Shortest Paths

- Dijkstra's algorithm
- implementation
- negative weights

References:
Algorithms in Java, Chapter 21
http://www.cs.princeton.edu/introalgsds/55dijkstra

Edsger W. Dijkstra: a few select quotes

The question of whether computers can think is like the question of whether submarines can swim.

Do only what only you can do.

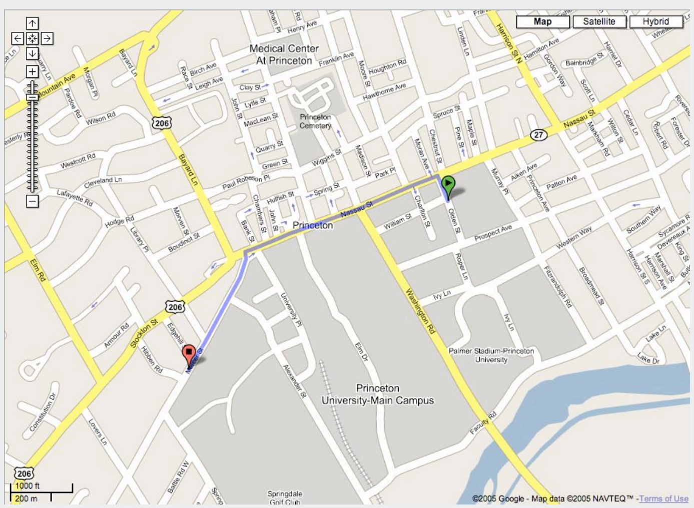
In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.

The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.

APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.

Edger Dijkstra
Turing award 1972

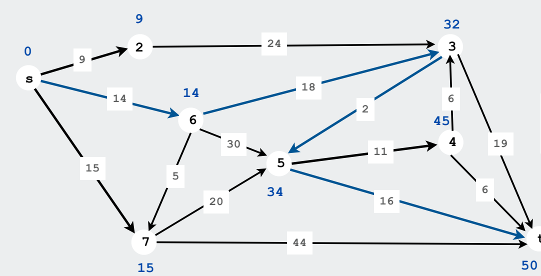
Shortest paths in a weighted digraph



Shortest paths in a weighted digraph

Given a weighted digraph, find the shortest directed path from s to t.

cost of path = sum of edge costs in path



Path: s→6→3→5→t
Cost: 14 + 18 + 2 + 16 = 50

Note: weights are arbitrary numbers
- not necessarily distances
- need not satisfy the triangle inequality
- Ex: airline fares [stay tuned for others]

Versions

- source-target (s-t)
- single source
- all pairs.
- nonnegative edge weights
- arbitrary weights
- Euclidean weights.

Early history of shortest paths algorithms

Shimbel (1955). Information networks.

Ford (1956). RAND, economics of transportation.

Leyzorek, Gray, Johnson, Ladew, Meaker, Petry, Seitz (1957).
Combat Development Dept. of the Army Electronic Proving Ground.

Dantzig (1958). Simplex method for linear programming.

Bellman (1958). Dynamic programming.

Moore (1959). Routing long-distance telephone calls for Bell Labs.

Dijkstra (1959). Simpler and faster version of Ford's algorithm.

Applications

Shortest-paths is a broadly useful problem-solving model

- Maps
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Subroutine in advanced algorithms.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Approximating piecewise linear functions.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

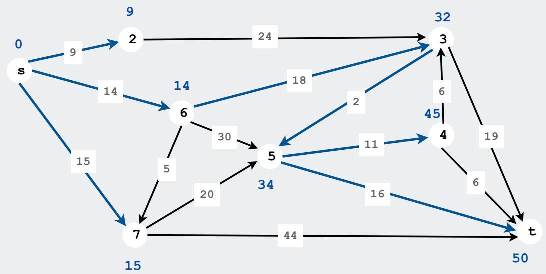
- **Dijkstra's algorithm**
- implementation
- negative weights

Single-source shortest-paths

Given. Weighted digraph, single source `s`.

Distance from s to v: length of the shortest path from s to v .

Goal. Find distance (and shortest path) from `s` to every other vertex.

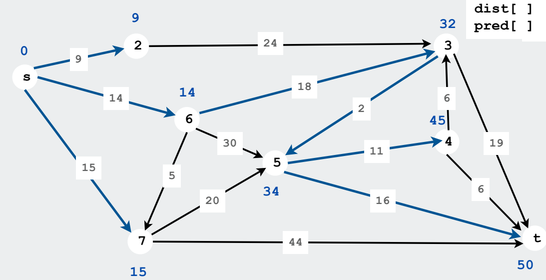


Shortest paths form a tree

Single-source shortest-paths: basic plan

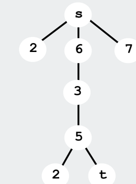
Goal: Find distance (and shortest path) from `s` to every other vertex.

Design pattern:
- **`ShortestPaths`** class (**`WeightedDigraph`** client)
- instance variables: vertex-indexed arrays **`dist[]`** and **`pred[]`**
- client query methods return distance and path iterator



v	s	2	3	4	5	6	7	t
dist[ ]	0	9	32	45	34	14	15	50
pred[ ]	0	0	6	5	3	0	0	5

shortest path tree
(parent-link representation)



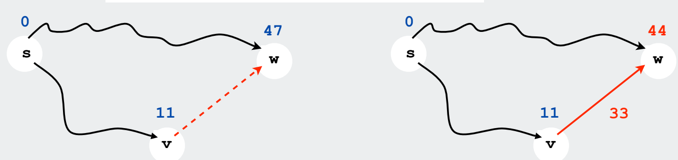
Note: Same pattern as Prim, DFS, BFS; BFS works when weights are all 1.

Edge relaxation

For all `v`, `dist[v]` is the length of some path from `s` to `v`.

Relaxation along edge `e` from `v` to `w`.
- `dist[v]` is length of some path from `s` to `v`
- `dist[w]` is length of some path from `s` to `w`
- if `v-w` gives a shorter path to `w` through `v`, update `dist[w]` and `pred[w]`

```
if (dist[w] > dist[v] + e.weight())
{
    dist[w] = dist[v] + e.weight());
    pred[w] = e;
}
```



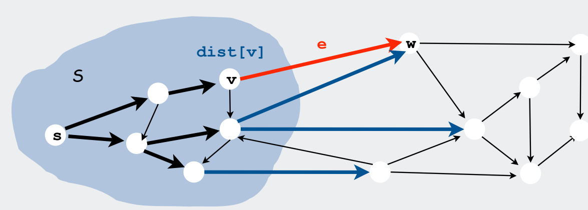
Relaxation sets `dist[w]` to the length of a shorter path from `s` to `w` (if `v-w` gives one)

Dijkstra's algorithm

S: set of vertices for which the shortest path length from `s` is known.

Invariant: for `v` in S, `dist[v]` is the length of the shortest path from `s` to `v`.

Initialize S to `s`, `dist[s]` to `0`, `dist[v]` to ∞ for all other `v`
Repeat until S contains all vertices connected to `s`
- find `e` with `v` in S and `w` in S' that minimizes `dist[v] + e.weight()`
- relax along that edge
- add `w` to S

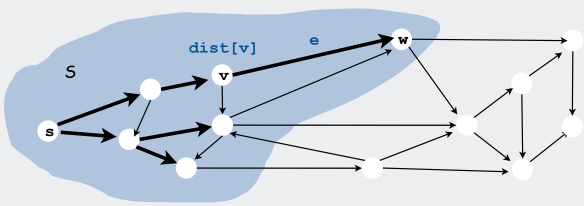


Dijkstra's algorithm

S: set of vertices for which the shortest path length from **s** is known.

Invariant: for **v** in S, **dist[v]** is the length of the shortest path from **s** to **v**.

Initialize S to **s**, **dist[s]** to **0**, **dist[v]** to ∞ for all other **v**
Repeat until S contains all vertices connected to **s**
- find **e** with **v** in S and **w** in S′ that minimizes **dist[v] + e.weight()**
- relax along that edge
- add **w** to S

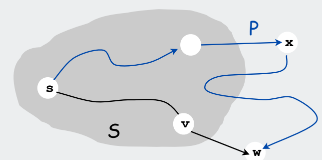


Dijkstra's algorithm proof of correctness

S: set of vertices for which the shortest path length from **s** is known.

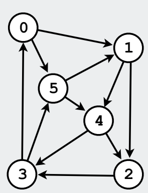
Invariant: for **v** in S, **dist[v]** is the length of the shortest path from **s** to **v**.

Pf. (by induction on |S|)
- Let **w** be next vertex added to S.
- Let P* be the **s-w** path through v.
- Consider any other **s-w** path P, and let **x** be first node on path outside S.
- P is already longer than P* as soon as it reaches **x** by greedy choice.



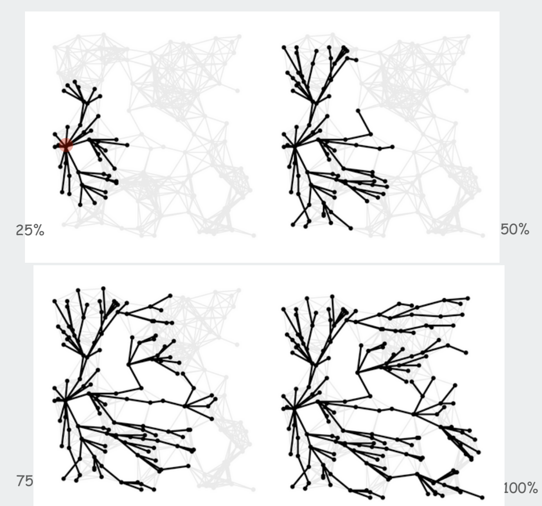
Dijkstra's algorithm example

Dijkstra's algorithm. [Dijkstra 1957]
Start with vertex 0 and greedily grow tree T. At each step,
add cheapest path ending in an edge that has exactly one endpoint in T.



0-1	0.41
0-5	0.29
1-2	0.51
1-4	0.32
2-3	0.50
3-0	0.45
3-5	0.38
4-2	0.32
4-3	0.36
5-1	0.29
5-4	0.21

Shortest Path Tree



- Dijkstra’s algorithm
- **implementation**
- negative weights

Weighted directed edge data type

```
public class Edge implements Comparable<Edge>
{
   public final int v, int w;
   public final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v; }

   public int to()
   {  return w; }

   public int weight()
   {  return weight; }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

code is the same as for (undirected) `WeightedGraph`

except `from()` and `to()` replace `either()` and `other()`

Weighted digraph data type

Identical to `WeightedGraph` but just one representation of each `Edge`.

```
public class WeightedDigraph
{
   private int V;
   private SET<Edge>[] adj;

   public Graph(int V)
   {
      this.V = V;
      adj = (SET<Edge>[]) new SET[V];
      for (int v = 0; v < V; v++)
         adj[v] = new SET<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.from();
      adj[v].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }

}
```


Dijkstra's algorithm: implementation approach

Initialize S to s, dist[s] to 0, dist[v] to ∞ for all other v

Repeat until S contains all vertices connected to s

- find `v-w` with `v` in S and `w` in S' that minimizes `dist[v] + weight[v-w]`
- relax along that edge
- add w to S

Idea 1 (easy): Try all edges

Total running time proportional to VE

Dijkstra's algorithm: implementation approach

Initialize S to s, dist[s] to 0, dist[v] to ∞ for all other v
Repeat until S contains all vertices connected to s
- find `v-w` with `v` in S and `w` in S' that minimizes `dist[v] + weight[v-w]`
- relax along that edge
- add w to S

Idea 2 (Dijkstra) : maintain these invariants
- for `v` in S, `dist[v]` is the length of the shortest path from `s` to `v`.
- for `w` in S', `dist[w]` minimizes `dist[v] + weight[v-w]`.

Two implications
- find `v-w` in V steps (smallest `dist[]` value among vertices in S')
- update `dist[]` in at most V steps (check neighbors of `w`)

Total running time proportional to V^2

Dijkstra's algorithm implementation

Initialize S to s, dist[s] to 0, dist[v] to ∞ for all other v
Repeat until S contains all vertices connected to s
- find `v-w` with `v` in S and `w` in S' that minimizes `dist[v] + weight[v-w]`
- relax along that edge
- add w to S

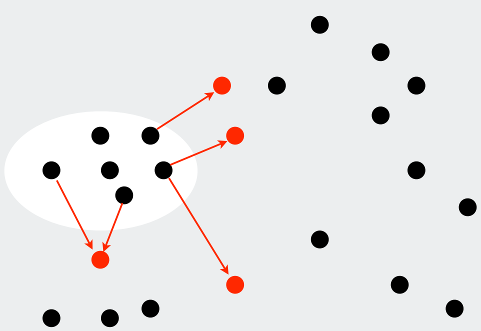
Idea 3 (modern implementations):
- for all v in S, `dist[v]` is the length of the shortest path from `s` to `v`.
- use a priority queue to find the edge to relax

	sparse	dense
easy	V^2	EV
Dijkstra	V^2	V^2
modern	E lg E	E lg E

Total running time proportional to E lg E

Dijkstra's algorithm implementation

Q. What goes onto the priority queue?
A. Fringe vertices connected by a single edge to a vertex in S



Starting to look familiar?

Lazy implementation of Prim's MST algorithm

```
public class LazyPrim
{
   Edge[] pred = new Edge[G.V()];
   public LazyPrim(WeightedGraph G)
   {
      boolean[] marked = new boolean[G.V()];
      double[] dist = new double[G.V()];
      for (int v = 0; v < G.V(); v++)
         dist[v] = Double.POSITIVE_INFINITY;
      MinPQplus<Double, Integer> pq;
      pq = new MinPQplus<Double, Integer>();
      dist[s] = 0.0;
      pq.put(dist[s], s);
      while (!pq.isEmpty())
      {
         int v = pq.delMin();
         if (marked[v]) continue;
         marked(v) = true;
         for (Edge e : G.adj(v))
         {
            int w = e.other(v);
            if (!marked[w] && (dist[w] > e.weight() ))
            {
                dist[w] = e.weight();
                pred[w] = e;
                pq.insert(dist[w], w);
            }
         }
      }
   }
}
```

marks vertices in MST
distance to MST
edges to MST
key-value PQ
get next vertex
ignore if already in MST
add to PQ any vertices brought closer to S by v

Lazy implementation of Dijkstra's SPT algorithm

```
public class LazyDijkstra
{
   double[] dist = new double[G.V()];
   Edge[] pred = new Edge[G.V()];
   public LazyDijkstra(WeightedDigraph G, int s)
   {
      boolean[] marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         dist[v] = Double.POSITIVE_INFINITY;
      MinPQplus<Double, Integer> pq;
      pq = new MinPQplus<Double, Integer>();
      dist[s] = 0.0;
      pq.put(dist[s], s);
      while (!pq.isEmpty())
      {
         int v = pq.delMin();
         if (marked[v]) continue;
         marked(v) = true;
         for (Edge e : G.adj(v))
         {
            int w = e.to();
            if (dist[w] > dist[v] + e.weight())
            {
               dist[w] = dist[v] + e.weight();
               pred[w] = e;
               pq.insert(dist[w], w);
            }
         }
      }
   }
}
```

code is the same as Prim's (!!)

except
- `WeightedDigraph`, not `WeightedGraph`
- weight is distance to s, not to tree
- add client query for distances

Eager implementation of Dijkstra's algorithm

Use indexed priority queue that supports
- contains: is there a key associated with value v in the priority queue?
- decrease key: decrease the key associated with value v

[more complicated data structure, see text]

Putative "benefit": reduces PQ size guarantee from E to V
- no signficant impact on time since lg E < 2lg V
- extra space not important for huge sparse graphs found in practice
  [PQ size is far smaller than E or even V in practice]
- widely used, but practical utility is debatable (as for Prim's)

Improvements to Dijkstra's algorithm

Use a d-way heap (Johnson, 1970s)
- easy to implement
- reduces costs to $E\ d \log_d V$
- indistinguishable from linear for huge sparse graphs found in practice

Use a Fibonacci heap (Sleator-Tarjan, 1980s)
- very difficult to implement
- reduces worst-case costs (in theory) to $E + V \lg V$
- not quite linear (in theory)
- practical utility questionable

Find an algorithm that provides a linear worst-case guarantee?
   [open problem]

Dijkstra's Algorithm: performance summary

Fringe implementation directly impacts performance

Best choice depends on sparsity of graph.
- 2,000 vertices, 1 million edges. heap 2-3x slower than array
- 100,000 vertices, 1 million edges. heap gives 500x speedup.
- 1 million vertices, 2 million edges. heap gives 10,000x speedup.

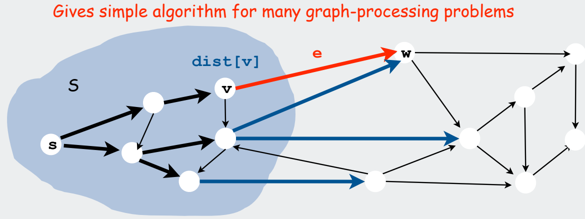
Bottom line.
- array implementation optimal for dense graphs
- binary heap far better for sparse graphs
- d-way heap worth the trouble in performance-critical situations
- Fibonacci heap best in theory, but not worth implementing

Priority-first search

Insight: All of our graph-search methods are the same algorithm!

Maintain a set of explored vertices S
Grow S by exploring edges with exactly one endpoint leaving S.

DFS. Take edge from vertex which was discovered most recently.
BFS. Take from vertex which was discovered least recently.
Prim. Take edge of minimum weight.
Dijkstra. Take edge to vertex that is closest to s.
... Gives simple algorithm for many graph-processing problems



Challenge: express this insight in (re)usable Java code

Priority-first search: application example

Shortest s-t paths in Euclidean graphs (maps)
- Vertices are points in the plane.
- Edge weights are Euclidean distances.

A sublinear algorithm.
- Assume graph is already in memory.
- Start Dijkstra at s.
- Stop when you reach t.

Even better: exploit geometry
- For edge v-w, use weight d(v, w) + d(w, t) – d(v, t).
- Proof of correctness for Dijkstra still applies.
- In practice only $O(V^{1/2})$ vertices examined.
- Special case of A* algorithm

Euclidean distance

[Practical map-processing programs precompute many of the paths.]

- Dijkstra's algorithm
- implementation
- negative weights

Shortest paths application: Currency conversion

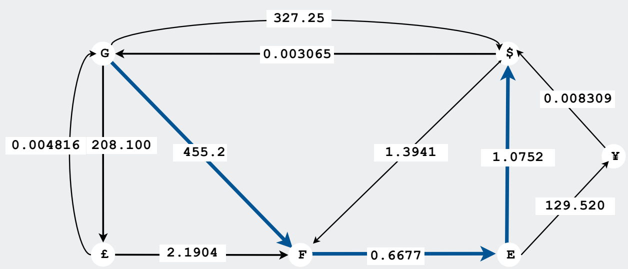
Currency conversion. Given currencies and exchange rates, what is best way to convert one ounce of gold to US dollars?
- 1 oz. gold ⇒ \$327.25.
- 1 oz. gold ⇒ £208.10 ⇒ ⇒ \$327.00. [208.10 × 1.5714]
- 1 oz. gold ⇒ 455.2 Francs ⇒ 304.39 Euros ⇒ \$327.28. [455.2 × .6677 × 1.0752]

Currency	£	Euro	¥	Franc	\$	Gold
UK Pound	1.0000	0.6853	0.005290	0.4569	0.6368	208.100
Euro	1.4599	1.0000	0.007721	0.6677	0.9303	304.028
Japanese Yen	189.050	129.520	1.0000	85.4694	120.400	39346.7
Swiss Franc	2.1904	1.4978	0.011574	1.0000	1.3941	455.200
US Dollar	1.5714	1.0752	0.008309	0.7182	1.0000	327.250
Gold (oz.)	0.004816	0.003295	0.0000255	0.002201	0.003065	1.0000

Shortest paths application: Currency conversion

Graph formulation.

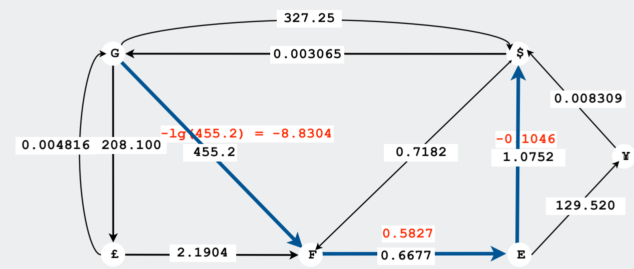
- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find path that maximizes product of weights.



Shortest paths application: Currency conversion

Reduce to shortest path problem by taking logs

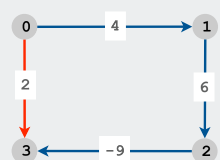
- Let weight(v-w) = – lg (exchange rate from currency v to w)
- multiplication turns to addition
- Shortest path with costs c corresponds to best exchange sequence.



Challenge. Solve shortest path problem with negative weights.

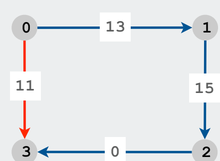
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Adding a constant to every edge weight also doesn't work.

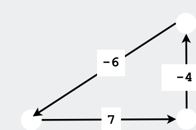


Adding 9 to each edge changes the shortest path because it adds 9 to each segment, wrong thing to do for paths with many segments.

Bad news: need a different algorithm.

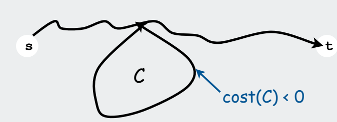
Shortest paths with negative weights: negative cycles

Negative cycle. Directed cycle whose sum of edge weights is negative.



Observations.

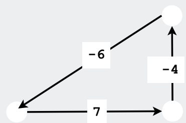
- If negative cycle C on path from `s` to `t`, then shortest path can be made arbitrarily negative by spinning around cycle
- There exists a shortest `s-t` path that is simple.



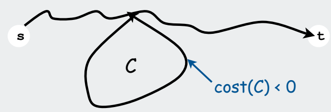
Worse news: need a different problem

Shortest paths with negative weights

Problem 1. Does a given digraph contain a negative cycle?



Problem 2. Find the shortest simple path from s to t.



Bad news: Problem 2 is intractable
Good news: Can solve problem 1 in O(VE) steps
Good news: Same algorithm solves problem 2 if no negative cycle

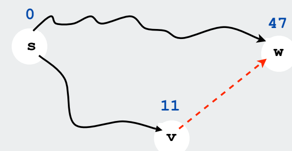
Bellman-Ford algorithm
- detects a negative cycle if any exist
- finds shortest simple path if no negative cycle exists

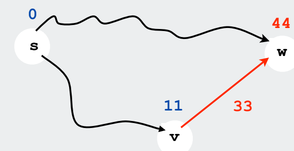
Edge relaxation

For all `v`, `dist[v]` is the length of some path from `s` to `v`.

Relaxation along edge `e` from `v` to `w`.
- `dist[v]` is length of some path from `s` to `v`
- `dist[w]` is length of some path from `s` to `w`
- if `v-w` gives a shorter path to `w` through `v`, update `dist[w]` and `pred[w]`

```
if (dist[w] > dist[v] + e.weight())
{
    dist[w] = dist[v] + e.weight();
    pred[w] = e;
}
```





Relaxation sets `dist[w]` to the length of a shorter path from `s` to `w` (if `v-w` gives one)

Shortest paths with negative weights: dynamic programming algorithm

A simple solution that works!
- Initialize `dist[v] = ∞`, `dist[s]= 0`.
- Repeat V times: relax each edge `e`.

phase i

```
for (int i = 1; i <= G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (Edge e : G.adj(v))
      {
         int w = e.to();
         if (dist[w] > dist[v] + e.weight())
         {
             dist[w] = dist[v] + e.weight())
             pred[w] = e;
         }
      }
```

relax v-w

Shortest paths with negative weights: dynamic programming algorithm

Running time proportional to E V

Invariant. At end of phase `i`, `dist[v]` ≤ length of any path from `s` to `v` using at most `i` edges.

Theorem. If there are no negative cycles, upon termination `dist[v]` is the length of the shortest path from from `s` to `v`.

and `pred[]` gives the shortest paths

Shortest paths with negative weights: Bellman-Ford-Moore algorithm

Observation. If `dist[v]` doesn't change during phase `i`, no need to relax any edge leaving `v` in phase `i+1`.

FIFO implementation.
Maintain queue of vertices whose distance changed.

be careful to keep at most one copy of each vertex on queue

Running time.
- still could be proportional to EV in worst case
- much faster than that in practice

Shortest paths with negative weights: Bellman-Ford-Moore algorithm

Initialize `dist[v] = ∞` and `marked[v]= false` for all vertices `v`.

```
Queue<Integer> q = new Queue<Integer>();
marked[s] = true;
dist[s] = 0;
q.enqueue(s);

while (!q.isEmpty())
{
   int v = q.dequeue();
   marked[v] = false;
   for (Edge e : G.adj(v))
   {
      int w = e.target();
      if (dist[w] > dist[v] + e.weight())
      {
         dist[w] = dist[v] + e.weight();
         pred[w] = e;
         if (!marked[w])
         {
            marked[w] = true;
            q.enqueue(w);
         }
      }
   }
}
```

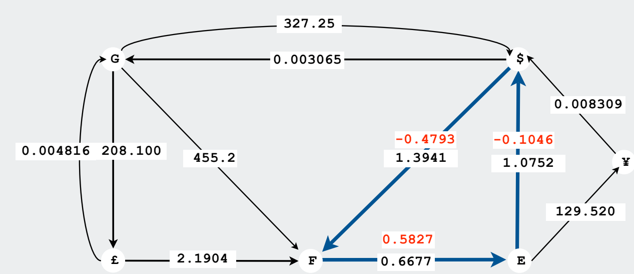

Single Source Shortest Paths Implementation: Cost Summary

	algorithm	worst case	typical case
nonnegative costs	Dijkstra (classic)	V^2	V^2
	Dijkstra (heap)	E lg E	E
no negative cycles	Dynamic programming	EV	EV
	Bellman-Ford-Moore	EV	E

Remark 1. Negative weights makes the problem harder.
Remark 2. Negative cycles makes the problem intractable.

Shortest paths application: arbitrage

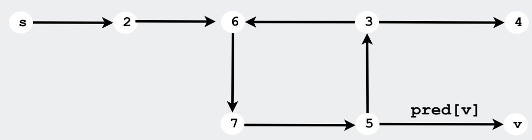
Is there an arbitrage opportunity in currency graph?
- Ex: \$1 ⇒ 1.3941 Francs ⇒ 0.9308 Euros ⇒ \$1.00084.
- Is there a negative cost cycle?
- Fastest algorithm is valuable!



-0.4793 + 0.5827 - 0.1046 < 0

Negative cycle detection

If there is a negative cycle reachable from s.
Bellman-Ford-Moore gets stuck in loop, updating vertices in cycle.

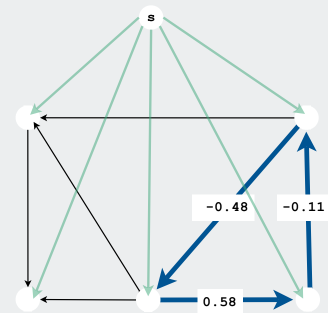


Finding a negative cycle. If any vertex `v` is updated in phase `v`, there exists a negative cycle, and we can trace back `pred[v]` to find it.

Negative cycle detection

Goal. Identify a negative cycle (reachable from any vertex).

Solution. Add 0-weight edge from artificial source `s` to each vertex `v`.
Run Bellman-Ford from vertex `s`.



Shortest paths summary

Dijkstra's algorithm
- easy and optimal for dense digraphs
- PQ/ST data type gives near optimal for sparse graphs

Priority-first search
- generalization of Dijkstra's algorithm
- encompasses DFS, BFS, and Prim
- enables easy solution to many graph-processing problems

Negative weights
- arise in applications
- make problem intractable in presence of negative cycles (!)
- easy solution using old algorithms otherwise

Shortest-paths is a broadly useful problem-solving model