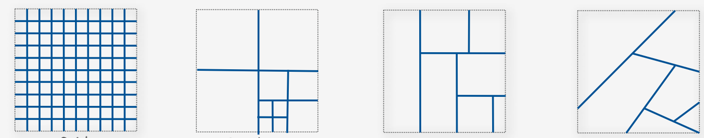


- range search
- **space partitioning trees**
- intersection search

Space-partitioning trees

Use a tree to represent a recursive subdivision of 2D space.

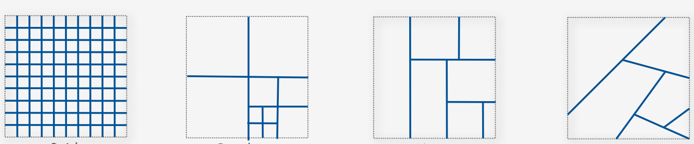
Quadtree. Recursively divide space into four quadrants.
2D tree. Recursively divide space into two halfplanes.
BSP tree. Recursively divide space into two regions.



Grid   Quadtree   2D tree   BSP tree

Space-partitioning trees: applications

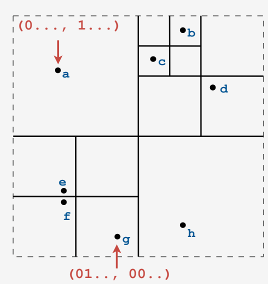
Applications.
- Ray tracing.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.

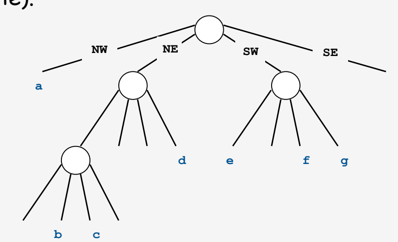


Grid   Quadtree   2D tree   BSP tree

Quadtree

Idea. Recursively divide space into 4 quadrants.
Implementation. 4-way tree (actually a trie).

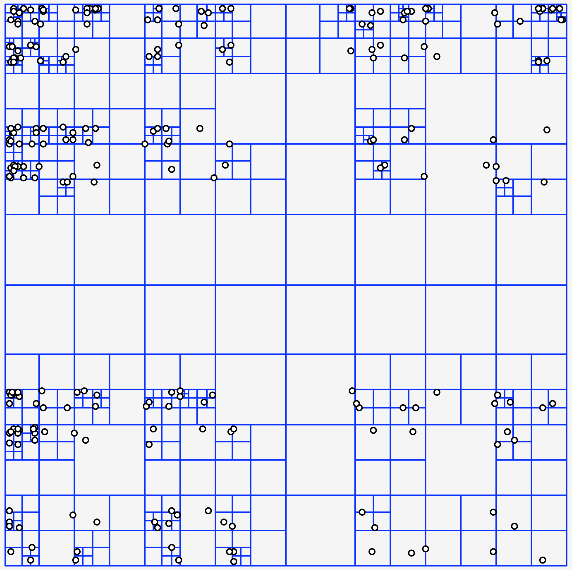





```
public class QuadTree
{
   private Quad quad;
   private Value val;
   private QuadTree NW, NE, SW, SE;
}
```

Benefit. Good performance in the presence of clustering.
Drawback. Arbitrary depth!

Quadtree: larger example

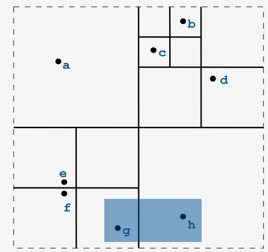


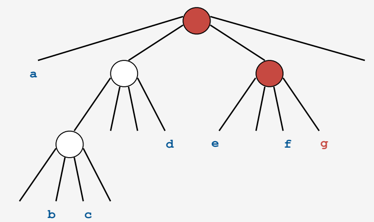
http://en.wikipedia.org/wiki/Image:Point_quadtree.svg

Quadtree: 2D range search

Range search. Find all keys in a given 2D range.

- Recursively find all keys in NE quad (if any could fall in range).
- Recursively find all keys in NW quad (if any could fall in range).
- Recursively find all keys in SE quad (if any could fall in range).
- Recursively find all keys in SW quad (if any could fall in range).

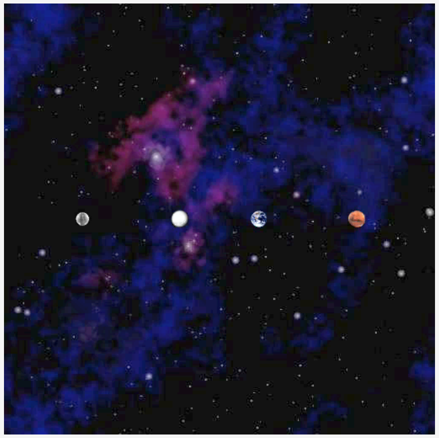




Typical running time. R + log N.

N-body simulation

Goal. Simulate the motion of N particles, mutually affected by gravity.

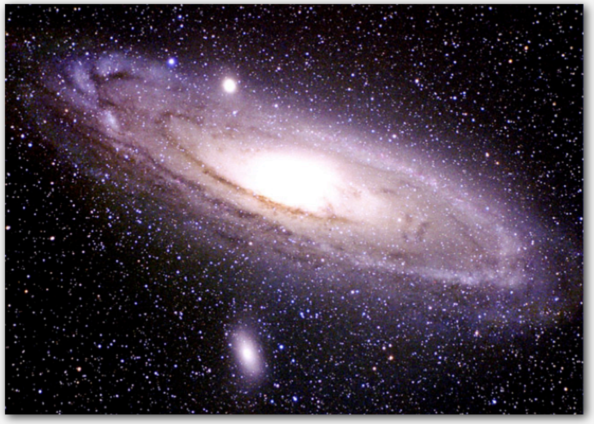


Brute force. For each pair of particles, compute force. $F = \frac{G\, m_1\, m_2}{r^2}$

Subquadratic N-body simulation

Key idea. Suppose particle is far, far away from cluster of particles.

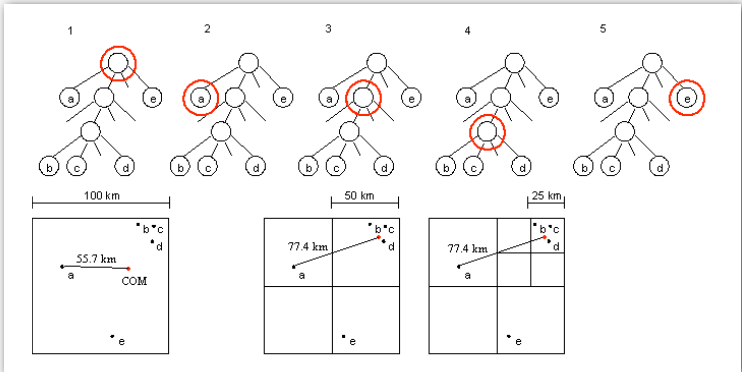
- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and center of mass of aggregate particle.



Barnes-Hut algorithm

Algorithm.

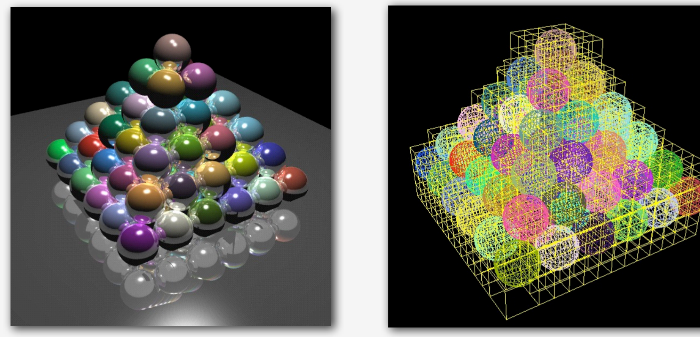
- Build quadtree with N particles as external nodes.
- Store center-of-mass of subtree in each internal node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to quad is sufficiently large.



Curse of dimensionality

Range search / nearest neighbor in k dimensions?
Main application. Multi-dimensional databases.

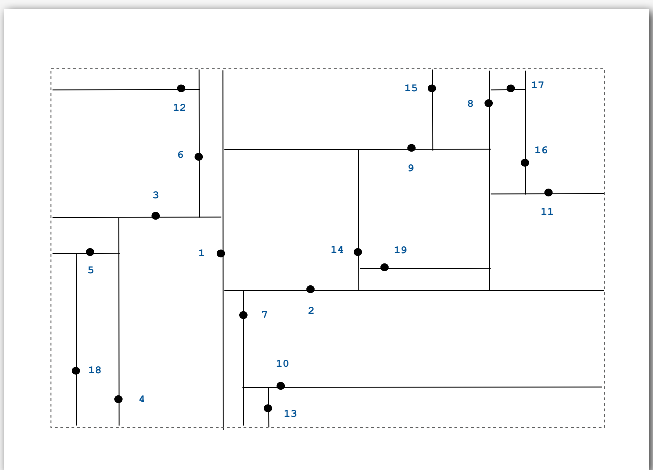
3D space. Octrees: recursively divide 3D space into 8 octants.
100D space. Centrees: recursively divide 100D space into 2^{100} centrants???



Raytracing with octrees
http://graphics.cs.ucdavis.edu/~gregorsk/graphics/275.html

2D tree

Recursively partition plane into two halfplanes.

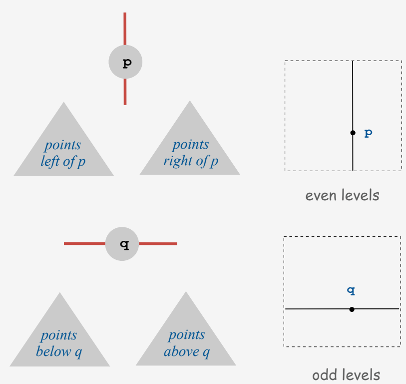


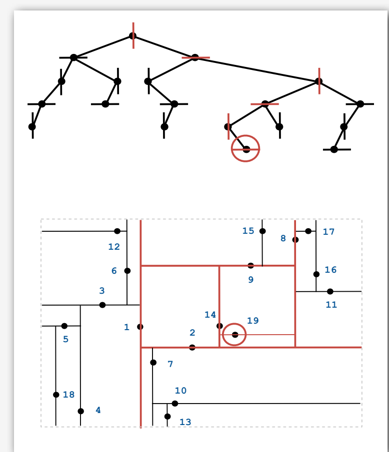
2D tree

Recursively partition plane into two halfplanes.

Implementation. BST, but alternate using x- and y-coordinates as key.

- Search gives rectangle containing point.
- Insert further subdivides the plane.

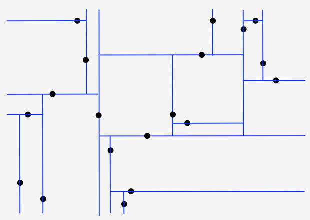


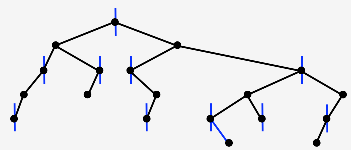


2D tree: 2D range search

Range search. Find all keys in a given 2D range.

- Check if point in node lies in given range.
- Recursively find all keys in left/top subdivision (if any could fall in range).
- Recursively find all keys in right/bottom subdivision (if any could fall in range).





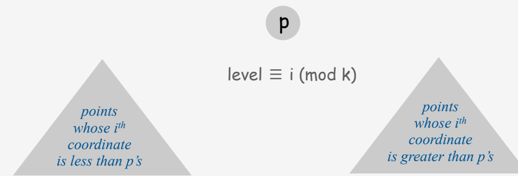
Worst case (assuming tree is balanced). $R + \sqrt{N}$.

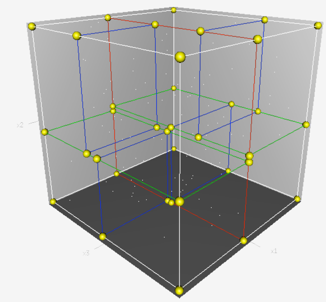
Typical case. $R + \log N$

kD Tree

kD tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2D trees.



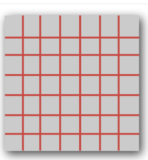
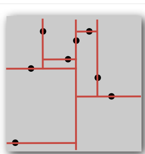
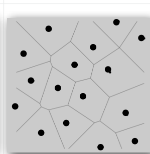
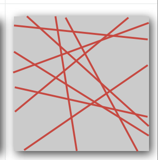


Efficient, simple data structure for processing k-dimensional data.

- Widely used.
- Discovered by an undergrad in an algorithms class!
- Adapts well to high-dimensional and clustered data.

Summary

Basis of many geometric algorithms. Search in a planar subdivision.

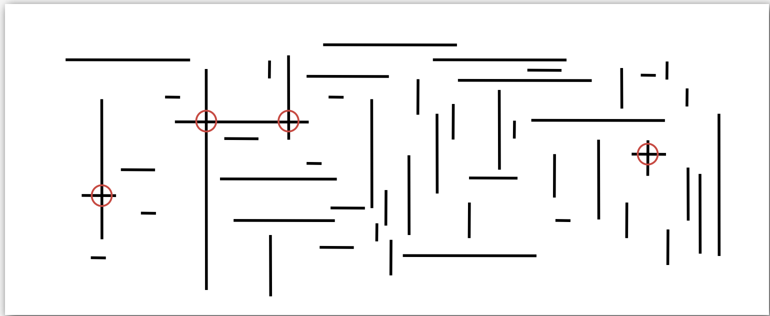
	grid	2D tree	Voronoi diagram	intersecting lines
basis	√N h-v lines	N points	N points	√N lines
representation	2D array of N lists	N-node BST	N-node multilist	~N-node BST
cells	~N squares	N rectangles	N polygons	~N triangles
search cost	1	log N	log N	log N
extends to kD	too many cells	easy	cells too complicated	use (k-1)D hyperplane
picture				

- range search
- space partitioning trees
- **intersection search**

Search for intersections

Problem. Find all intersecting pairs among N geometric objects.
Applications. CAD, games, movies, virtual reality.

Simple version. 2D, all objects are horizontal or vertical line segments.

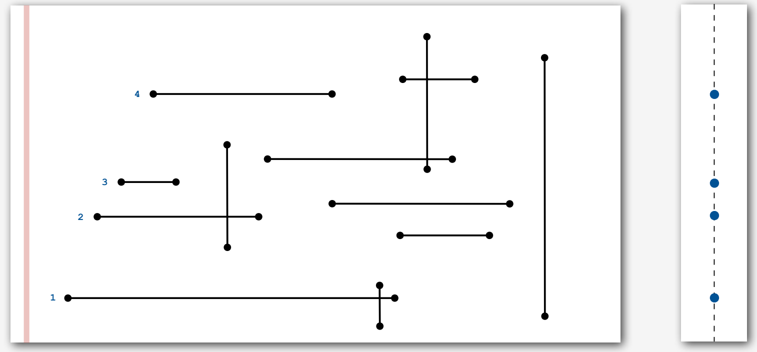


Brute force. Test all $\Theta(N^2)$ pairs of line segments for intersection.

Orthogonal segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- Left endpoint of h-segment: insert y-coordinate into ST.

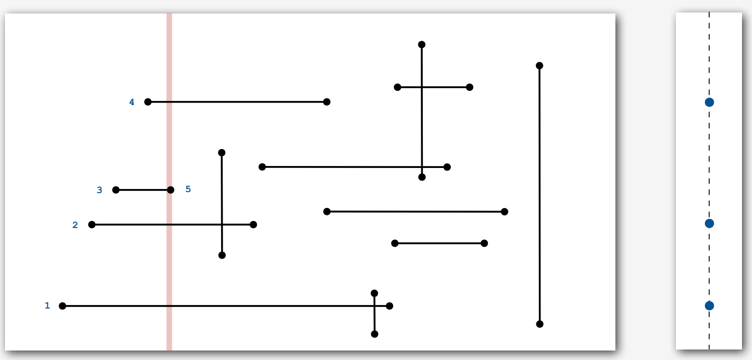


y-coordinates

Orthogonal segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- Left endpoint of h-segment: insert y-coordinate into ST.
- Right endpoint of h-segment: remove y-coordinate from ST.

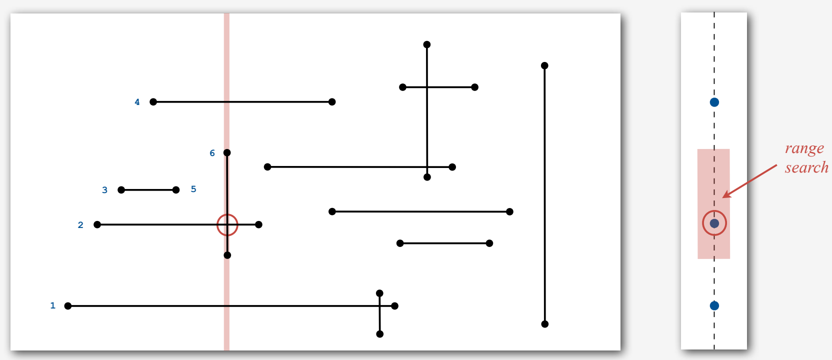


y-coordinates

Orthogonal segment intersection search: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- Left endpoint of h-segment: insert y-coordinate into ST.
- Right endpoint of h-segment: remove y-coordinate from ST.
- v-segment: range search for interval of y endpoints.



y-coordinates

Orthogonal segment intersection search: sweep-line algorithm

Reduces 2D orthogonal segment intersection search to 1D range search!

Running time of sweep line algorithm.

- Put x-coordinates on a PQ (or sort). O(N log N)
- Insert y-coordinate into ST. O(N log N)
- Delete y-coordinate from ST. O(N log N)
- Range search. O(R + N log N)

N = # line segments
R = # intersections

Efficiency relies on judicious use of data structures.

Remark. Sweep-line solution extends to 3D and more general shapes.

Immutable h-v segment data type

```
public final class SegmentHV implements Comparable<SegmentHV>
{
   public final int x1, y1;
   public final int x2, y2;

   public SegmentHV(int x1, int y1, int x2, int y2)
   {  ...  }

   public boolean isHorizontal()
   {  ...  }
   public boolean isVertical()
   {  ...  }

   public int compareTo(SegmentHV b)
   {  ...  }
}
```

constructor

is segment horizontal?
is segment vertical?

compare by x-coordinate;
break ties by y-coordinate

horizontal segment   *vertical segment*

Sweep-line event subclass

```
private class Event implements Comparable<Event>
{
   private int time;
   private SegmentHV segment;

   public Event(int time, SegmentHV segment)
   {
      this.time    = time;
      this.segment = segment;
   }

   public int compareTo(Event that)
   {  return this.time - that.time;  }
}
```


Sweep-line algorithm: initialize events

```
MinPQ<Event> pq = new MinPQ<Event>();

for (int i = 0; i < N; i++)
{
   if (segments[i].isVertical())
   {
      Event e = new Event(segments[i].x1, segments[i]);
      pq.insert(e);
   }

   else if (segments[i].isHorizontal())
   {
      Event e1 = new Event(segments[i].x1, segments[i]);
      Event e2 = new Event(segments[i].x2, segments[i]);
      pq.insert(e1);
      pq.insert(e2);
   }
}
```

initialize PQ

vertical
segment

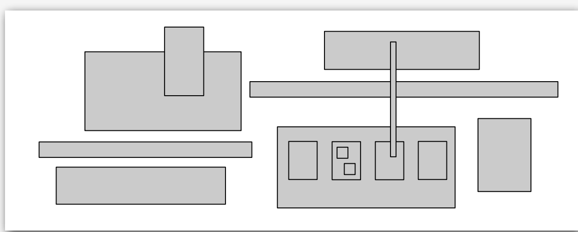
horizontal
segment

Sweep-line algorithm: simulate the sweep line

```
int INF = Integer.MAX_VALUE;

SET<SegmentHV> set = new SET<SegmentHV>();

while (!pq.isEmpty())
{
   Event event = pq.delMin();
   int sweep = event.time;
   SegmentHV segment = event.segment;

   if (segment.isVertical())
   {
      SegmentHV seg1, seg2;
      seg1 = new SegmentHV(-INF, segment.y1, -INF, segment.y1);
      seg2 = new SegmentHV(+INF, segment.y2, +INF, segment.y2);
      for (SegmentHV seg : set.range(seg1, seg2))
          StdOut.println(segment + " intersects " + seg);
   }

   else if (sweep == segment.x1) set.add(segment);
   else if (sweep == segment.x2) set.remove(segment);
}
```


Rectangle intersection search

Goal. Find all intersections among h-v rectangles.

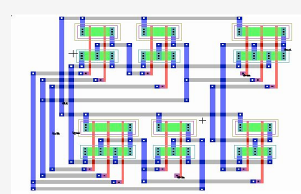


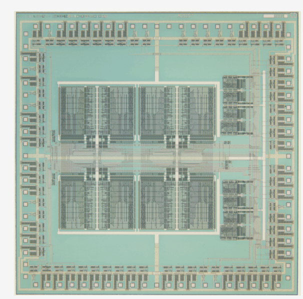
Application. Design-rule checking in VLSI circuits.

Microprocessors and geometry

Early 1970s. microprocessor design became a geometric problem.
- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.
- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = rectangle intersection search.





Algorithms and Moore's law

"Moore's law." Processing power doubles every 18 months.

- 197x: need to check N rectangles.
- 197(x+1.5): need to check 2N rectangles on a 2x-faster computer.

Bootstrapping. We get to use the faster computer for bigger circuits.

But bootstrapping is not enough if using a quadratic algorithm:

- 197x: takes M days.
- 197(x+1.5): takes (4M)/2 = 2M days. (!)

quadratic algorithm

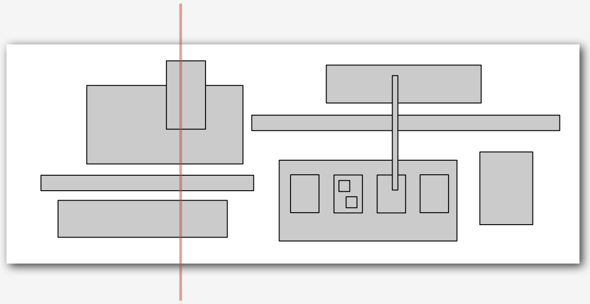
2x-faster computer

Bottom line. Linearithmic CAD algorithm is necessary to sustain Moore's Law.

Rectangle intersection search

Sweep vertical line from left to right.

- x-coordinates of rectangles define events.
- Maintain set of y-intervals intersecting sweep line.
- Left endpoint: search set for y-interval; insert y-interval.
- Right endpoint: delete y-interval.

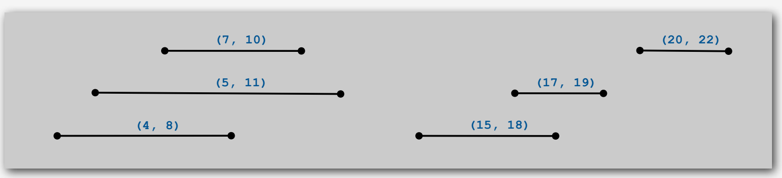


Interval search trees

operation	brute	interval search tree	best in theory
insert interval	1	log N	log N
delete interval	N	log N	log N
find an interval that intersects (lo, hi)	N	log N	log N
find all intervals that intersects (lo, hi)	N	R log N	R + log N

augmented red-black tree

N = # intervals
R = # intersections



Rectangle intersection search: costs summary

Reduces 2D orthogonal rectangle intersection search to 1D interval search!

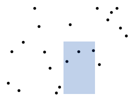
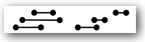
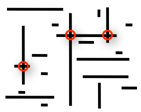
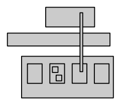
Running time of sweep line algorithm.

- Put x-coordinates on a PQ (or sort). O(N log N)
- Insert y-interval into ST. O(N log N)
- Delete y-interval from ST. O(N log N)
- Interval search. O(R + N log N)

N = # rectangles
R = # intersections

Efficiency relies on judicious use of data structures.

Geometric search summary: algorithms of the day

1D range search		BST
kD range search		kD tree
1D interval intersection search		interval search tree
2D orthogonal line intersection search		sweep line reduces to 1D range search
2D orthogonal rectangle intersection search		sweep line reduces to 1D interval intersection search