Radix Sorts

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- longest repeated substring

References:
Algorithms in Java, Chapter 10
`http://www.cs.princeton.edu/algs4/61radix`

Review: summary of the performance of sorting algorithms

Number of operations on keys = compares.

algorithm	guarantee	average	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	no	yes	`compareTo()`
mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	no	no	`compareTo()`

Lower bound. ~ N lg N compares are required by any compare-based algorithm.

Q. Can we do better (despite the lower bound)?
A. Yes, if we don't depend on compares.

Digital keys

Digital key. Sequence of digits over fixed alphabet.
Radix. Number of digits in alphabet.

Applications.
- DNA: sequence of a, c, g, t.
- IPv6 address: sequence of 128 bits.
- ASCII: sequence of 7-bit characters.
- Protein: sequence of amino acids A, C, ..., Y.
- English words: sequence of lowercase letters.
- Credit card number: sequence of 16 decimal digits.
- International words: sequence of Unicode characters.
- Library call numbers: sequence of letters, numbers, periods.

This lecture. `String` of ASCII characters.

- **key-indexed counting**
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- longest repeated substring

Key-indexed counting: assumptions about keys

Assumption. Keys are integers between 0 and R-1.
Implication. Can use key as an array index.

Applications.
- Sort string by first letter.
- Sort class roster by precept.
- Sort phone numbers by area code.
- Subroutine in a sorting algorithm.

Remark. Keys have associated data ⇒ can't just count up number of keys of each value.

Key-indexed counting

Goal. Sort an array `a[]` of `N` integers between `0` and `R-1`.
- Count frequencies of each letter using key as index.
-
-
-

count frequencies

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; k++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   temp[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = temp[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

offset by 1
[stay tuned]

r	count[r]
a	0
b	2
c	3
d	1
e	2
f	1
-	3

Key-indexed counting

Goal. Sort an array `a[]` of `N` integers between `0` and `R-1`.
- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
-
-

compute cumulates

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   temp[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = temp[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

6 keys < d, 8 keys < e
so d's go in `a[6]` and `a[7]`

Key-indexed counting

Goal. Sort an array `a[]` of `N` integers between `0` and `R-1`.
- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move records.
-

move records

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; k++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   temp[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = temp[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	temp[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting

Goal. Sort an array `a[]` of `N` integers between `0` and `R-1`.
- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move records.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   temp[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = temp[i];
```

copy back

i	a[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	temp[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting: analysis

Assumption. Keys are integers between 0 and R-1.

Running time. Takes time proportional to N + R.

Extra space.
- Array of size R (for counting).
- Array of size N (for rearrangement).

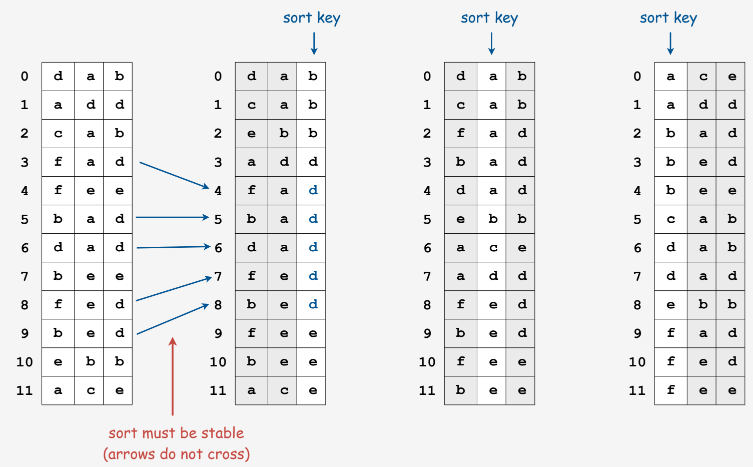
inplace version is possible and practical

Stable? Yes!

- key-indexed counting
- **LSD radix sort**
- MSD radix sort
- 3-way radix quicksort
- longest repeated substring

Least-significant-digit-first radix sort

LSD radix sort.
- Consider characters from right to left.
- Stably sort using dth character as the key via key-indexed counting.



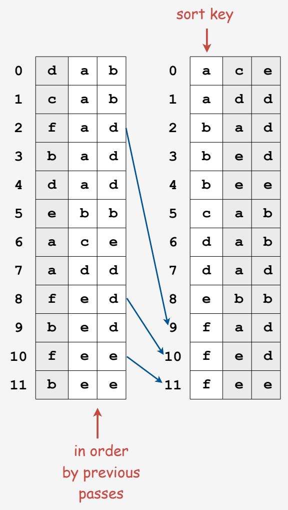
i	input	i	sort key: 3rd char	i	sort key: 2nd char	i	sort key: 1st char
0	d a b	0	d a b	0	d a b	0	a c e
1	a d d	1	c a b	1	c a b	1	a d d
2	c a b	2	e b b	2	f a d	2	b a d
3	f a d	3	a d d	3	b a d	3	b e d
4	f e e	4	f a d	4	d a d	4	b e e
5	b a d	5	b a d	5	e b b	5	c a b
6	d a d	6	d a d	6	a c e	6	d a b
7	b e e	7	f e d	7	a d d	7	d a d
8	f e d	8	b e d	8	f e d	8	e b b
9	b e d	9	f e e	9	b e d	9	f a d
10	e b b	10	b e e	10	f e e	10	f e d
11	a c e	11	a c e	11	b e e	11	f e e

LSD radix sort: correctness proof

Proposition. LSD sorts fixed-length strings in ascending order.

Pf. [thinking about the future]

- If the characters not yet examined differ, it doesn't matter what we do now.
- If the characters not yet examined agree, stability ensures later pass won't affect order.



0	d	a	b
1	c	a	b
2	f	a	d
3	b	a	d
4	d	a	d
5	e	b	b
6	a	c	e
7	a	d	d
8	f	e	d
9	b	e	d
10	f	e	e
11	b	e	e

0	a	c	e
1	a	d	d
2	b	a	d
3	b	e	d
4	b	e	e
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	d
11	f	e	e

LSD radix sort: Java implementation

Assumption. Radix R, fixed-length W keys.

```
public static void lsd(String[] a)
{
   int N = a.length;
   String[] temp = new String[N];
   for (int d = W-1; d >= 0; d--)
   {
      int[] count = new int[R+1];
      for (int i = 0; i < N; i++)
         count[a[i].charAt(d) + 1]++;
      for (int r = 0; r < R; r++)
         count[r+1] += count[r];
      for (int i = 0; i < N; i++)
         temp[count[a[i].charAt(d)]++] = a[i];
      for (int i = 0; i < N; i++)
         a[i] = temp[i];
   }
}
```

do key-indexed counting for each digit from right to left

key-indexed counting

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	average	extra space	stable?	operations on keys
insertion sort	$N^2/2$	$N^2/4$	no	yes	compareTo()
mergesort	N lg N	N lg N	N	yes	compareTo()
quicksort	1.39 N lg N	1.39 N lg N	c lg N	no	compareTo()
heapsort	2 N lg N	2 N lg N	no	no	compareTo()
LSD †	W (N + R)	W (N + R)	N + R	yes	charAt()

† fixed-length W keys

Sorting challenge

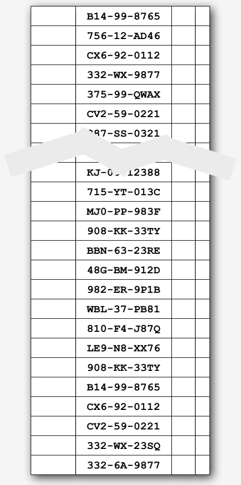
Problem. Sort a huge commercial database on a fixed-length key field.

Ex. Account number, date, SS number, ...

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD radix sort.

256 (or 65536) counters;
Fixed-length strings sort in W passes.



B14-99-8765
756-12-AD46
CX6-92-0112
332-WX-9877
375-99-QWAX
CV2-59-0221
387-SS-0321
KJ-0 12388
715-YT-013C
MJ0-PP-983F
908-KK-33TY
BBN-63-23RE
48G-BM-912D
982-ER-9P1B
WBL-37-PB81
810-F4-J87Q
LE9-N8-XX76
908-KK-33TY
B14-99-8765
CX6-92-0112
CV2-59-0221
332-WX-23SQ
332-6A-9877

Sorting challenge

Problem. Sort huge files of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD radix sort.

Divide each word into eight 16-bit "chars."
2^{16} = 65536 counters.
Sort in 8 passes.

LSD radix sort: a moment in history (1960s)

card punch

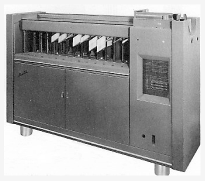
punched cards

card reader

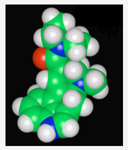
mainframe

line printer

To sort a card deck
- start on right column
- put cards into hopper
- machine distributes into bins
- pick up cards (stable)
- move left one column
- continue until sorted

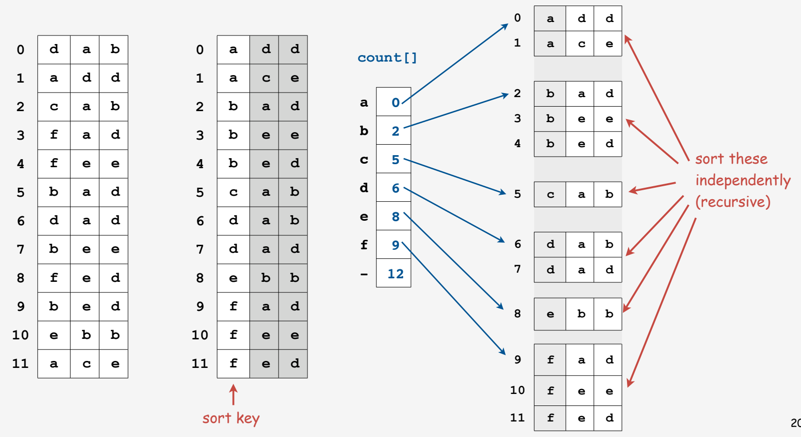

card sorter

not related to sorting


Lysergic Acid Diethylamide
(Lucy in the Sky with Diamonds)

Most-significant-digit-first radix sort

MSD radix sort.
- Partition file into R pieces according to first character (use key-indexed counting) .
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort key

count[]

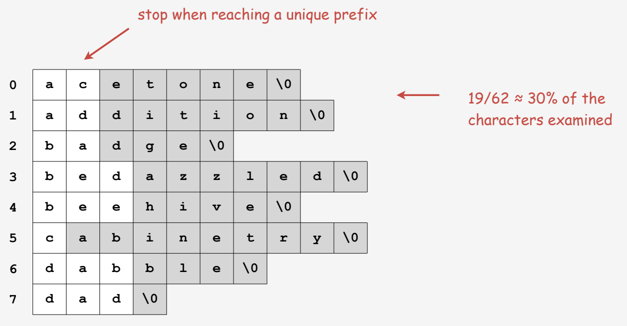
a	0
b	2
c	5
d	6
e	8
f	9
-	12

sort these independently (recursive)

MSD radix sort bonuses

Bonus 1. May not have to examine all of the keys.

Bonus 2. Works for variable-length keys, e.g., `'\0'`-terminated `String`.



Implication. Sublinear-time sort (!) [assuming data is already in memory]

MSD radix sort: Java implementation

```
public static void msd(String[] a)
{  msd(a, 0, a.length, 0);  }

private static void msd(String[] a, int lo, int hi, int d)
{
   if (hi <= lo + 1) return;

   int[] count = new int[R+1];
   for (int i = lo; i < hi; i++)
      count[a[i].charAt(d) + 1]++;
   for (int r = 0; r < R; r++)
      count[r+1] += count[r];
   for (int i = lo; i < hi; i++)
      temp[count[a[i].charAt(d)]++] = a[i];
   for (int i = lo; i < hi; i++)
      a[i] = temp[i - lo];

   for (int r = 1; r < R; r++)
      msd(a, lo + count[r], lo + count[r+1], d+1);
}
```

key-indexed counting

recursively sort R-1 subarrays

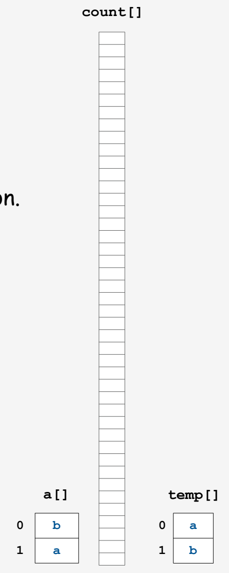
assumes strings are `'\0'` terminated; don't sort substrings that start with `'\0'`

MSD radix sort: potential for disastrous performance

Observation 1. Much too slow for small files.
- The `count[]` array must be re-initialized.
- ASCII (256 counts): 100x slower than copy pass for N = 2.
- Unicode (65536 counts): 32,000x slower for N = 2.

Observation 2. Huge number of small files because of recursion.

Solution. Cutoff to insertion sort for small N.



Summary of the performance of sorting algorithms

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mergesort	N lg N	N lg N	N	yes	`compareTo()`
quicksort	1.39 N lg N	1.39 N lg N	c lg N	no	`compareTo()`
heapsort	2 N lg N	2 N lg N	no	no	`compareTo()`
LSD †	W (N + R)	W (N + R)	N + R	yes	`charAt()`
MSD ‡	W N	c N log N	D (N + R)	yes	`charAt()`

just enough to distinguish N ASCII strings: $N \log_{128} N$

stack depth D = length of longest prefix match

† fixed-length W keys
‡ average-length W keys

MSD radix sort vs. quicksort for strings

Disadvantages of MSD radix sort.

- Accesses memory "randomly" (cache inefficient).
- Inner loop has a lot of instructions.
- Extra space for `count[]`.
- Extra space for `temp[]`.

Disadvantage of quicksort.

- Linearithmic (not linear).
- Has to rescan long keys for compares.
  [but stay tuned]

Sorting challenge

Problem. Sort 1 million 32-bit integers.
Ex. Google interview or presidential debate.

Which sorting method to use?

- Bubblesort.
- Mergesort.
- Quicksort.
- LSD radix sort.
- MSD radix sort.

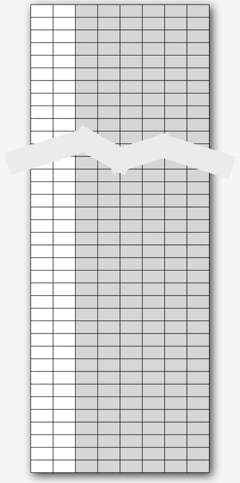
Sorting challenge (revisited)

Problem. Sort huge files of random 128-bit numbers.
Ex. Supercomputer sort, internet router.

Which sorting method to use?

- ✓ Insertion sort.
- Mergesort.
- Quicksort.
- ✓ LSD radix sort.
- MSD radix sort.

Divide each word into 16-bit “chars”
2^{16} = 65536 counters
LSD sort on leading 32 bits in 2 passes
Finish with insertion sort
Examines only ~25% of the data



- key-indexed counting
- LSD radix sort
- MSD radix sort
- **3-way radix quicksort**
- longest repeated substring

3-way radix quicksort (Bentley and Sedgewick, 1997)

Idea. Do 3-way partitioning on the dth character.

- Cheaper than R-way partitioning of MSD radix sort.
- Need not examine again chars equal to the partitioning char.

`qsortX(0, 12, 0)`

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	a	c	e
10	e	b	b
11	b	e	d

3-way partition
0th char on b

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	f	a	d
6	d	a	d
7	c	a	b
8	f	e	d
9	d	a	b
10	e	b	b
11	f	e	e

result of 3-way partition

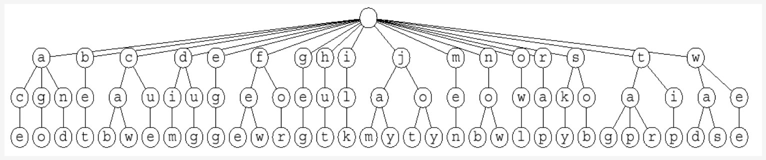
0	a	d	d	`qsortX(0, 2, 0)`
1	a	c	e	
2	b	a	d	`qsortX(2, 5, 1)`
3	b	e	e	
4	b	e	d	
5	f	a	d	`qsortX(5, 12, 0)`
6	d	a	d	
7	c	a	b	
8	f	e	d	
9	d	a	b	
10	e	b	b	
11	f	e	e	

only examine string starting at index 1

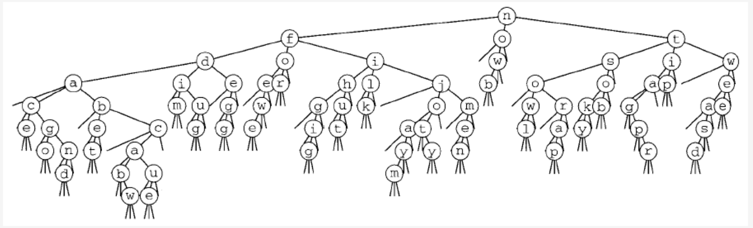
recursively sort 3 pieces

Recursive structure: MSD radix sort vs. 3-way radix quicksort

3-way radix quicksort collapses empty links in MSD recursion tree.



MSD radix sort recursion tree
(1035 null links, not shown)



3-way radix quicksort recursion tree
(155 null links)

3-way radix quicksort vs. standard quicksort

Standard quicksort.

- Uses 2N ln N string compares on average.
- Costly for long keys that differ only at the end (and this is a common case!)

3-way radix quicksort.

- Uses 2 N ln N character compares on average for random strings.
- Avoids recomparing initial parts of the string.
- Adapts to data: uses just "enough" characters to resolve order.
- Sublinear when strings are long.

Proposition. Quicksort with 3-way partitioning is optimal (to within a constant factor); no sorting algorithm can (asymptotically) examine fewer chars.

Pf. Ties cost to entropy. Beyond scope of 226.

3-way radix quicksort vs. MSD radix sort

MSD radix sort.

- Has a long inner loop.
- Is cache-inefficient.
- Too much overhead reinitializing `count[]` and `temp[]` for long keys that differ only at end (and this is a common case!)

3-way radix quicksort.

- Has a short inner loop.
- Is cache-friendly.
- Is in-place.

library call numbers

```
WUS-------10706-----7---10
WUS-------12692-----4---27
WLSOC------2542----30
LTK--6015-P-63-1988
LDS---361-H-4
...
```

Bottom line. 3-way radix quicksort is the method of choice for sorting strings.

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- **longest repeated substring**

String processing

String. Sequence of characters.

Important fundamental abstraction.

- Java programs.
- Natural languages.
- Genomic sequences.
- …

" *The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology.* " — *M. V. Olson*

Using strings in Java: review

String concatenation. Append one string to end of another string.

Substring. Extract a contiguous sequence of characters from a string.

s	t	r	i	n	g	s
0	1	2	3	4	5	6

```
String s = "strings";              // s = "strings"
char   c = s.charAt(2);            // c = 'r'
String t = s.substring(2, 6);      // t = "ring"
String u = s + t;                  // u = "stringsring"
```

Implementing strings in Java

Memory. 40 + 2N bytes for a virgin string!

use byte array instead of `String` to save space

`java.lang.String`

```
public final class String implements Comparable<String>
{
   private char[] value;   // characters
   private int offset;     // index of first char into array
   private int count;      // length of string
   private int hash;       // cache of hashCode()

   private String(int offset, int count, char[] value)
   {
       this.offset = offset;
       this.count  = count;
       this.value  = value;
   }
   public String substring(int from, int to)
   {
      return new String(offset + from, to - from, value);   }
   …
}
```

`String` vs. `StringBuilder`

`String`. [immutable] Constant substring, linear concatenation.

`StringBuilder`. [mutable] Linear substring, constant (amortized) append.

Ex. Reverse a `String`.

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
         rev += s.charAt(i);
   return rev;
}
```

quadratic time

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

linear time

Warmup: longest common prefix

LCP. Given two strings, find the longest substring that is a prefix of both.

p	r	e	f	e	t	c	h
0	1	2	3	4	5	6	7
p	r	e	f	i	x		

```
public static String lcp(String s, String t)
{
   int n = Math.min(s.length(), t.length());
   for (int i = 0; i < n; i++)
   {
      if (s.charAt(i) != t.charAt(i))
         return s.substring(0, i);
   }
   return s.substring(0, n);
}
```

Running time. Linear-time in length of prefix match.

Space. Constant extra space.

Longest repeated substring

LRS. Given a string of N characters, find the longest repeated substring.

Ex.

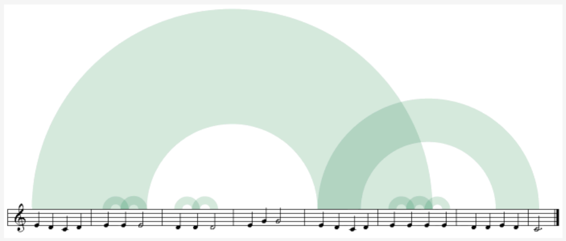
```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a a c t c t a t a t c t a t a a a a
```

Applications. Bioinformatics, cryptanalysis, data compression, ...

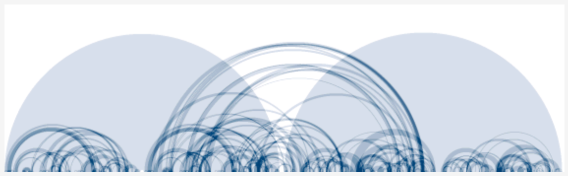
Longest repeated substring: a musical application

Visualize repetitions in music. `http://www.bewitched.com`

Mary Had a Little Lamb



Goldbach Variations

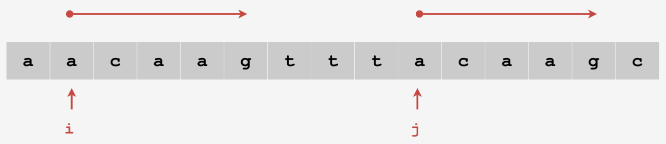


Longest repeated substring

LRS. Given a string of N characters, find the longest repeated substring.

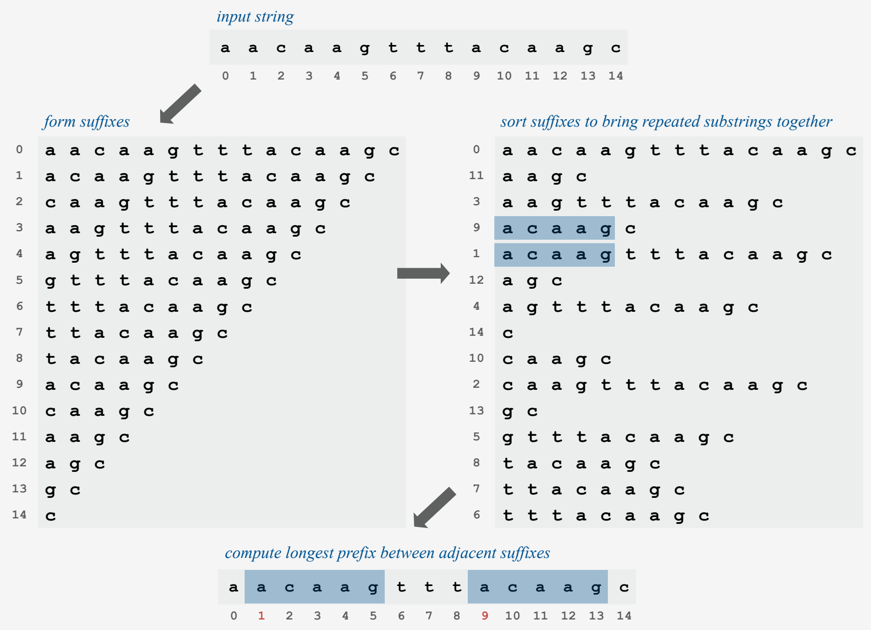
Brute force algorithm.

- Try all indices `i` and `j` for start of possible match.
- Compute longest common prefix (LCP) for each pair.



Analysis. Running time ≤ M N^2, where M is length of longest match.

Longest repeated substring: a sorting solution


input string

```
a  a  c  a  a  g  t  t  t  a  c  a  a  g  c
0  1  2  3  4  5  6  7  8  9  10 11 12 13 14
```

form suffixes

```
 0  a a c a a g t t t a c a a g c
 1  a c a a g t t t a c a a g c
 2  c a a g t t t a c a a g c
 3  a a g t t t a c a a g c
 4  a g t t t a c a a g c
 5  g t t t a c a a g c
 6  t t t a c a a g c
 7  t t a c a a g c
 8  t a c a a g c
 9  a c a a g c
10  c a a g c
11  a a g c
12  a g c
13  g c
14  c
```

sort suffixes to bring repeated substrings together

```
 0  a a c a a g t t t a c a a g c
11  a a g c
 3  a a g t t t a c a a g c
 9  a c a a g c
 1  a c a a g t t t a c a a g c
12  a g c
 4  a g t t t a c a a g c
14  c
10  c a a g c
 2  c a a g t t t a c a a g c
13  g c
 5  g t t t a c a a g c
 8  t a c a a g c
 7  t t a c a a g c
 6  t t t a c a a g c
```

compute longest prefix between adjacent suffixes

```
a  a  c  a  a  g  t  t  t  a  c  a  a  g  c
0  1  2  3  4  5  6  7  8  9  10 11 12 13 14
```

Longest repeated substring: Java implementation

```
public String lrs(String s)
{
  int N = s.length();

  String[] suffixes = new String[N];
  for (int i = 0; i < N; i++)
     suffixes[i] = s.substring(i, N);

  Arrays.sort(suffixes);

  String lrs = "";
  for (int i = 0; i < N-1; i++)
  {
     String x = lcp(suffixes[i], suffixes[i+1]);
     if (x.length() > lrs.length()) lrs = x;
  }
  return lrs;
}
```

create suffixes (linear time and space)

sort suffixes

find LCP between suffixes that are adjacent after sorting

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```

Sorting challenge

Problem. Five scientists A, B, C, D, and E are looking for long repeated substring in a genome with over 1 billion nucleotides.

- A has a grad student do it by hand.
- B uses brute force (check all pairs).
- C uses suffix sorting solution with insertion sort.
- D uses suffix sorting solution with LSD radix sort.
- ✓ E uses suffix sorting solution with 3-way radix quicksort.

only if LRS is not long (!)

Q. Which one is more likely to lead to a cure cancer?

Longest repeated substring: empirical analysis

input file	characters	brute	suffix sort	length of LRS
`LRS.java`	2,162	0.6 sec	0.14 sec	73
`amendments.txt`	18,369	37 sec	0.25 sec	216
`aesop.txt`	191,945	1.2 hours	1.0 sec	58
`mobydick.txt`	1.2 million	43 hours †	7.6 sec	79
`chromosome11.txt`	7.1 million	2 months †	61 sec	12,567
`pi.txt`	10 million	4 months †	84 sec	14

† estimated

Suffix sorting: worst-case input

Longest repeated substring not long. Hard to beat 3-way radix quicksort.

Longestest repeated substring very long.
- Radix sorts are quadratic in the length of the longest match.
- Ex: two copies of Aesop's fables.

```
abcdefgh
abcdefghabcdefgh
bcdefgh
bcdefghabcdefgh
cdefgh
cdefghabcdefgh
defgh
efghabcdefgh
efgh
fghabcdefgh
fgh
ghabcdefgh
fh
habcdefgh
h
```

`abcdefghabcdefgh`

	time to suffix sort (seconds)	
algorithm	`mobydick.txt`	`aesopaesop.txt`
brute-force	36.000 †	4000 †
quicksort	9.5	167
LSD	not fixed length	not fixed length
MSD	395	out of memory
MSD with cutoff	6.8	162
3-way radix quicksort	2.8	400

† estimated

Suffix sorting challenge

Problem. Suffix sort an arbitrary string of length N.

Q. What is worst-case running time of best algorithm for problem?
- Quadratic.
- ✓ Linearithmic. ← Manber's algorithm
- ✓ Linear. ← suffix trees (see COS 423)
- Nobody knows.

Suffix sorting in linearithmic time

Manber's MSD algorithm.
- Phase 0: sort on first character using key-indexed sort.
- Phase i: given list of suffixes sorted on first 2^{i-1} characters, create list of suffixes sorted on first 2^i characters.

Worst-case running time. N log N.
- Finishes after lg N phases.
- Can perform a phase in linear time. [stay tuned]

Linearithmic suffix sort example: phase 0

	original suffixes		*index sort (first character)*		*inverse*
0	b a b a a a a b c b a b a a a a a 0	17	0	0	12
1	a b a a a a b c b a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	1	1
2	b a a a a b c b a b a a a a a 0	16	a 0	2	16
3	a a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	3	3
4	a a a b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	4	4
5	a a b c b a b a a a a a 0	5	a a b c b a b a a a a a 0	5	5
6	a b c b a b a a a a a 0	6	a b c b a b a a a a a 0	6	6
7	b c b a b a a a a a 0	15	a a 0	7	15
8	c b a b a a a a a 0	14	a a a 0	8	17
9	b a b a a a a a 0	13	a a a a 0	9	13
10	a b a a a a a 0	12	a a a a a 0	10	11
11	b a a a a a 0	10	a b a a a a a 0	11	14
12	a a a a a 0	0	b a b a a a a b c b a b a a a a a 0	12	10
13	a a a a 0	9	b a b a a a a a 0	13	9
14	a a a 0	11	b a a a a a 0	14	8
15	a a 0	7	b c b a b a a a a a 0	15	7
16	a 0	2	b a a a a b c b a b a a a a a 0	16	2
17	0	8	c b a b a a a a a 0	17	0

↑ *sorted*

Linearithmic suffix sort example: phase 1

	original suffixes		*index sort (first two characters)*		*inverse*
0	b a b a a a a b c b a b a a a a a 0	17	0	0	12
1	a b a a a a b c b a b a a a a a 0	16	a 0	1	10
2	b a a a a b c b a b a a a a a 0	12	a a a a a 0	2	15
3	a a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	3	3
4	a a a b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	4	4
5	a a b c b a b a a a a a 0	5	a a b c b a b a a a a a 0	5	5
6	a b c b a b a a a a a 0	13	a a a a 0	6	9
7	b c b a b a a a a a 0	15	a a 0	7	16
8	c b a b a a a a a 0	14	a a a 0	8	17
9	b a b a a a a a 0	6	a b c b a b a a a a a 0	9	13
10	a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	10	11
11	b a a a a a 0	10	a b a a a a a 0	11	14
12	a a a a a 0	0	b a b a a a a b c b a b a a a a a 0	12	2
13	a a a a 0	9	b a b a a a a a 0	13	6
14	a a a 0	11	b a a a a a 0	14	8
15	a a 0	2	b a a a a b c b a b a a a a a 0	15	7
16	a 0	7	b c b a b a a a a a 0	16	1
17	0	8	c b a b a a a a a 0	17	0

↑ *sorted*

Linearithmic suffix sort example: phase 2

	original suffixes		*index sort (first four characters)*		*inverse*
0	b a b a a a a b c b a b a a a a a 0	17	0	0	14
1	a b a a a a b c b a b a a a a a 0	16	a 0	1	9
2	b a a a a b c b a b a a a a a 0	15	a a 0	2	12
3	a a a a b c b a b a a a a a 0	14	a a a 0	3	4
4	a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	4	7
5	a a b c b a b a a a a a 0	12	a a a a a 0	5	8
6	a b c b a b a a a a a 0	13	a a a a 0	6	11
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	7	16
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0	8	17
9	b a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	9	15
10	a b a a a a a 0	10	a b a a a a a 0	10	10
11	b a a a a a 0	6	a b c b a b a a a a a 0	11	13
12	a a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0	12	5
13	a a a a 0	11	b a a a a a 0	13	6
14	a a a 0	0	b a b a a a a b c b a b a a a a a 0	14	3
15	a a 0	9	b a b a a a a a 0	15	2
16	a 0	7	b c b a b a a a a a 0	16	1
17	0	8	c b a b a a a a a 0	17	0

↑ *sorted*

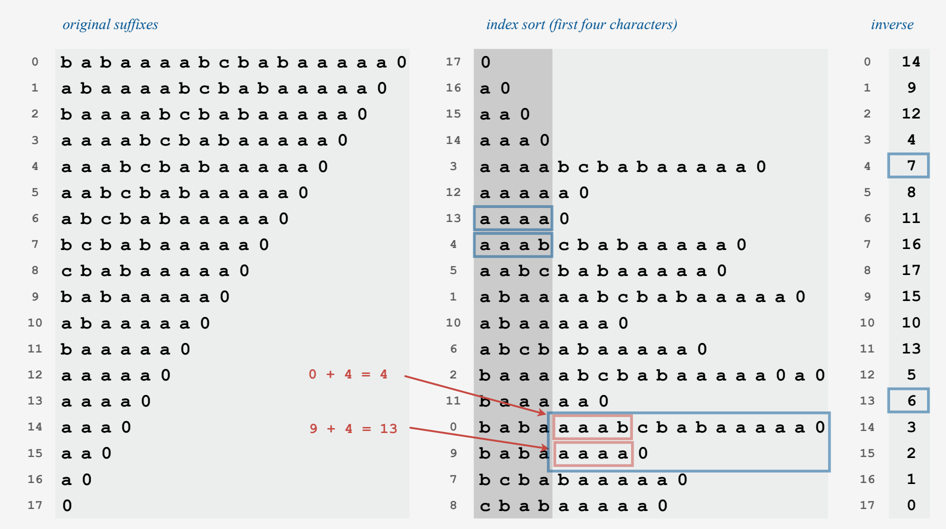
Linearithmic suffix sort example: phase 3

	original suffixes		*index sort (first eight characters)*		*inverse*
0	b a b a a a a b c b a b a a a a a 0	17	0	0	15
1	a b a a a a b c b a b a a a a a 0	16	a 0	1	10
2	b a a a a b c b a b a a a a a 0	15	a a 0	2	13
3	a a a a b c b a b a a a a a 0	14	a a a 0	3	4
4	a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	4	7
5	a a b c b a b a a a a a 0	13	a a a a 0	5	8
6	a b c b a b a a a a a 0	12	a a a a a 0	6	11
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	7	16
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0	8	17
9	b a b a a a a a 0	10	a b a a a a a 0	9	14
10	a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	10	9
11	b a a a a a 0	6	a b c b a b a a a a a 0	11	12
12	a a a a a 0	11	b a a a a a 0	12	6
13	a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0	13	5
14	a a a 0	9	b a b a a a a a 0	14	3
15	a a 0	0	b a b a a a a b c b a b a a a a a 0	15	2
16	a 0	7	b c b a b a a a a a 0	16	1
17	0	8	c b a b a a a a a 0	17	0

↑ *sorted*

FINISHED! (no equal keys)

Achieve constant-time string compare by indexing into inverse



	original suffixes		*index sort (first four characters)*		*inverse*
0	b a b a a a a b c b a b a a a a a 0	17	0	0	14
1	a b a a a a b c b a b a a a a a 0	16	a 0	1	9
2	b a a a a b c b a b a a a a a 0	15	a a 0	2	12
3	a a a a b c b a b a a a a a 0	14	a a a 0	3	4
4	a a a b c b a b a a a a a 0	3	a a a a b c b a b a a a a a 0	4	7
5	a a b c b a b a a a a a 0	12	a a a a a 0	5	8
6	a b c b a b a a a a a 0	13	a a a a 0	6	11
7	b c b a b a a a a a 0	4	a a a b c b a b a a a a a 0	7	16
8	c b a b a a a a a 0	5	a a b c b a b a a a a a 0	8	17
9	b a b a a a a a 0	1	a b a a a a b c b a b a a a a a 0	9	15
10	a b a a a a a 0	10	a b a a a a a 0	10	10
11	b a a a a a 0	6	a b c b a b a a a a a 0	11	13
12	a a a a a 0	2	b a a a a b c b a b a a a a a 0 a 0	12	5
13	a a a a 0	11	b a a a a a 0	13	6
14	a a a 0	0	b a b a a a a b c b a b a a a a a 0	14	3
15	a a 0	9	b a b a a a a a 0	15	2
16	a 0	7	b c b a b a a a a a 0	16	1
17	0	8	c b a b a a a a a 0	17	0

`suffixes₄[13] ≤ suffixes₄[4]` (*because* `inverse[13] < inverse[4]`)

so `suffixes₈[9] ≤ suffixes₈[0]`

Suffix sort: experimental results

	time to suffix sort (seconds)	
algorithm	mobydick.txt	aesopaesop.txt
brute-force	36.000 †	4000 †
quicksort	9.5	167
LSD	not fixed length	not fixed length
MSD	395	out of memory
MSD with cutoff	6.8	162
3-way radix quicksort	2.8	400
Manber MSD	17	8.5

† estimated

Radix sort summary

We can develop linear-time sorts.
- Compares not necessary for some types of keys.
- Use keys to index an array.

We can develop sublinear-time sorts.
- Should measure amount of data in keys, not number of keys.
- Not all of the data has to be examined.

No algorithm can asymptotically examine fewer chars than 3-way radix quicksort.
- 1.39 N lg N chars for random data.

Long strings are rarely random in practice.
- Goal is often to learn the structure!
- May need specialized algorithms.