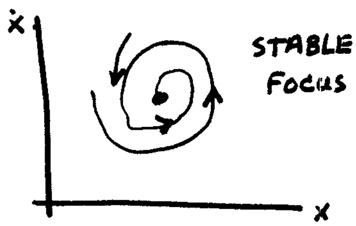
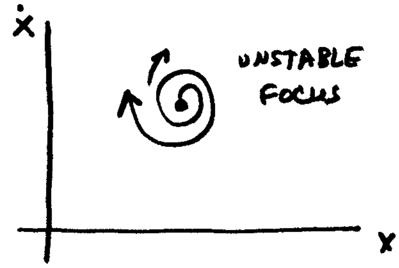
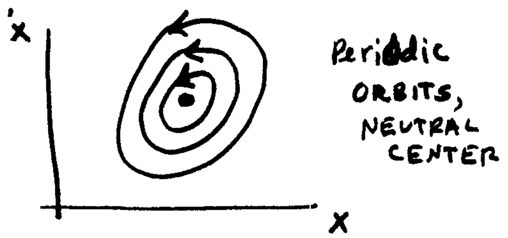
Review of ODE behavior in State Space.

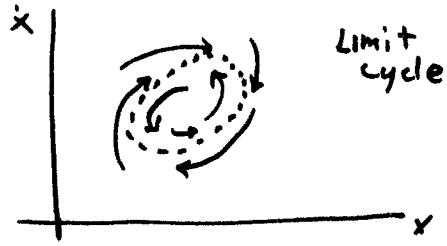
Sources
[Dic 91]
[EK88]

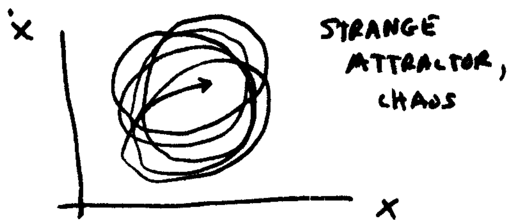
(2-d examples)

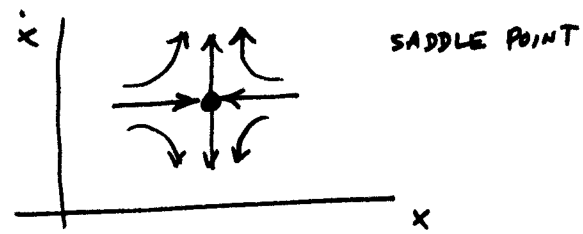


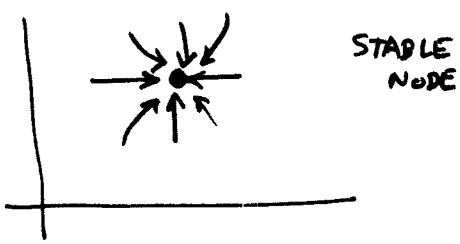


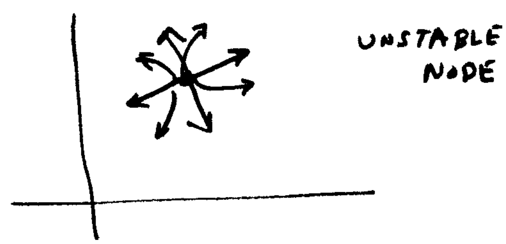




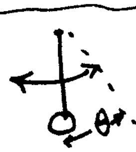




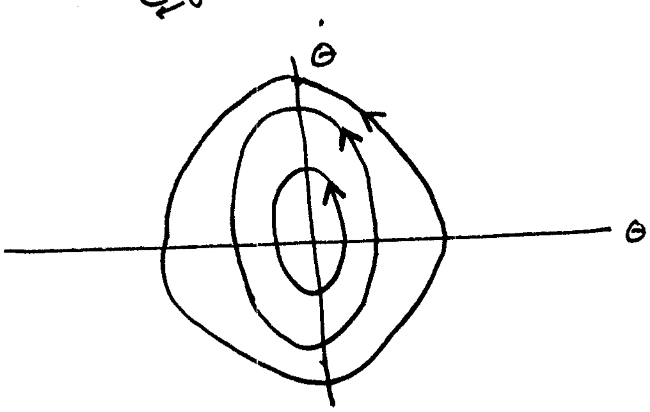




PENDULUM:



$$\frac{d^2\theta}{dt^2} + f^2 \sin\theta = 0$$



and then...?

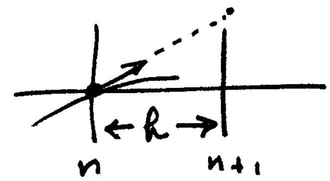
[PTVF 92]

Why is Runge-Kutta better?

Let's analyze Euler & 2^{nd} Order R-K.

Euler $\quad \dot{x} = f(x) \quad$ (x is in general a vector)

$$\hat{x}_{n+1} = x_n + h\,\underbrace{\dot{x}_n}_{f(x_n)}$$

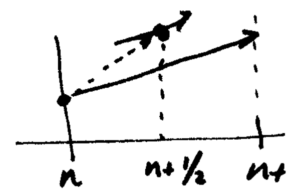


$h = \Delta t$

true $\quad x_{n+1} = x_n + h\,\dot{x}_n + O(h^2) \quad$ ← Taylor's Series

$$\boxed{\text{Local truncation Error} = x_{n+1} - \hat{x}_{n+1} = O(h^2)}$$

2^{nd}-Order R-K



uses f at Euler half-step

$$\hat{x}_{n+1} = x_n + h \cdot f(\hat{x}_{n+1/2})$$

$$= x_n + h \cdot \left[f(x_{n+1/2}) + O(h^2) \right] \quad \leftarrow \text{half-Euler step}$$

$$= x_n + h \cdot \left[f(x_n) + \frac{h}{2}\dot{f}(x_n) + O(h^2) \right] \quad \text{Taylor's series}$$

$$= x_n + h f(x_n) + \frac{h^2}{2}\dot{f}(x_n) + O(h^3)$$

true $\quad x_{n+1} = x_n + h f(x_n) + \frac{h^2}{2}\dot{f}(x_n) + O(h^3) \quad$ Taylor's Series

$$\boxed{x_{n+1} - \hat{x}_{n+1} = O(h^3) \quad \text{Local truncation Error}}$$

(4^{th}-Order R-K is $O(h^5)$)

this is the local truncation error — at one step. Error accumulates from step to step.

In general, if equation is "well behaved" (smooth derivatives, etc.), then

$$\text{total error} \sim \#\text{steps} \cdot (\text{local error})$$
$$\sim \frac{1}{h} \cdot (\text{local error})$$

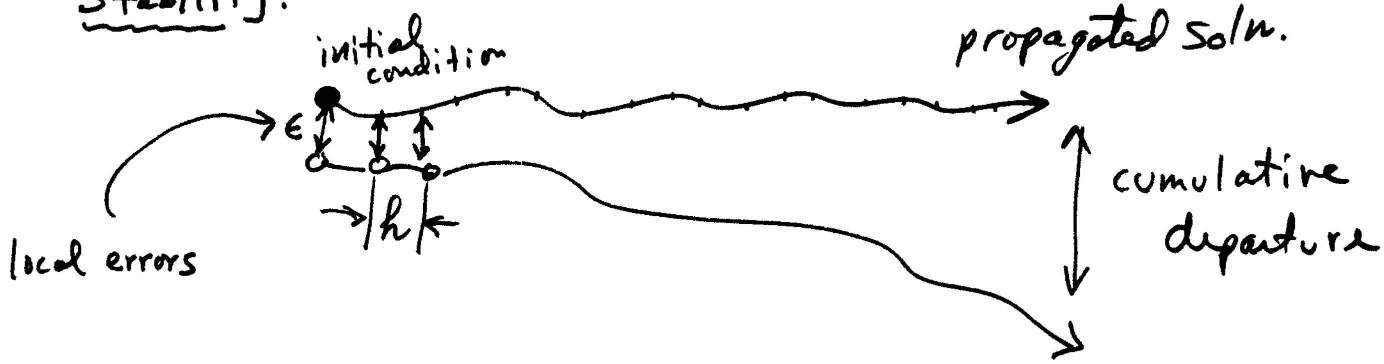
$$\therefore \begin{array}{l}\text{total error} \\ \text{for Euler's method}\end{array} \sim O(h)$$

$$\begin{array}{l}\text{total error} \\ \text{for } 4^{th} \text{ order} \\ \text{R-K}\end{array} \sim O(h^4)$$

See [Atk93, p. 331]

Stability and Convergence

Stability:



Said to be pointwise stable if cumulative departure

$\to 0$ as $|\epsilon\text{'s}| \to 0$

& increases no faster than $O(1/h^K)$ as $h \to 0$

(not exponential)

Convergence: If soln. of discrete problem

$\longrightarrow$ soln. of continuous problem

as $h \to 0$ (no errors)

For ODE's, in general, $\boxed{\text{Stability} \Leftrightarrow \text{Convergence}}$

Not always so for PDE's.

We Require an additional condition —

"Consistency"

<u>Consistent</u> if

Local truncation Error $\rightarrow 0$ as grid size(s) $\rightarrow 0$

[not always true for PDE's!]

<u>LAX'S EQUIVALENCE THEOREM</u>

Given an initial boundary value problem & a finite-difference approximation that is <u>consistent</u>, then

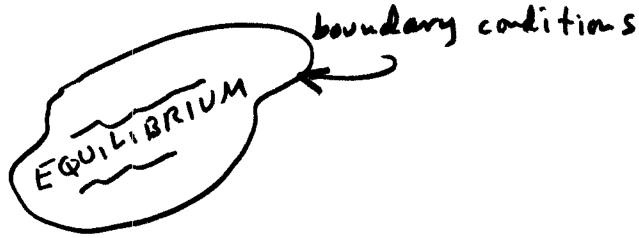
$$\boxed{\text{Stability} \Longleftrightarrow \text{convergence}}$$

[AMES 92]
[SMITH 85]

Partial Differential Equations

Fall into three "physical" classes

Elliptic



STEADY-STATE TEMPERATURE
ELECTROSTATIC POTENTIAL
GRAVITATIONAL POTENTIAL

EXAMPLES LAPLACE'S EQUATION

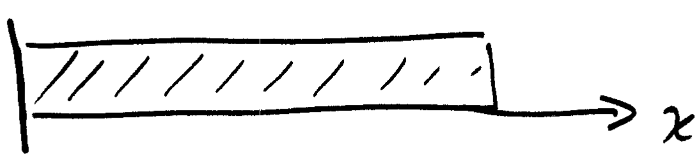
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (\nabla^2 \phi = 0)$$

POISSON'S EQUATION

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + g = 0 \qquad (\nabla^2 \phi = -g)$$

PARABOLIC (t is one of the independent variables)

Simplest example: heat conduction along bar



Given initial Temperature distribution, find temperature at later times.

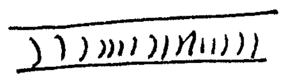
$$\partial u / \partial t = k \, \partial^2 u / \partial x^2$$

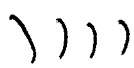
Hyperbolic Equations (also involve time)

typically, vibration, propagation of shockwaves

simplest is $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$ wave eqn.

vibrating string


vibrating air in pipe


electromagnetic waves

Classification really a physical classification based on "characteristics" (See [Ames 92], e.g.)

for 2nd Order Equations:

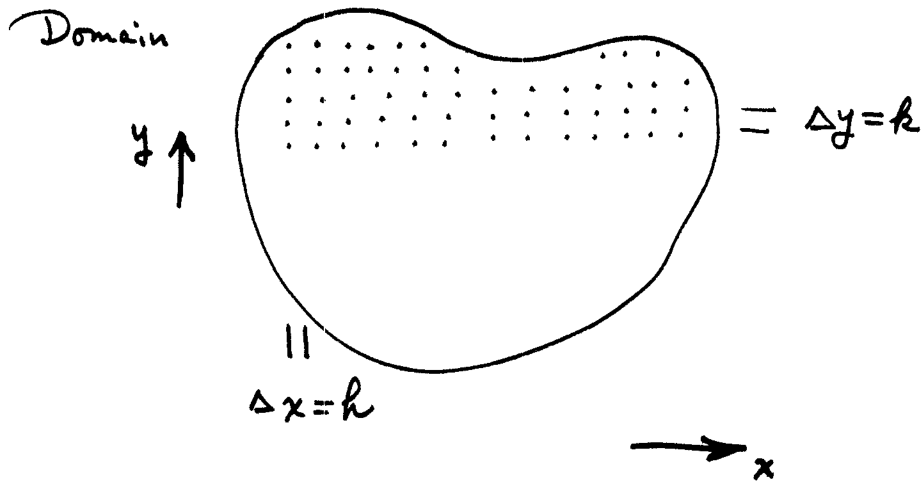
$$\underbrace{a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2}} + d \frac{\partial \phi}{\partial x} + e \frac{\partial \phi}{\partial y} + f \phi + g = 0$$

$b^2 - 4ac < 0 \Rightarrow$ Elliptic

$b^2 - 4ac = 0 \Rightarrow$ parabolic

$b^2 - 4ac > 0 \Rightarrow$ hyperbolic

Finite Differences & Computational Molecules [Ames 92]



Notation Solution $u(x,y)$ (continuous)

$$u_{ij} = u(ih, jk) \quad \text{(discrete approximation)}$$

As always, start with Taylor Series (here in x)

$$u(x+\Delta x, y) = u(x,y) + \Delta x \frac{\partial u(x,y)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}(x,y) + O[(\Delta x)^4]$$

Divide by Δx

$$\frac{\partial u}{\partial x}(x,y) = \frac{u(x+\Delta x) - u(x,y)}{\Delta x} + O(\Delta x)$$

OR

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{ij} = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h)}$$

forward difference

Taylor Series for $x - \Delta x$:

$$u(x-\Delta x, y) = u(x,y) - \frac{\partial u}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 - \frac{1}{3!}\frac{\partial^3 u}{\partial x^3}(\Delta x)^3 + O[(\Delta x)^4]$$

Divide by Δx as before

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{ij} - u_{i-1,j}}{h} + O(h)}$$

backward difference

Or, subtract the two Taylor Series above:

$$u(x+\Delta x, y) - u(x-\Delta x, y) = 2\Delta x\frac{\partial u}{\partial x} + 2\frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \underbrace{O[(\Delta x)^5]}_{\text{even powers cancel}}$$

Divide by $(2\Delta x)$

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2)}$$

Second-order approximation to $\partial u/\partial x$

or, add the two Taylor Series:

$$\boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + O(h^2)}$$

approximation to $\partial^2 u/\partial x^2$

Etc.

"Molecules"

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{1}{h}\left\{ \underset{i-1,j}{-1} \;\text{—}\; \underset{i,j}{1} \right\} + O(h) \qquad \text{backward diff.}$$

$$= \frac{1}{h}\left\{ \underset{i,j}{-1} \;\text{—}\; \underset{i+1,j}{1} \right\} + O(h) \qquad \text{forward diff.}$$

$$= \frac{1}{2h}\left\{ \underset{i-1,j}{-1} \;\text{—}\; \underset{i,j}{0} \;\text{—}\; \underset{i+1,j}{1} \right\} + O(h^2)$$

$$\left.\frac{\partial u}{\partial y}\right|_{ij} = \frac{1}{2k}\left\{ \begin{array}{ll} 1 & i,j+1 \\ | & \\ 0 & i,j \\ | & \\ -1 & i,j-1 \end{array} \right\} + O(k^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} = \frac{1}{h^2}\left\{ \underset{i-1,j}{1} \;\text{—}\; \underset{i,j}{-2} \;\text{—}\; \underset{i+1,j}{1} \right\} + O(h^2)$$

$$\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{ij} = \frac{1}{4h^2}\left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 0 & \underset{ij}{0} & 0 \\ 1 & 0 & -1 \end{array} \right\} + O(h^2) \qquad (h=k)$$

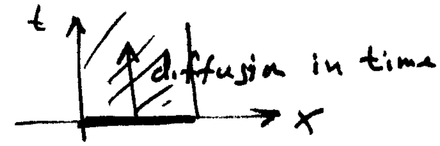
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\left\{ \begin{array}{ccc} & 1 & \\ 1 & \underset{ij}{-4} & 1 \\ & 1 & \end{array} \right\} + O(h^2)$$

these can be used to generate algorithms for obtaining approximate solutions, in analogy with Euler's Method or Runge-Kutta for ODE'S.

<u>EXAMPLE</u>: diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(normalized so const. = 1)



forward difference

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] + O(k) = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + O(h^2)$$

multiply by k & rearrange

$$\boxed{u_{i,j+1} = r\,u_{i-1,j} + (1-2r)\,u_{i,j} + r\,u_{i+1,j} + O(k^2 + kh^2) \quad \text{where } r = \frac{k}{h^2}}$$

this is an <u>explicit</u> <u>marching</u> algorithm