Priority Queues

- API
- elementary implementations
- binary heaps
- heapsort
- event-based simulation

Priority queue API

Keys. Items that can be compared.

public class MaxPQ<Key extends Comparable<Key>>		
	MaxPQ()	*create an empty priority queue*
boolean	isEmpty()	*is the priority queue empty*
void	insert(Key key)	*insert a key*
Key	delMax()	*delete and return the maximum key*
Key	max()	*return the maximum key*
int	size()	*return the number of keys*

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Computational number theory. [sum of powers]
- Artificial intelligence. [A* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Generalizes: stack, queue, randomized queue.

Priority queue client example

Problem. Find the largest M of a stream of N elements.

- Fraud detection: isolate $$ transactions.
- File maintenance: find biggest files or directories.

Constraint. Not enough memory to store N elements.

Solution. Use a min-oriented priority queue.

implementation	time	space
sort	N log N	N
elementary PQ	M N	M
binary heap	N log M	M
best in theory	N	M

cost of finding the largest M in a stream of N items

```
MinPQ<String> pq = new MinPQ<String>();

while(!StdIn.isEmpty())
{
   String s = StdIn.readString();
   pq.insert(s);
   if (pq.size() > M)
      pq.delMin();
}

while (!pq.isEmpty())
   System.out.println(pq.delMin());
```

- API
- **elementary implementations**
- binary heaps
- heapsort
- event-based simulation

Priority queue: unordered and ordered array implementation

operation	argument	return value	size	contents (unordered)	contents (ordered)
insert	P		1	P	P
insert	Q		2	P Q	P Q
insert	E		3	P Q E	E P Q
remove max		Q	2	P E	E P
insert	X		3	P E X	E P X
insert	A		4	P E X A	A E P X
insert	M		5	P E X A M	A E M P X
remove max		X	4	P E M A	A E M P
insert	P		5	P E M A P	A E M P P
insert	L		6	P E M A P L	A E L M P P
insert	E		7	P E M A P L E	A E E L M P P
remove max		P	6	E M A P L E	A E E L M P

A sequence of operations on a priority queue

Priority queue: unordered array implementation

```
public class UnorderedMaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;   // pq[i] = ith element on pq
   private int N;      // number of elements on pq

   public UnorderedPQ(int capacity)
   {  pq = (Key[]) new Comparable[capacity];  }

   public boolean isEmpty()
   {  return N == 0; }

   public void insert(Key x)
   {  pq[N++] = x;  }

   public Key delMax()
   {
      int max = 0;
      for (int i = 1; i < N; i++)
         if (less(max, i)) max = i;
      exch(max, N-1);
      return pq[--N];
   }
}
```

no generic array creation

`less()` and `exch()` as for sorting

Priority queue elementary implementations

Challenge. Implement all operations efficiently.

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

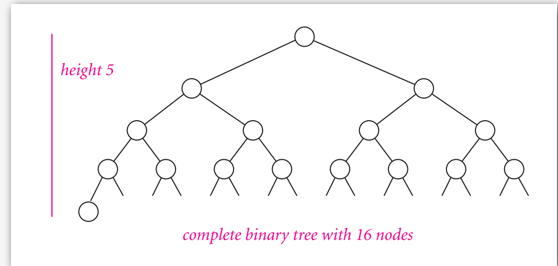
order-of-growth running time for PQ with N items

- API
- elementary implementations
- **binary heaps**
- heapsort
- event-based simulation

Binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



Property. Height of binary heap with N nodes is $1 + \lfloor \lg N \rfloor$.

Pf. Height only increases when N is exactly a power of 2.

Binary heap

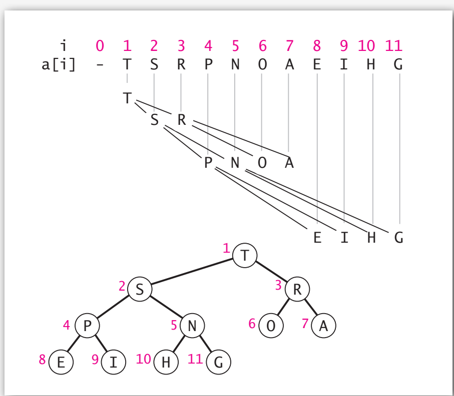
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- No smaller than children's keys.

Array representation.

- Take nodes in level order.
- No explicit links needed!



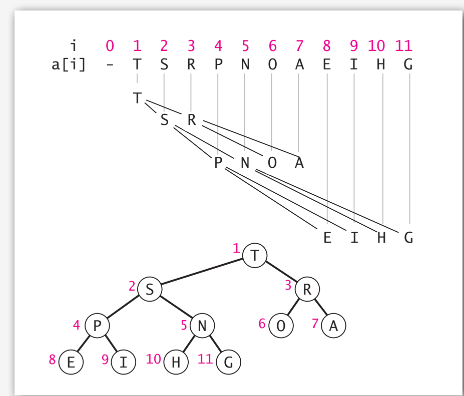
Binary heap properties

Property A. Largest key is at root.

indices start at 1

Property B. Can use array indices to move through tree.

- Parent of node at k is at k/2.
- Children of node at k are at 2k and 2k+1.



Promotion in a heap

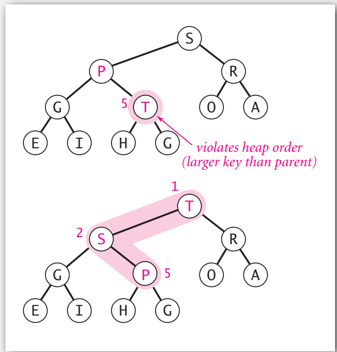
Scenario. Exactly one node has a larger key than its parent.

To eliminate the violation:

- Exchange with its parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2

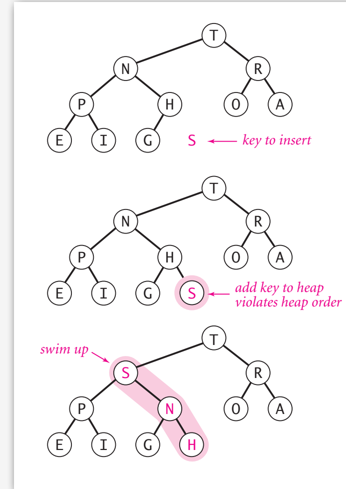


Peter principle. Node promoted to level of incompetence.

Insertion in a heap

Insert. Add node at end, then promote.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Demotion in a heap

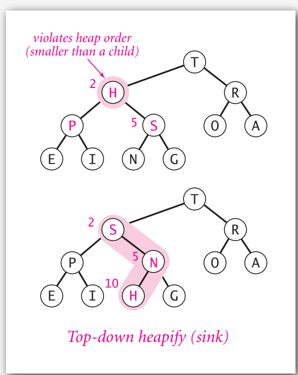
Scenario. Exactly one node has a smaller key than does a child.

To eliminate the violation:

- Exchange with larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2k and 2k+1



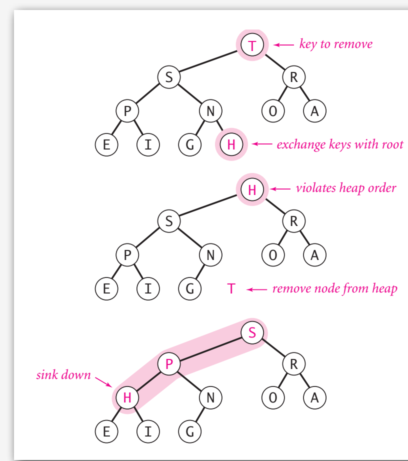
Power struggle. Better subordinate promoted.

Delete the maximum in a heap

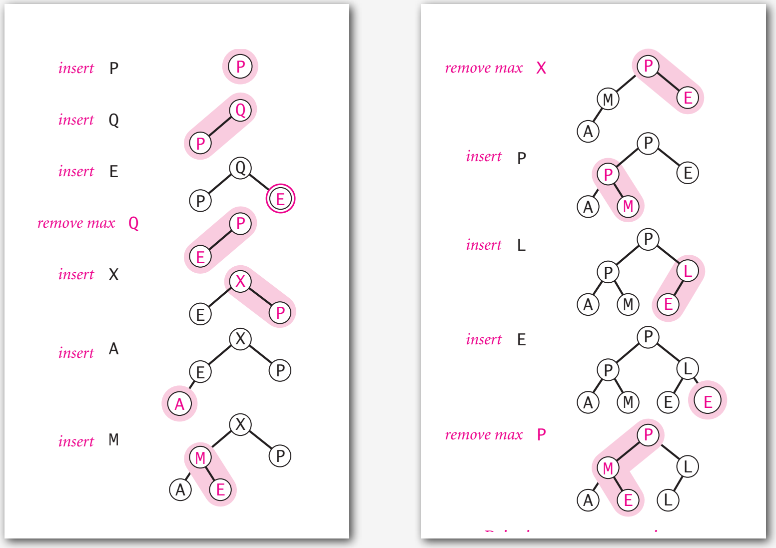
Delete max. Exchange root with node at end, then demote.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;
   return max;
}
```

prevent loitering



Heap operations



Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int N;

   public MaxPQ(int capacity)
   {   ...   }

   public boolean isEmpty()
   {   return N == 0;   }
   public void insert(Key key)
   {   /* see previous code */  }
   public Key delMax()
   {   /* see previous code */  }

   private void swim(int k)
   {   /* see previous code */  }
   private void sink(int k)
   {   /* see previous code */  }

   private boolean less(int i, int j)
   {   return pq[i].compareTo(pq[j]) < 0;   }
   private void exch(int i, int j)
   {   Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;   }
}
```

← same as array-based PQ, but allocate one extra element

← PQ ops

← heap helper functions

← array helper functions

Binary heap considerations

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Dynamic array resizing.

- Add no-arg constructor.
- Apply repeated doubling and shrinking. ← leads to O(log N) amortized time per op

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

← easy to implement with `sink()` and `swim()` [stay tuned]

Priority queues implementation cost summary

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	log N	log N	1

order-of-growth running time for PQ with N items

Hopeless challenge. Make all operations constant time.

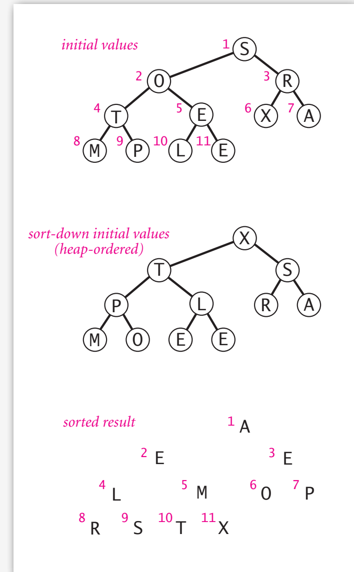
Q. Why hopeless?

- API
- elementary implementations
- binary heaps
- **heapsort**
- event-based simulation

Heapsort

Basic plan for in-place sort.

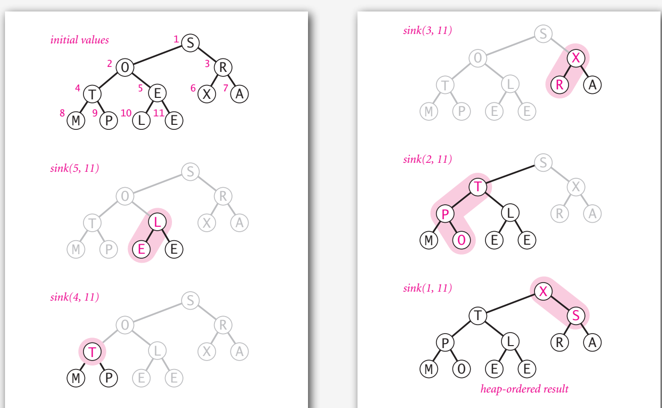
- Create max-heap with all N keys.
- Repeatedly remove the maximum key.



Heapsort

First pass. Build heap using bottom-up method.

```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```

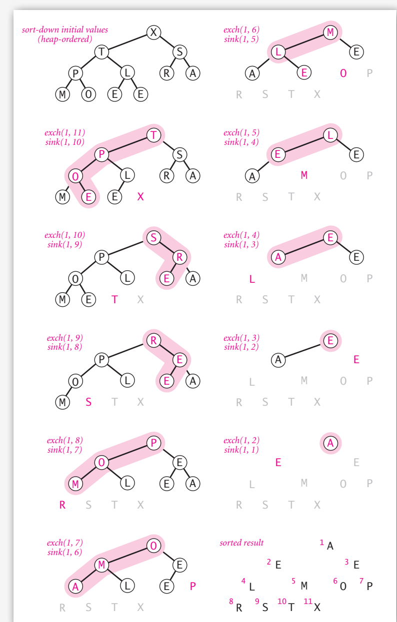


Heapsort

Second pass. Sort.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] pq)
   {
      int N = pq.length;
      for (int k = N/2; k >= 1; k--)
         sink(pq, k, N);
       while (N > 1)
       {
          exch(pq, 1, N);
          sink(pq, 1, --N);
       }
   }

   private static void sink(Comparable[] pq, int k, int N)
   {  /* as before */  }

   private static boolean less(Comparable[] pq, int i, int j)
   {  /* as before */  }

   private static void exch(Comparable[] pq, int i, int j)
   {  /* as before */  }

}
```

but use 1-based indexing

Heapsort: trace

N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

a[i]

entries in black are compared

entries in red are exchanged

entries in gray are not touched

Heapsort trace (array contents after each sink)

Heapsort: mathematical analysis

Property D. At most 2 N lg N compares.

Significance. Sort in N log N worst-case without using extra memory.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case. ← N log N worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache memory.

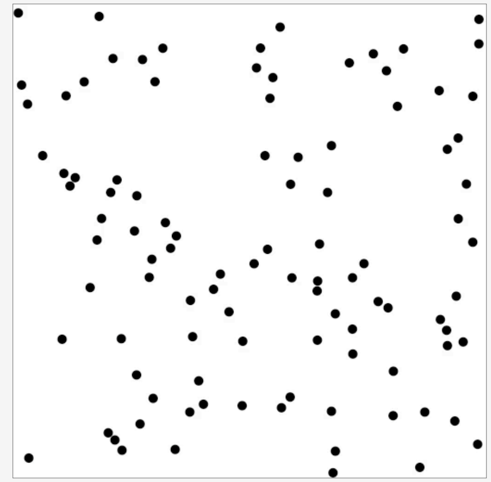
Sorting algorithms: summary

	inplace?	stable?	worst	average	best	remarks
selection	×		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	N exchanges
insertion	×	×	$N^2 / 2$	$N^2 / 4$	N	use for small N or partially ordered
shell	×		?	?	N	tight code, subquadratic
quick	×		$N^2 / 2$	$2 N \ln N$	$N \lg N$	$N \log N$ probabilistic guarantee fastest in practice
3-way quick	×		$N^2 / 2$	$2 N \ln N$	N	improves quicksort in presence of duplicate keys
merge		×	$N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee, stable
heap	×		$2 N \lg N$	$2 N \lg N$	$N \lg N$	$N \log N$ guarantee, in-place
???	×	×	$N \lg N$	$N \lg N$	$N \lg N$	holy sorting grail

- API
- elementary implementations
- binary heaps
- heapsort
- **event-based simulation**

Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.



Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard disc model.
- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces are exerted.

temperature, pressure, diffusion constant

motion of individual atoms and molecules

Significance. Relates macroscopic observables to microscopic dynamics.
- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

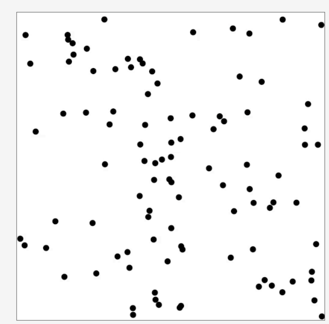
Warmup: bouncing balls

Time-driven simulation. N bouncing balls in the unit square.

```
public class BouncingBalls
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Ball balls[] = new Ball[N];
      for (int i = 0; i < N; i++)
         balls[i] = new Ball();
      while(true)
      {
         StdDraw.clear();
         for (int i = 0; i < N; i++)
         {
            balls[i].move(0.5);
            balls[i].draw();
         }
         StdDraw.show(50);
      }
   }
}
```

main simulation loop

```
% java BouncingBalls 100
```



Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;        // position
    private double vx, vy;        // velocity
    private final double radius;  // radius
    public Ball()
    {  /* initialize position and velocity */  }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }
    public void draw()
    {  StdDraw.filledCircle(rx, ry, radius);  }
}
```

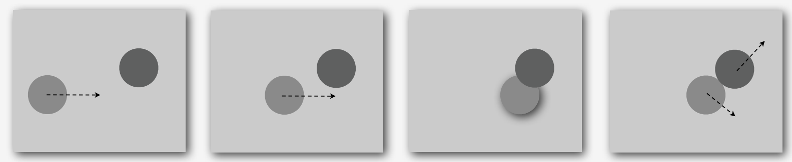
check for collision with walls

Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: what object does the checks? too many checks?

Time-driven simulation

- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.

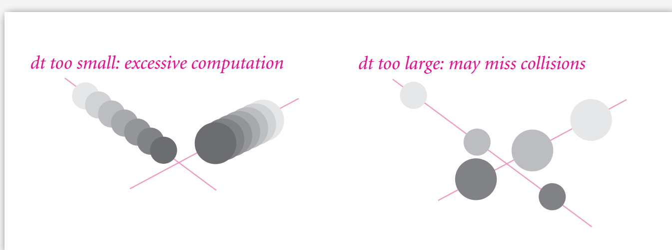


t | t + dt | t + 2 dt (collision detected) | t + Δt (roll back clock)

Time-driven simulation

Main drawbacks.

- ~ $N^2/2$ overlap checks per time quantum.
- Simulation is too slow if dt is very small.
- May miss collisions if dt is too large and colliding particles fail to overlap when we are looking.



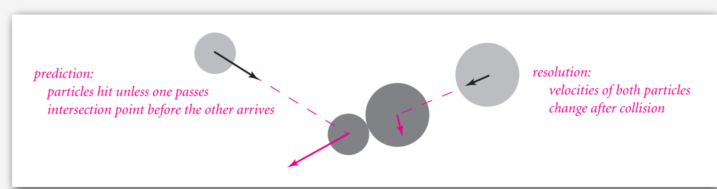
Event-driven simulation

Change state only when something happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Remove the min = get next collision.

Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

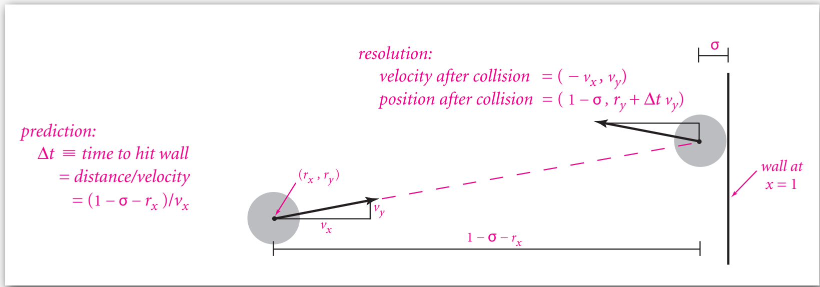
Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.



Particle-wall collision

Collision prediction and resolution.

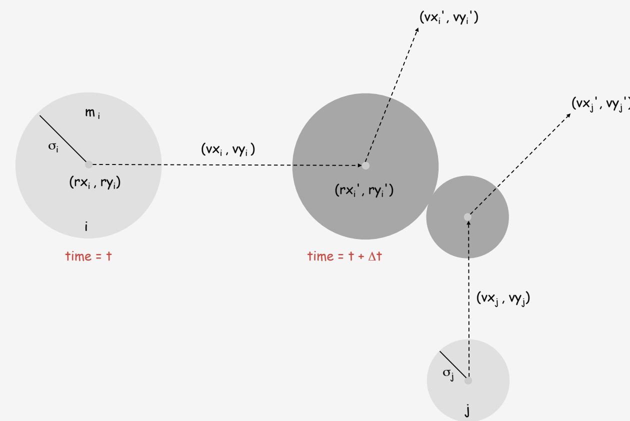
- Particle of radius σ at position (rx, ry).
- Particle moving in unit box with velocity (vx, vy).
- Will it collide with a vertical wall? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius σ_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius σ_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius σ_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius σ_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \ge 0 \\ \infty & \text{if } d < 0 \\ -\dfrac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v)\,(\Delta r \cdot \Delta r - \sigma^2) \qquad \sigma = \sigma_i + \sigma_j$$

$$\Delta v = (\Delta vx, \Delta vy) = (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx, \Delta ry) = (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$\begin{aligned} vx_i' &= vx_i + Jx / m_i \\ vy_i' &= vy_i + Jy / m_i \\ vx_j' &= vx_j - Jx / m_j \\ vy_j' &= vx_j - Jy / m_j \end{aligned}$$

← Newton's second law (momentum form)

$$Jx = \frac{J\,\Delta rx}{\sigma}, \quad Jy = \frac{J\,\Delta ry}{\sigma}, \quad J = \frac{2\, m_i\, m_j\, (\Delta v \cdot \Delta r)}{\sigma (m_i + m_j)}$$

impulse due to normal force
(conservation of energy, conservation of momentum)

Particle data type skeleton

```
public class Particle
{
    private double rx, ry;        // position
    private double vx, vy;        // velocity
    private final double radius;  // radius
    private final double mass;    // mass
    private int count;            // number of collisions

    public Particle(...) { }

    public void move(double dt) { }
    public void draw()          { }

    public double dt(Particle that) { }
    public double dtX() { }
    public double dtY() { }

    public void bounce(Particle that) { }
    public void bounceX() { }
    public void bounceY() { }

}
```

predict collision with particle or wall (dt, dtX, dtY)

resolve collision with particle or wall (bounce, bounceX, bounceY)

Particle-particle collision and resolution implementation

```
public double dt(Particle that)
{
   if (this == that) return INFINITY;
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx; dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   if( dvdr > 0) return INFINITY;
   double dvdv = dvx*dvx + dvy*dvy;
   double drdr = dx*dx + dy*dy;
   double sigma = this.radius + that.radius;
   double d = (dvdr*dvdr) - dvdv * (drdr - sigma*sigma);
   if (d < 0) return INFINITY;
   return -(dvdr + Math.sqrt(d)) / dvdv;
}
```

no collision (`if( dvdr > 0) return INFINITY;` and `if (d < 0) return INFINITY;`)

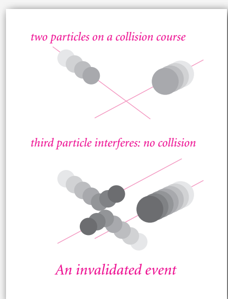
```
public void bounce(Particle that)
{
   double dx  = that.rx - this.rx, dy  = that.ry - this.ry;
   double dvx = that.vx - this.vx; dvy = that.vy - this.vy;
   double dvdr = dx*dvx + dy*dvy;
   double dist = this.radius + that.radius;
   double J = 2 * this.mass * that.mass * dvdr / ((this.mass + that.mass) * dist);
   double Jx = J * dx / dist;
   double Jy = J * dy / dist;
   this.vx += Jx / this.mass;
   this.vy += Jy / this.mass;
   that.vx -= Jx / that.mass;
   that.vy -= Jy / that.mass;
   this.count++;
   that.count++;
}
```


Collision system: event-driven simulation main loop

Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

"potential" since collision may not happen if some other collision intervenes



An invalidated event

Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event has been invalidated, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

Event data type

Conventions.

- Neither particle `null` ⇒ particle-particle collision.
- One particle `null` ⇒ particle-wall collision.
- Both particles `null` ⇒ redraw event.

```
public class Event implements Comparable<Event>
{
    private double time;          // time of event
    private Particle a, b;        // particles involved in event
    private int countA, countB;   // collision counts for a and b

    public Event(double t, Particle a, Particle b) { }

    public double time()    { return time; }
    public Particle a()     { return a;    }
    public Particle b()     { return b;    }

    public int compareTo(Event that)
    {   return this.time - that.time;    }

    public boolean isValid()
    {   }
}
```

create event

accessor methods

ordered by time

invalid if intervening collision

Collision system implementation: skeleton

```
public class CollisionSystem
{
    private MinPQ<Event> pq;          // the priority queue
    private double t  = 0.0;          // simulation clock time
    private Particle[] particles;     // the array of particles

    public CollisionSystem(Particle[] particles) { }

    private void predict(Particle a)
    {
       if (a == null) return;
       for (int i = 0; i < N; i++)
       {
          double dt = a.dt(particles[i]);
          pq.insert(new Event(t + dt, a, particles[i]));
       }
       pq.insert(new Event(t + a.dtX(), a, null));
       pq.insert(new Event(t + a.dtY(), null, a));
 }

    private void redraw()  { }

    public void simulate() {  /* see next slide */  }
}
```

add all particle-wall and particle-particle collisions involving this particle to the PQ

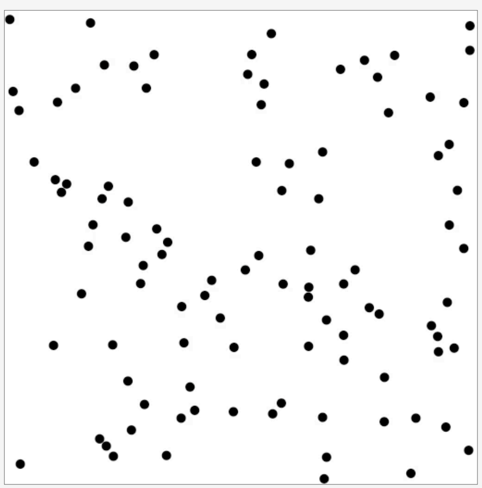
Collision system implementation: main event-driven simulation loop

```
public void simulate()
{
   pq = new MinPQ<Event>();
   for(int i = 0; i < N; i++) predict(particles[i]);
   pq.insert(new Event(0, null, null));

   while(!pq.isEmpty())
   {
      Event event = pq.delMin();
      if(!event.isValid()) continue;
      Particle a = event.a();
      Particle b = event.b();

      for(int i = 0; i < N; i++)
         particles[i].move(event.time() - t);
      t = event.time();

      if      (a != null && b != null) a.bounce(b);
      else if (a != null && b == null) a.bounceX()
      else if (a == null && b != null) b.bounceY();
      else if (a == null && b == null) redraw();

      predict(a);
      predict(b);
   }
}
```

- initialize PQ with collision events and redraw event
- get next event
- update positions and time
- process event
- predict new events based on changes

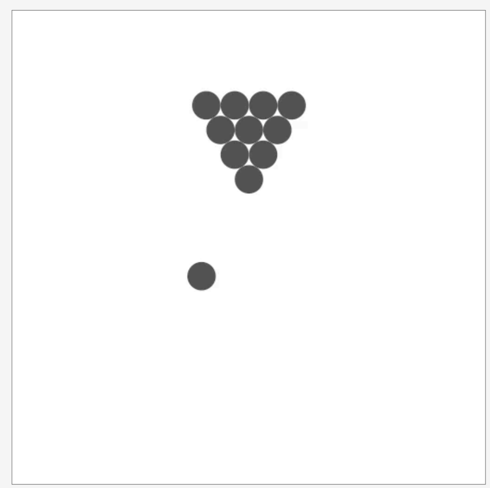
Simulation example 1

```
% java CollisionSystem 100
```



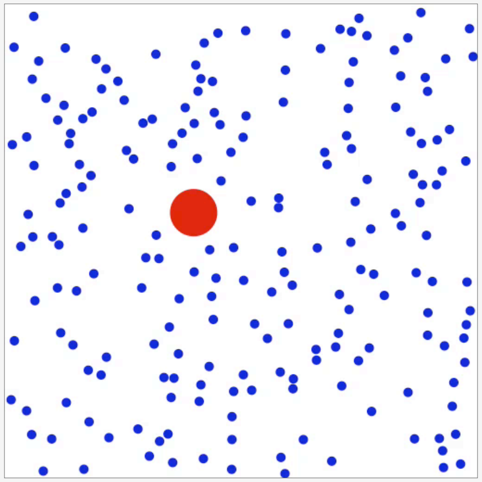
Simulation example 2

```
% java CollisionSystem < billiards.txt
```



Simulation example 3

% java CollisionSystem < brownian.txt



Simulation example 4

% java CollisionSystem < diffusion.txt