Linear Algebra Review

Most slides are courtesy of Prof. Octavia I. Camps, Penn State University, with some modification

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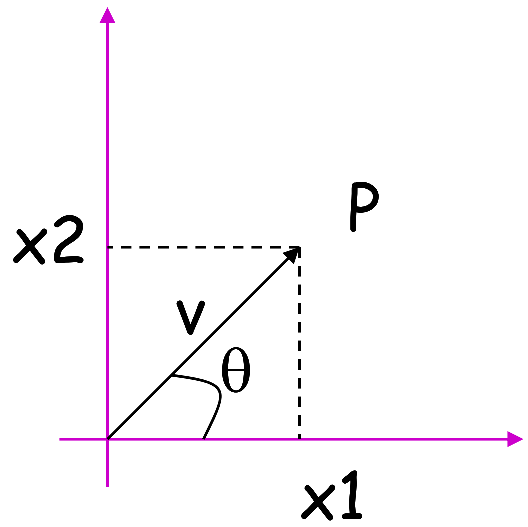
Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

2D Vector

$$\mathbf{v} = (x_1, x_2)$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

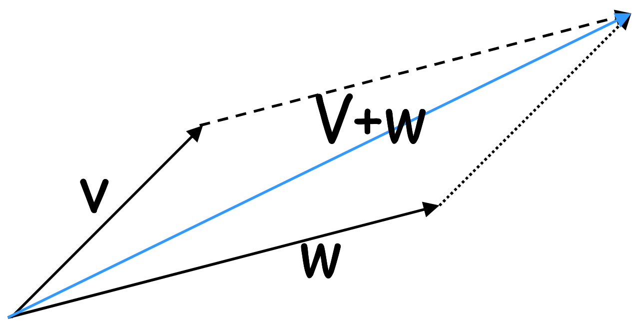
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ Is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right)$$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1} \right)$

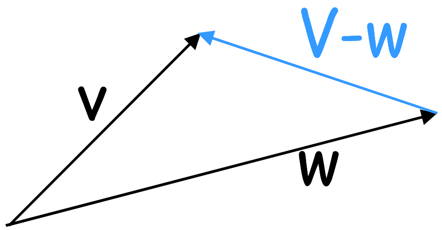
Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



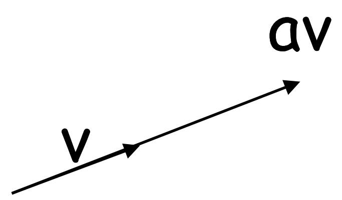
Vector Subtraction

$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$



Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Linearly independent vectors

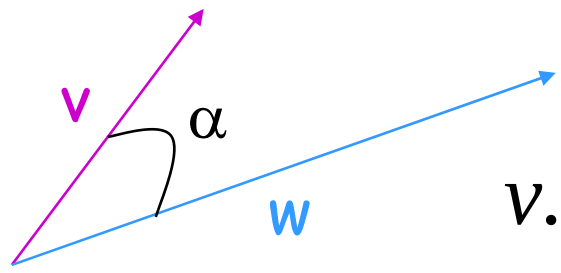
- No vector is a linear combination of others,
- or equivalently

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \ldots + \lambda_i v_i = 0 \quad \text{only for } \lambda_1 = \lambda_2 = \ldots = \lambda_i = 0$$

Basis

- span(V): span of a set of vectors V is all linear combinations of vectors v_i , i.e. a vector space.
- Basis of a vector space: a set of vectors that are linearly independent and that span the space.

Inner (dot) Product



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2 . y_2$$

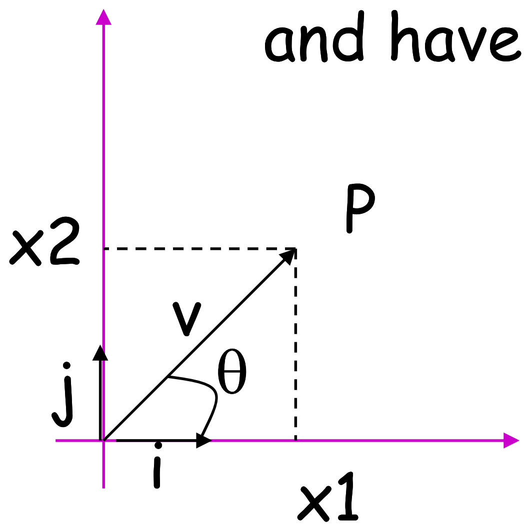
The inner product is a SCALAR!

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

Orthonormal Basis

basis vectors are perpendicular to each other and have unit length.



$$\mathbf{i} = (1,0) \quad \|\mathbf{i}\| = 1$$

$$\mathbf{j} = (0,1) \quad \|\mathbf{j}\| = 1$$

$$\mathbf{i} \cdot \mathbf{j} = 0$$

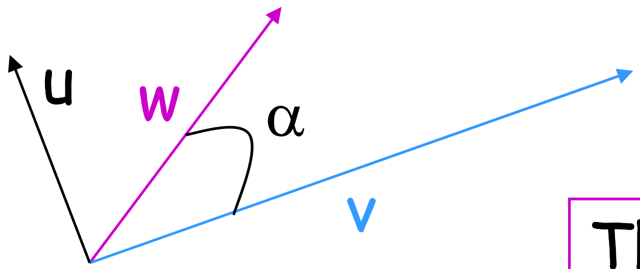
$$\mathbf{v} = (x_1, x_2)$$

$$\mathbf{v} = x_1.\mathbf{i} + x_2.\mathbf{j}$$

$$\mathbf{v}.\mathbf{i} = (x_1.\mathbf{i} + x_2.\mathbf{j}).\mathbf{i} = x_1.1 + x_2.0 = x_1$$

$$\mathbf{v}.\mathbf{j} = (x_1.\mathbf{i} + x_2.\mathbf{j}).\mathbf{j} = x_1.0 + x_2.1 = x_2$$

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\| u \| = \| v.w \| = \| v \| \| w \| \sin\alpha$

Orientation:

$$u \perp v \Rightarrow u \cdot v = (v \times w) \cdot v = 0$$

$$u \perp w \Rightarrow u \cdot w = (v \times w) \cdot w = 0$$

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Vector Product Computation

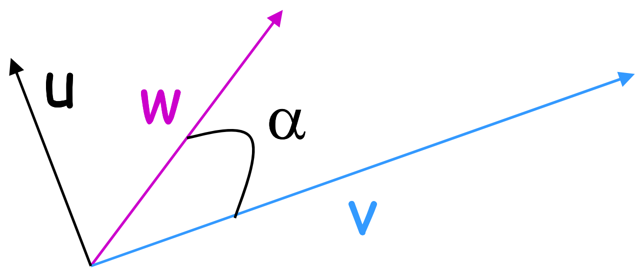
$$\mathbf{i} = (1,0,0) \qquad \|\mathbf{i}\| = 1$$

$$\mathbf{j} = (0,1,0) \qquad \|\mathbf{j}\| = 1 \qquad \mathbf{i}\cdot\mathbf{j} = 0, \mathbf{i}\cdot\mathbf{k} = 0, \mathbf{j}\cdot\mathbf{k} = 0$$

$$\mathbf{k} = (0,0,1) \qquad \|\mathbf{k}\| = 1$$

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$



$$= (x_2 y_3 - x_3 y_2)\mathbf{i} + (x_3 y_1 - x_1 y_3)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse: A must be square

$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

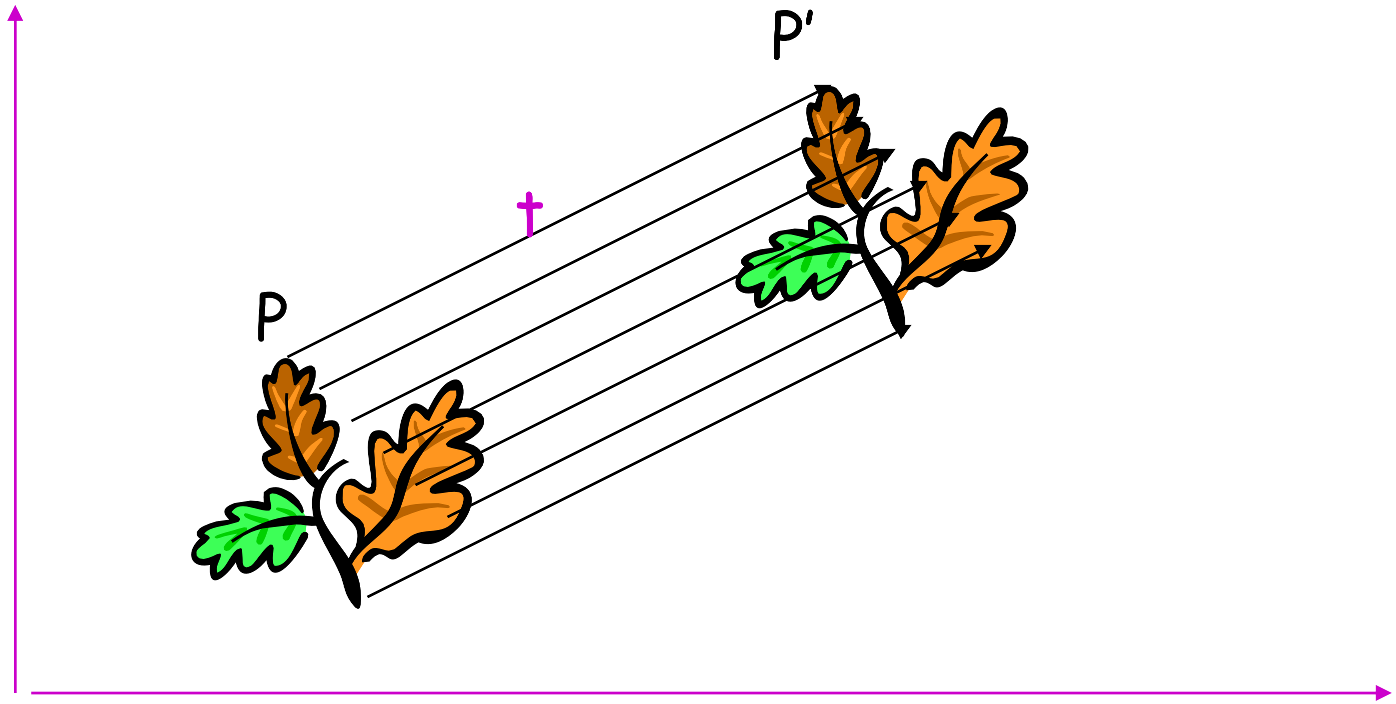
Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

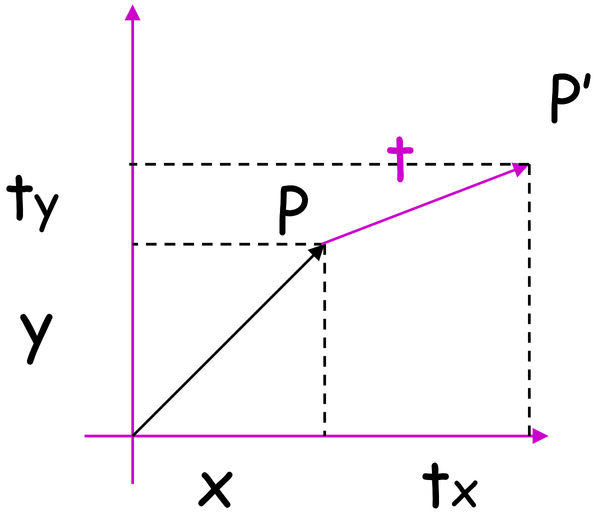
$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} . \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} . \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D Geometrical Transformations

2D Translation



2D Translation Equation

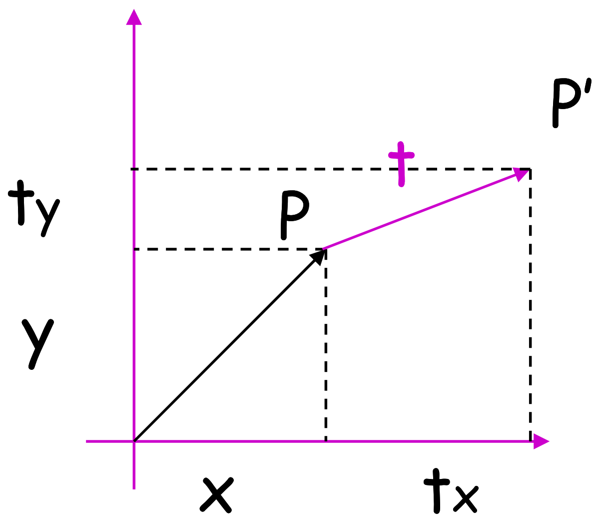


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

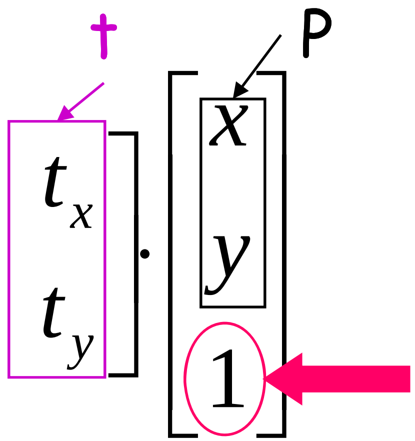
$$\mathbf{P'} = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$



$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \to (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \to (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

- NOTE: If the scalar is 1, there is no need for the multiplication!

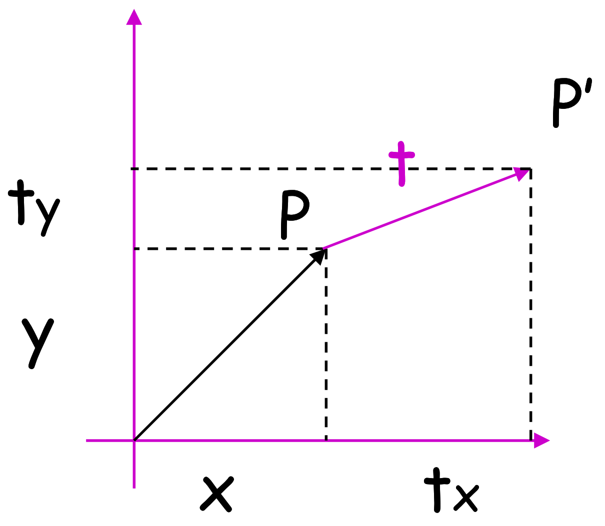
Back to Cartesian Coordinates:

- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

2D Translation using Homogeneous Coordinates



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

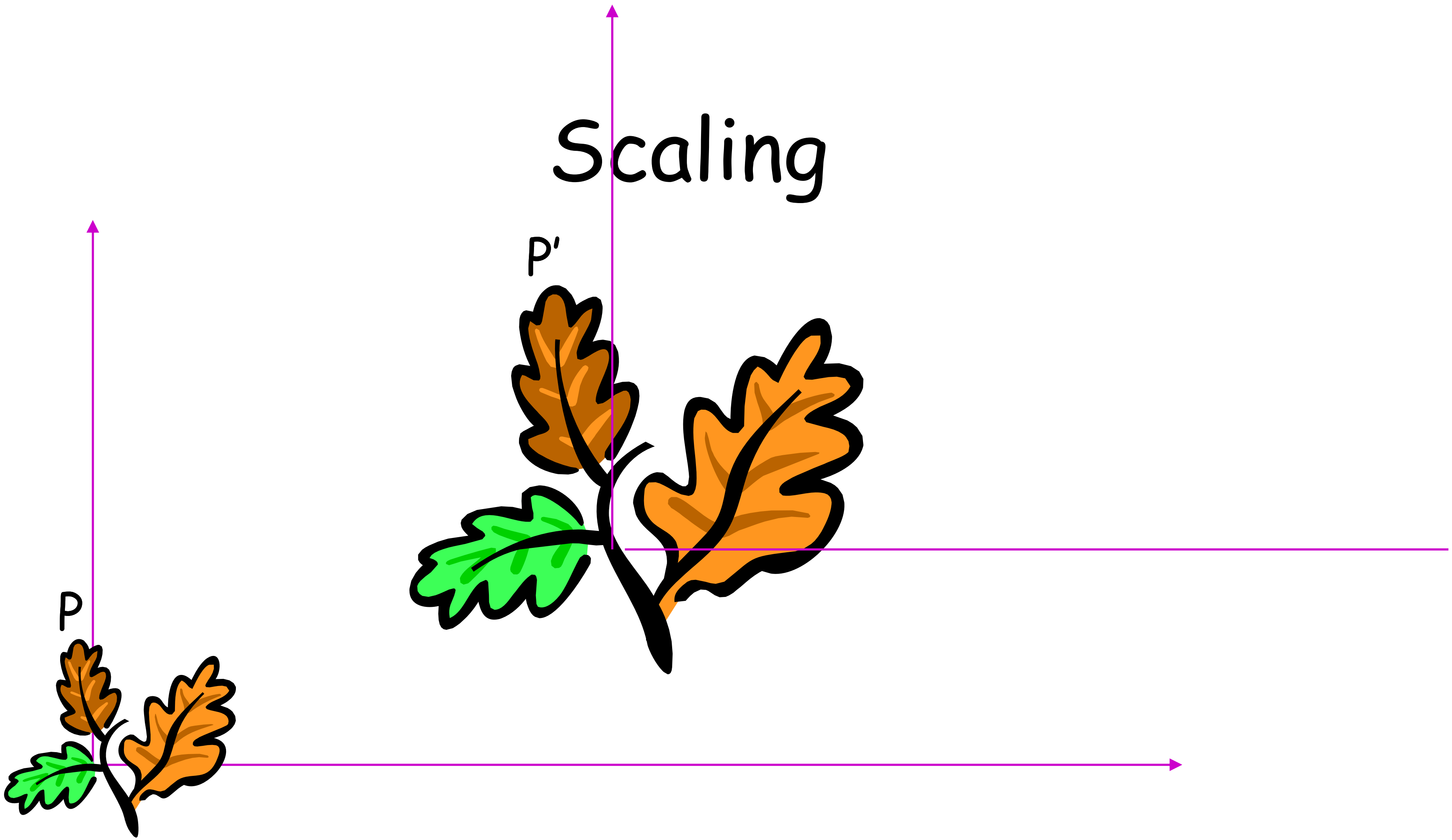
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

t

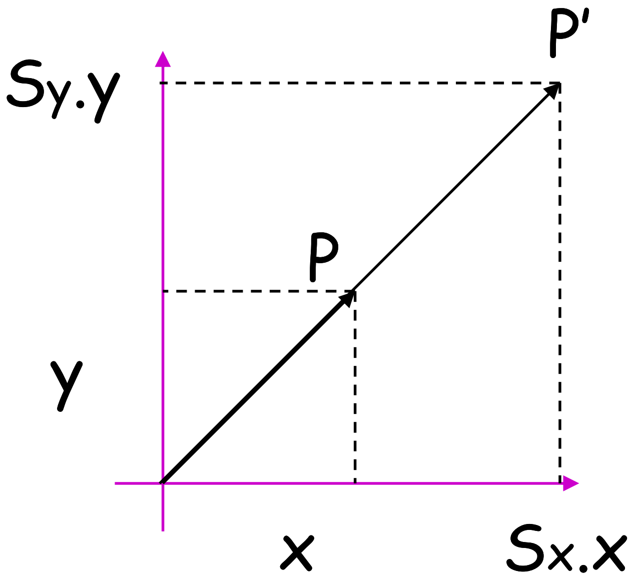
P

$$\mathbf{P}' = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



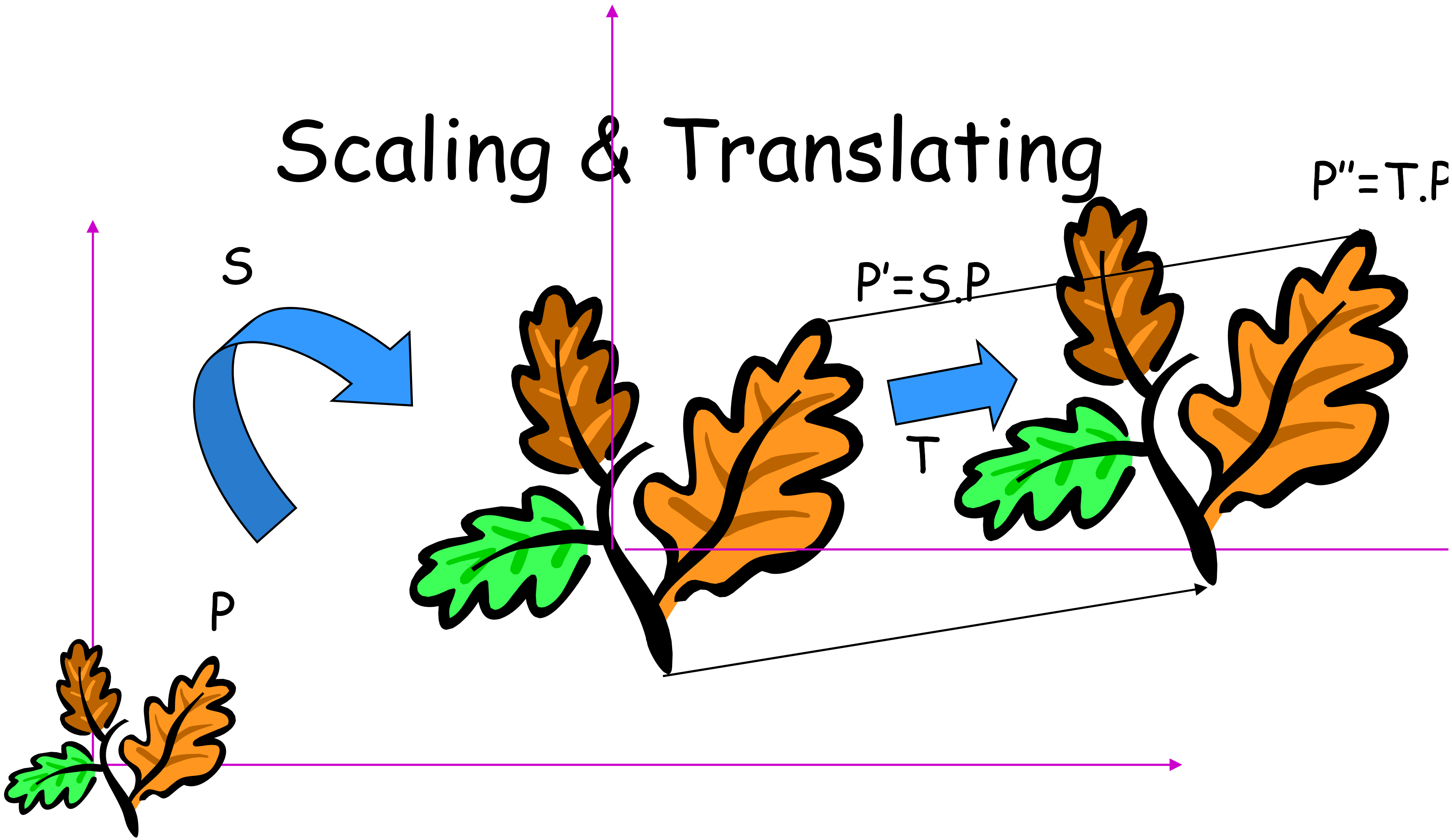
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



P''=T.P'=T.(S.P)=(T.S).P

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Scaling & Translating

P''=T.P'=T.(S.P)=(T.S).P

$$\mathbf{P''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

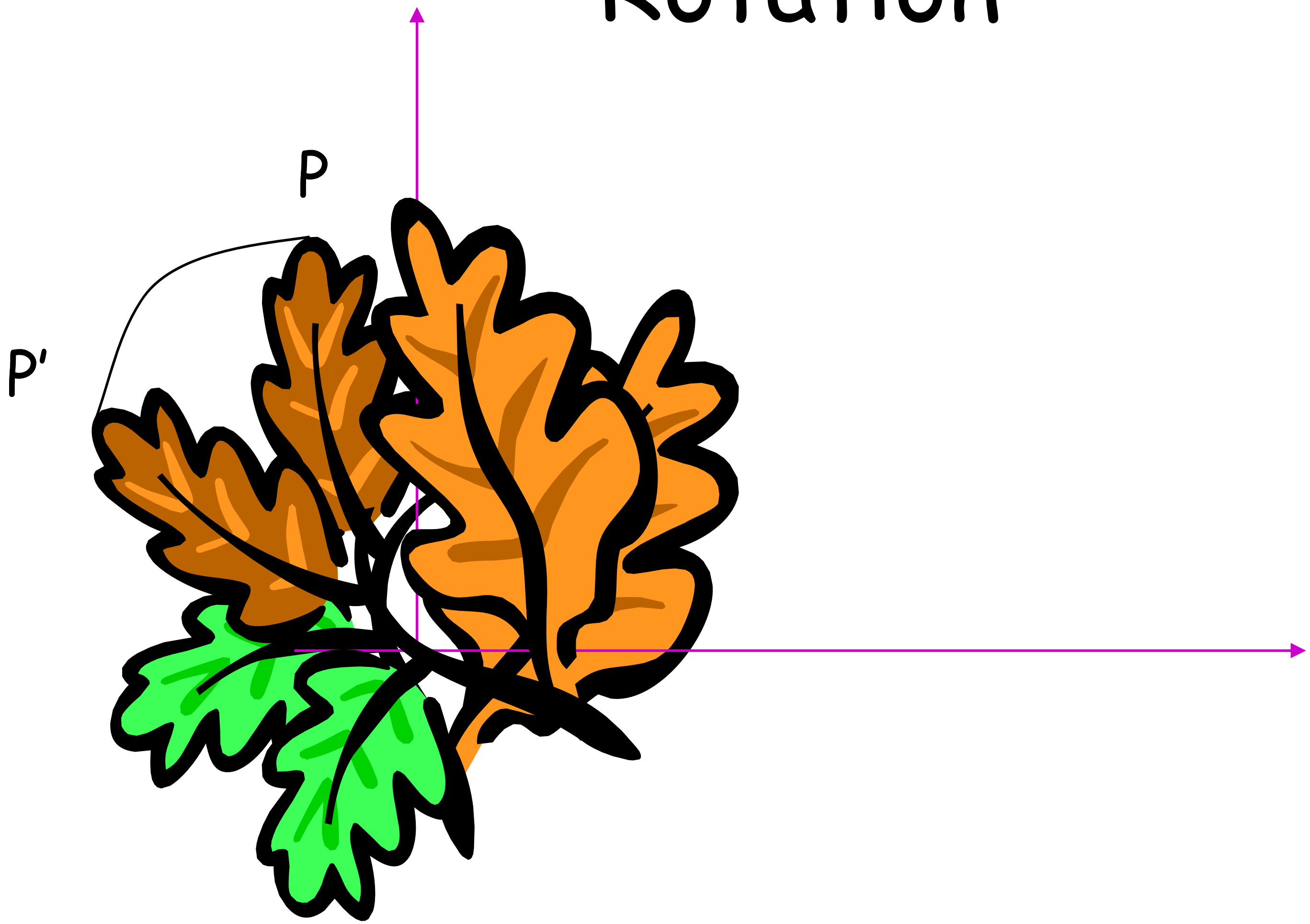
Translating & Scaling $\neq$ Scaling & Translating

P''=S.P'=S.(T.P)=(S.T).P

$$\mathbf{P}''=\mathbf{S}\cdot\mathbf{T}\cdot\mathbf{P}=\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=$$

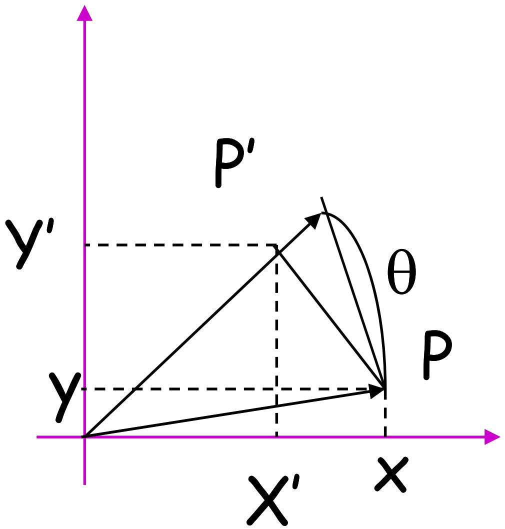
$$=\begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R.P}$$

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Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 ⟹ 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

i.e. R is an orthogonal matrix

Orthogonal matrix

For square matrix *A*,

A is an orthogonal matrix

iff $AA^{\mathbf{T}} = A^{\mathbf{T}}A = \mathbf{I}$

iff columns of *A* are an orthonomal basis

iff rows of *A* are an orthonomal basis

Orthogonal matrix preserves length

$$\| Ax \| = \| x \|$$

Scaling, Translating & Rotating

Order matters!

$P' = S.P$

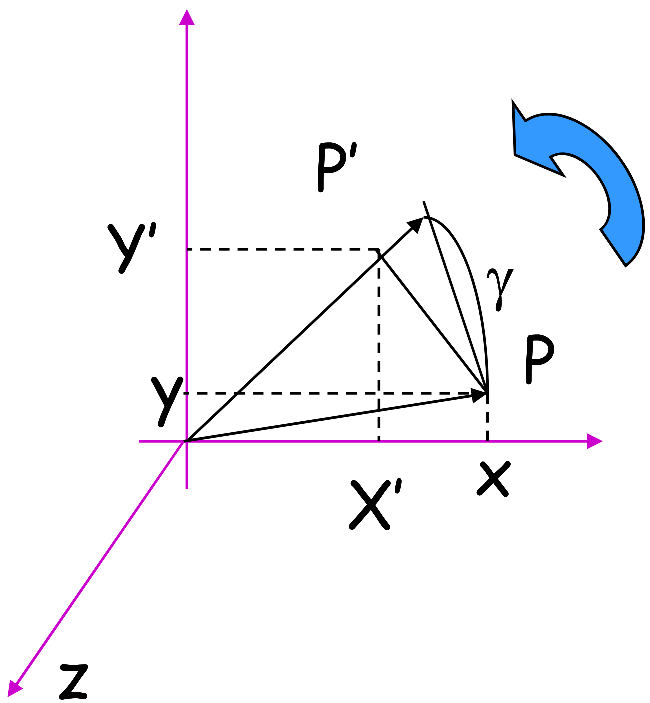
$P'' = T.P' = (T.S).P$

$P''' = R.P'' = R.(T.S).P = (R.T.S).P$

$R.T.S \neq R.S.T \neq T.S.R \ldots$

3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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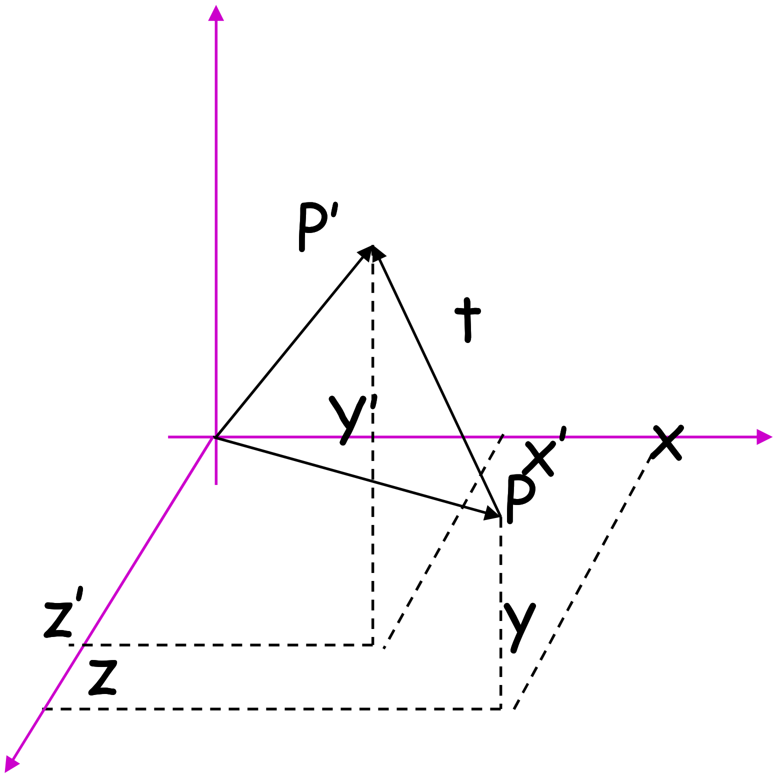
3D Rotation (axis & angle)

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T, \ \|n\|=1, \ \text{angle } \theta,$$

$$\mathbf{R} = \mathbf{I}\cos\theta + \mathbf{I}(1-\cos\theta)\begin{bmatrix} n_1^{\ 2} & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^{\ 2} & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^{\ 2} \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

3D Translation of Points

Translate by a vector $t=(t_x,t_y,t_x)^T$:



$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change of basis

Linear transform: y = Ax (A is square)

Express x and y in a new basis E

x' = Px

y' = Py

P: change of basis matrix

The same transform represented in E

$$y' = PAP^{-1}x'$$

Matrix A and B are *similar* if $B = PAP^{-1}$

Eigenvalues and Eigenvectors

- Square matrix possesses its own natural basis.
- Eigen relation

$$\mathbf{Au}=\lambda\mathbf{u}$$

- Matrix **A** acts on vector **u** and produces a scaled version of the vector.
- Eigen is a German word meaning "proper" or "specific"
- **u** is the eigenvector while λ is the eigenvalue.
  - If **u** is an eigenvector so is $\alpha\mathbf{u}$
  - If $||\mathbf{u}||=1$ then we call it a normal eigenvector
  - λ is like a measure of the "strength" of **A** in the direction of **u**
- Set of all eigenvalues and eigenvectors of **A** is called the "spectrum of A"

$$Ax = \lambda x \rightarrow (\lambda I - A)x = 0 \quad \text{hence } (\lambda I - A) \text{ is singular}$$

The eigenvalues of A are the roots of the characteristic equation

$$p(\lambda) = \det(\lambda I - A) = 0$$

If A and B are similar, i.e., $B = PAP^{-1}$

(1)A and B have the same eigenvalues

(2)if x is an eigenvector of A, Px is an eigenvector of B

Spectral theorem:

if A is symmetric

(1)all eigenvalues of A are real.

(2)there is an orthonormal basis consisting of eigenvectors of A

$$A = U \Lambda U^T = U \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \lambda_N \end{bmatrix} U^T$$

U is orthogonal.

Columns of U are eigenvectors of A.

A is positive (semi)definite:

for any nonzero vector x, $x^T Ax > (\geq) 0$

if A is symmetric and positive (semi)definite

all eigenvalues of A are positive(nonnegative)

Rank and Nullspace

$$\underset{m\times n}{A}\;\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Range of a $m\times n$ dimensional matrix **A**

  Range (**A**) = $\{\mathbf{y}\in\mathbb{R}^m\text{: }\mathbf{y}=\mathbf{A}\mathbf{x}\text{ for some }\mathbf{x}\in\mathbb{R}^n\}$

- Null space of **A** is the set of vectors which it takes to zero.

  Null(**A**) = $\{\mathbf{x}\in\mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\}$

- Rank of a matrix is the dimension of its range.

  Rank (**A**) = Rank ($\mathbf{A}^t$)

  – Maximal number of independent rows **or** columns

- Dimension of Null(**A**)+Rank(**A**) = n

Least Squares

$$Ax = b$$

- More equations than unknowns
- Look for solution which minimizes ||Ax-b|| = $(Ax-b)^T(Ax-b)$
- Solve $$\frac{\partial (Ax-b)^T(Ax-b)}{\partial x_i} = 0$$
- Same as the solution to $$A^T A x = A^T b$$
- LS solution $x = (A^T A)^{-1} A^T b$ when $A^T A$ is invertible

Singular Value Decomposition

- Chief tool for dealing with *m* by *n* systems and singular systems.
- **Singular values**: Non negative square roots of the eigenvalues of $\mathbf{A}^t\mathbf{A}$. Denoted $\sigma_i,\ i=1,\dots,n$
  - $\mathbf{A}^t\mathbf{A}$ is symmetric → eigenvalues and singular values are real.
- SVD: If **A** is a real *m* by *n* matrix then there exist orthogonal matrices **U** ($\in\mathbb{R}^{m\times m}$) and **V** ($\in\mathbb{R}^{n\times n}$) such that $\mathbf{U}^t\mathbf{A}\mathbf{V}=\Sigma=\text{diag}(\sigma_1,\ \sigma_2,\dots,\ \sigma_p)$ $\qquad p=\min\{m,n\}$

$$\mathbf{A}=\mathbf{U}\,\Sigma\,\mathbf{V}^t$$

- Geometrically, singular values are the lengths of the hyperellipsoid defined by $E=\{\mathbf{Ax}:\ \|\mathbf{x}\|_2=1\}$
- Singular values arranged in decreasing order.

Properties of the SVD

- Suppose we know the singular values of **A** and we know *r* are non zero

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \sigma_{r+1} = \ldots = \sigma_p = 0$$

  - Rank(**A**) = *r*.
  - Null(**A**) = span$\{\mathbf{v}_{r+1},\ldots,\mathbf{v}_n\}$
  - Range(**A**)=span$\{\mathbf{u}_1,\ldots,\mathbf{u}_r\}$
- $||A||_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_p^2$ $\qquad$ $||A||_2 = \sigma_1$
- *Numerical rank:* If *k* singular values of *A* are larger than a given number ε. Then the ε rank of A is *k*.
- Distance of a matrix of rank *n* from being a matrix of rank $k = \sigma_{k+1}$

Properties of SVD

σ_i^2 are eigenvalues of A^TA

Columns of U (u_1 , u_2 , u_3) are eigenvectors of AA^T

Columns of V (v_1 , v_2 , v_3) are eigenvectors of A^TA

$$\| A \|_F = \sum_{i,j} a_{i,j}^2 = \sum \sigma_i^2$$

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Why is it useful?

- Square matrix may be singular due to round-off errors. Can compute a "regularized" solution
  - $$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=(\mathbf{U}\Sigma\mathbf{V}^{\mathbf{t}})^{-1}\mathbf{b} = \sum_{i=1}^{n}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i$$
- If σ_i is small (vanishes) the solution "blows up"
- Given a tolerance ε we can determine a solution that is "closest" to the solution of the original equation, but that does not "blow up" $$\mathbf{x}_r = \sum_{i=1}^{k}\frac{\mathbf{u}_i^t\mathbf{b}}{\sigma_i}\mathbf{v}_i \qquad \sigma_k > \varepsilon, \;\; \sigma_{k+1} \le \varepsilon$$
- Least squares solution is the x that satisfies
  $\mathbf{A}^t\mathbf{A}\mathbf{x}=\mathbf{A}^t\mathbf{b}$
- can be effectively solved using SVD

Solving $(A^t A)x = A^t b$ when rank(A) < n

$A^+ = V\Sigma^+ U^T$ pseudoinverse of A

$$\Sigma^+_{ij} = \begin{cases} \dfrac{1}{\Sigma_{ii}}, & if\ i = j\ and\ \Sigma_{ii} \neq 0 \\ 0 & \end{cases}$$

$x = A^+ b$ is the solution with minimum ||x||

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Least squares solution of homogeneous equation Ax=0

$$\text{Minimize} \| Ax \| \quad \text{subject to} \quad \| x \| = 1$$

$$A = UDV^T$$

$$\| UDV^T x \| = \| DV^T x \| \quad \text{and} \quad \| x \| = \| \mathrm{V}^{\mathrm{T}} x \|$$

$$y = V^T x \quad \rightarrow \quad \text{minimize} \quad \| \mathrm{DV}^{\mathrm{T}} x \| \text{subject to} \| \mathrm{V}^{\mathrm{T}} x \| = 1$$

$$\text{or} \quad \| \mathrm{Dy} \| \text{subject to} \| \mathrm{y} \| = 1$$

diagonal elements of D in descending order

$$\mathrm{y} = \begin{pmatrix} 0 \\ 0 \\ . \\ 1 \end{pmatrix} \quad \rightarrow \quad x = Vy \rightarrow \boxed{\text{last column of } V}$$

Enforce orthonormality constraints on an estimated rotation matrix R'

$$R' = UDV^T$$

replace by $R = UIV^T$ $\qquad$ I is identity matrix

Gauss-Newton iteration

Minimize $\| f(x) \|$, where

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$$

Approximate f(x) by

$$f(x) = f(x_k) + J_k \Delta_k,$$

$$J_k = \nabla f(x_k), \Delta_k = x - x_k$$

Minimize $\| f(x) \| \Leftrightarrow J_k^T J_k \Delta_k = -J_k^T f(x_k)$

Levenberg Marquardt iteration

Change $J_k^T J_k \Delta_k = -J_k^T f(x_k)$

To $(J_k^T J_k + \lambda I)\Delta_k = -J_k^T f(x_k)$

- Avoid singular $J_k^T J_k$
- Control step size
  - When $\| f(x) \|$ reduces rapidly, decrease λ
  - Otherwise increase λ

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