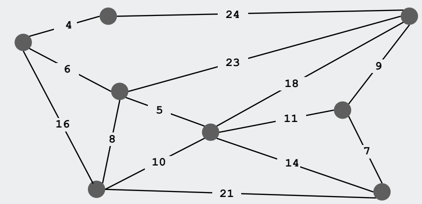
Minimum Spanning Trees

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

References:
Algorithms in Java, Chapter 20
http://www.cs.princeton.edu/introalgsds/54mst

Minimum Spanning Tree

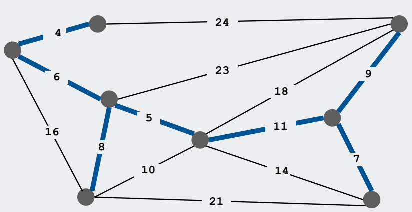
Given. Undirected graph G with positive edge weights (connected).
Goal. Find a min weight set of edges that connects all of the vertices.



G

Minimum Spanning Tree

Given. Undirected graph G with positive edge weights (connected).
Goal. Find a min weight set of edges that connects all of the vertices.



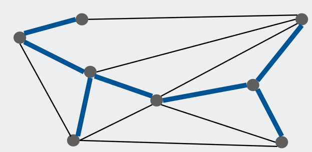
weight(T) = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Brute force. Try all possible spanning trees.
- Problem 1: not so easy to implement.
- Problem 2: far too many of them.

V^{V-2} spanning trees on the complete graph on V vertices [Cayley 1889]

MST Origin

Otakar Boruvka (1926).
- Electrical Power Company of Western Moravia in Brno.
- Most economical construction of electrical power network.
- Concrete engineering problem is now a cornerstone problem-solving model in combinatorial optimization.



Otakar Boruvka

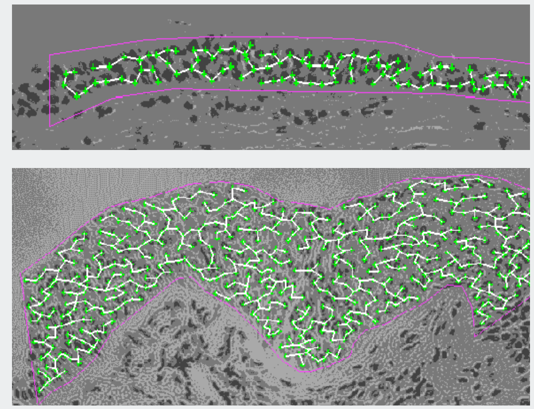
Applications

MST is fundamental problem with diverse applications.

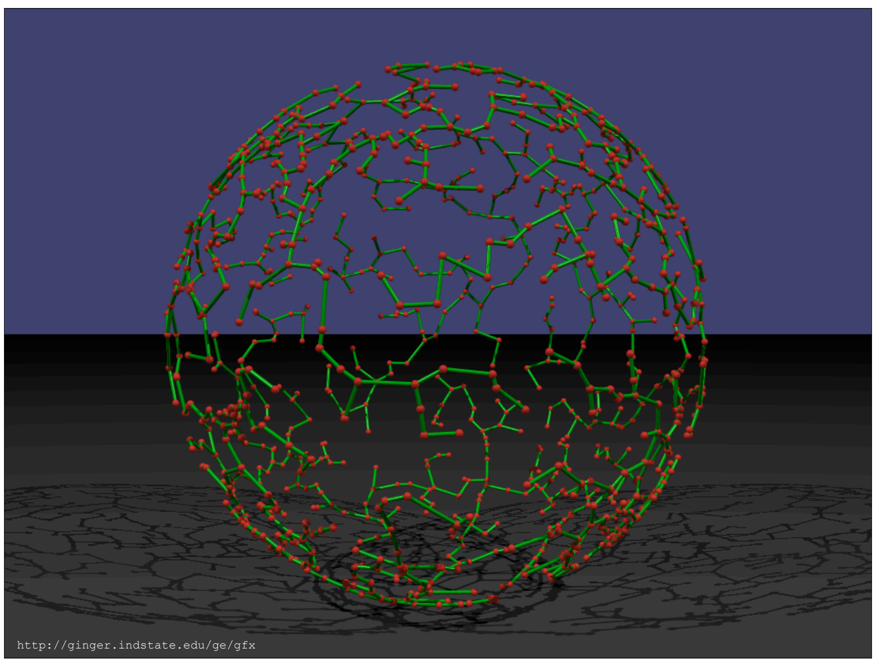
- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Medical Image Processing

MST describes arrangement of nuclei in the epithelium for cancer research



http://www.bccrc.ca/ci/ta01_archlevel.html



Two Greedy Algorithms

Kruskal's algorithm. Consider edges in ascending order of weight. Add to T the next edge unless doing so would create a cycle.

Prim's algorithm. Start with any vertex s and greedily grow a tree T from s. At each step, add to T the edge of min weight that has exactly one endpoint in T.

> *" Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit. "* - Gordon Gecko

Proposition. Both greedy algorithms compute an MST.

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Weighted graph and Edge APIs

`public class WeightedGraph`		
	`WeightedGraph(int V)`	create an empty graph with V vertices
`void`	`insert(Edge e)`	insert edge e
`Iterable<Edge>`	`adj(int v)`	return an iterator over edges incident to v
`int`	`V()`	return the number of vertices
`String`	`toString()`	return a string representation

Edge abstraction needed for weights

`public class Edge implements Comparable<Edge>`		
	`Edge(int v, int w, double weight)`	create an edge v-w with given weight
`int`	`either()`	return either endpoint
`int`	`other(int v)`	return the endpoint that's not v
`double`	`weight()`	return the weight
`String`	`toString()`	return a string representation

Weighted graph client

`public class WeightedGraph`		
	`WeightedGraph(int V)`	create an empty graph with V vertices
`void`	`insert(Edge e)`	insert edge e
`Iterable<Edge>`	`adj(int v)`	return an iterator over edges incident to v
`int`	`V()`	return the number of vertices
`String`	`toString()`	return a string representation

```
for (int v = 0; v < G.V(); v++)
{
   for (Edge e : G.adj(v))
   {
      // edge v-w
      int w = e.other(v);
   }
}
```

iterate through all edges
(once in each direction)

Weighted graph data type

Identical to `Graph.java` but use `Edge` adjacency sets instead of `int`.

```
public class WeightedGraph
{
   private final int V;
   private final SET<Edge>[] adj;

   public WeightedGraph(int V)
   {
      this.V = V;
      adj = (SET<Edge>[]) new SET[V];
      for (int v = 0; v < V; v++)
         adj[v] = new SET<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }

}
```

no parallel edges

Weighted edge data type

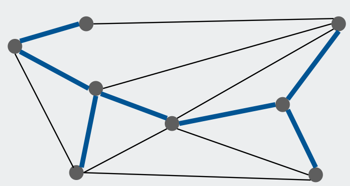
```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v; }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int weight()
   {  return weight; }

   // See facing box for compare methods.

}
```

```
// sorted by edge weight
public final static Comparator<Edge>
   BY_WEIGHT = new ByWeight();

private static class ByWeight
   implements Comparator<Edge>
{
   public int compare(Edge e, Edge f)
   {
      if (e.weight < f.weight) return -1;
      if (e.weight > f.weight) return +1;
      return 0;
   }
}


// sorted by edge endpoints
public int compareTo(Edge that)
{
    if (this.v < that.v) return -1;
    if (this.v > that.v) return +1;
    if (this.w < that.w) return -1;
    if (this.w > that.w) return +1;
    return 0;
}
```


Spanning Tree

MST. Given connected graph G with positive edge weights, find a min weight set of edges that connects all of the vertices.

Def. A spanning tree of a graph G is a subgraph T that is connected and acyclic.



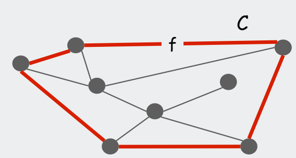
Property. MST of G is always a spanning tree.

Greedy Algorithms

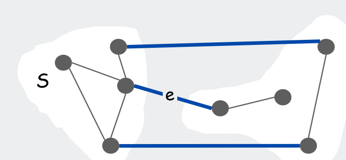
Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max weight edge belonging to C. Then the MST does not contain f.

Cut property. Let S be any subset of vertices, and let e be the min weight edge with exactly one endpoint in S. Then the MST contains e.



f is not in the MST



e is in the MST

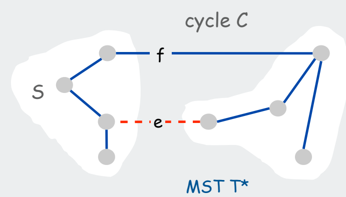
Cycle Property

Simplifying assumption. All edge weights w_e are distinct.

Cycle property. Let C be any cycle, and let f be the max weight edge belonging to C. Then the MST T* does not contain f.

Pf. [by contradiction]

- Suppose f belongs to T*. Let's see what happens.
- Deleting f from T* disconnects T*. Let S be one side of the cut.
- Some other edge in C, say e, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $w_e < w_f$, weight(T) < weight(T*).
- Contradicts minimality of T*. ▪



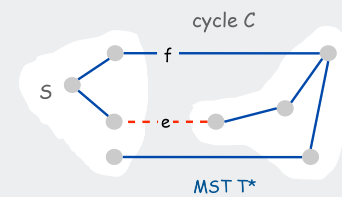
Cut Property

Simplifying assumption. All edge weights w_e are distinct.

Cut property. Let S be any subset of vertices, and let e be the min weight edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. [by contradiction]

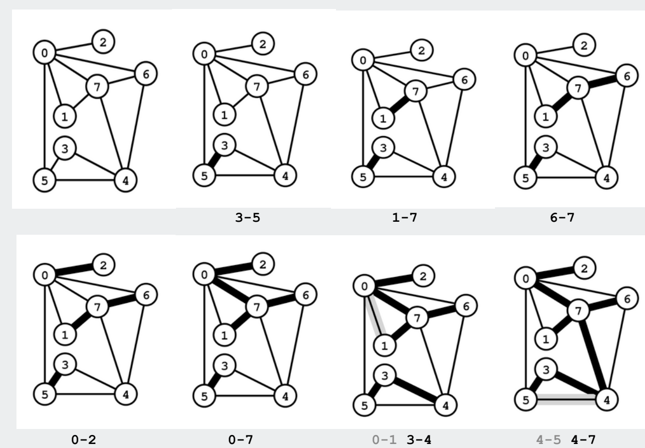
- Suppose e does not belong to T*. Let's see what happens.
- Adding e to T* creates a (unique) cycle C in T*.
- Some other edge in C, say f, has exactly one endpoint in S.
- $T = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $w_e < w_f$, weight(T) < weight(T*).
- Contradicts minimality of T*. ▪



- weighted graph API
- cycles and cuts
- **Kruskal's algorithm**
- Prim's algorithm
- advanced algorithms
- clustering

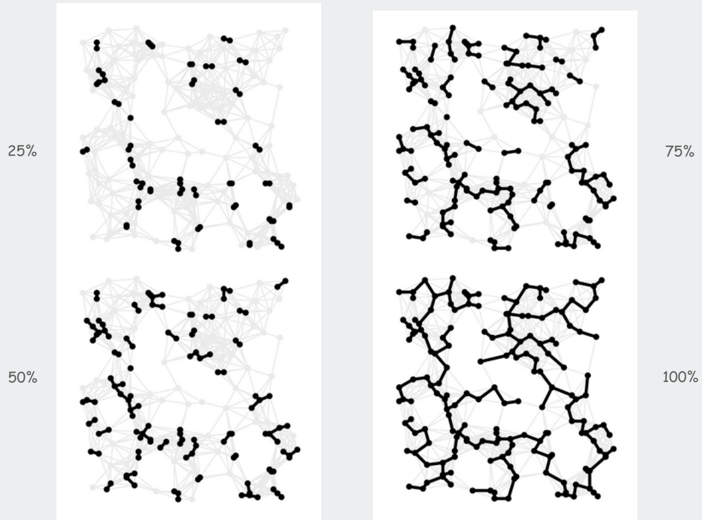
Kruskal's Algorithm: Example

Kruskal's algorithm. [Kruskal, 1956] Consider edges in ascending order of weight. Add the next edge to T unless doing so would create a cycle.



3-5	0.18
1-7	0.21
6-7	0.25
0-2	0.29
0-7	0.31
0-1	0.32
3-4	0.34
4-5	0.40
4-7	0.46
0-6	0.51
4-6	0.51
0-5	0.60

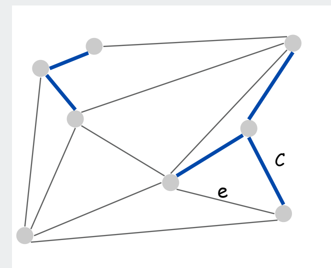
Kruskal's algorithm example



Kruskal's algorithm correctness proof

Proposition. Kruskal's algorithm computes the MST.

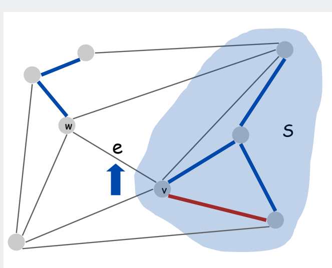
Pf. [case 1] Suppose that adding e to T creates a cycle C:
- e is the max weight edge in C (weights come in increasing order).
- e is not in the MST (cycle property).



Kruskal's algorithm correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. [case 2] Suppose that adding e = (v, w) to T does not create a cycle:
- let S be the vertices in v's connected component.
- w is not in S.
- e is the min weight edge with exactly one endpoint in S.
- e is in the MST (cut property). ▪



Kruskal's algorithm implementation

Q. How to check if adding an edge to T would create a cycle?

A1. Naïve solution: use DFS.
- O(V) time per (undirected) cycle check.
- O(E V) time overall.

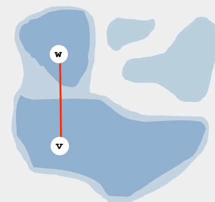
Kruskal's algorithm implementation

Q. How to check if adding an edge to T would create a cycle?

A2. Use the union-find data structure from lecture 1 (!).

- Maintain a set for each connected component.
- If v and w are in same component, then adding v-w creates a cycle.
- To add v-w to T, merge sets containing v and w.

Case 1: adding v-w creates a cycle



Case 2: add v-w to T and merge sets

Kruskal's algorithm: Java implementation

```
public class Kruskal
{
   private SET<Edge> mst = new SET<Edge>();

   public Kruskal(WeightedGraph G)
   {
      Edge[] edges = G.edges();
      Arrays.sort(edges, Edge.BY_WEIGHT);

      UnionFind uf = new UnionFind(G.V());
      for (Edge e: edges)
      {
         int v = e.either(), w = e.other(v);
         if (!uf.find(v ,w))
         {
            uf.unite(v, w);
            mst.add(edge);
         }
      }
   }

   public Iterable<Edge> mst()
   {  return mst;  }
}
```

sort edges by weight

greedily add edges to MST

return to client iterable sequence of edges

Easy speedup: Stop as soon as there are V-1 edges in MST.

Kruskal's algorithm running time

Proposition. Kruskal finds MST in time proportional to E log V.

Operation	Frequency	Time per op
sort	1	E log V
union	V	log* V †
find	E	log* V †

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

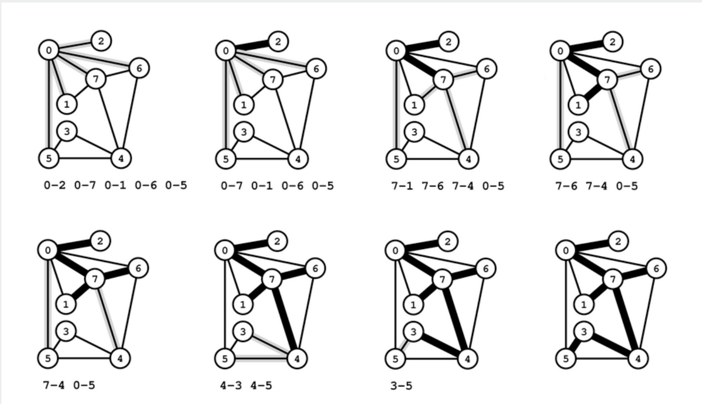
Remark 1. If edges are already sorted, time is proportional to E log* V

Remark 2. Linear in practice with PQ or quicksort partitioning (see book: don't need full sort)

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- **Prim's algorithm**
- advanced topics

Prim's algorithm example

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]
Start with vertex 0 and greedily grow tree T. At each step, add edge of min weight that has exactly one endpoint in T.



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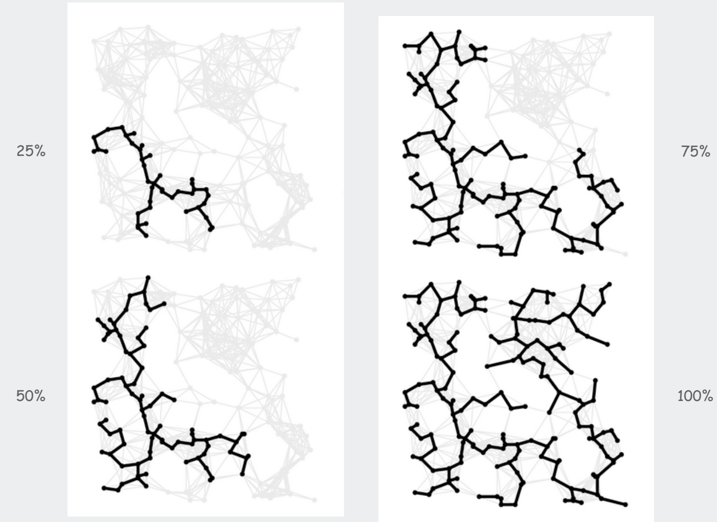
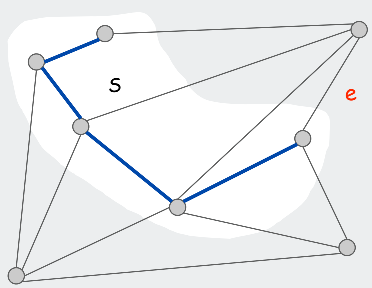
4-6	0.51
4-7	0.46
6-7	0.25

Prim's Algorithm example



Prim's algorithm correctness proof

Proposition. Prim's algorithm computes the MST.
Pf.
- Let S be the subset of vertices in current tree T.
- Prim adds the min weight edge e with exactly one endpoint in S.
- e is in the MST (cut property) ▪



Prim's algorithm implementation

Q. How to find min weight edge with exactly one endpoint in S?

A1. Brute force: try all edges.
- O(E) time per spanning tree edge.
- O(E V) time overall.

Prim's algorithm implementation

Q. How to find min weight edge with exactly one endpoint in S?

A2. Maintain a priority queue of vertices connected by an edge to S
- Delete min to determine next vertex v to add to S.
- Disregard v if already in S.
- Add to PQ any vertex brought closer to S by v.

Running time.
- log V steps per edge (using a binary heap).
- E log V steps overall.

Note: This is a lazy version of implementation in Algs in Java

lazy: put all adjacent vertices (that are not already in MST) on PQ
eager: first check whether vertex is already on PQ and decrease its key

Key-value priority queue

Associate a value with each key in a priority queue.

API:

`public class MinPQplus<Key extends Comparable<Key>, Value>`

	`MinPQplus()`	create a key-value priority queue
`void`	`put(Key key, Value val)`	put key-value pair into the priority queue
`Value`	`delMin()`	return value paired with minimal key

Implementation:
- start with same code as standard heap-based priority queue
- use a parallel array `vals[]` (value associated with `keys[i]` is `vals[i]`)
- modify `exch()` to maintain parallel arrays (do exch in `vals[]`)
- modify `delMin()` to return `Value`

Lazy implementation of Prim's algorithm

```
public class LazyPrim
{
   boolean[] marked;
   double[] dist;
   private Edge[] pred;

   public LazyPrim(WeightedGraph G)
   {
      marked = new boolean[G.V()];
      pred   = new Edge[G.V()];
      dist   = new double[G.V()];
      for (int v = 0; v < G.V(); v++)
         dist[v] = Double.POSITIVE_INFINITY;
      prim(G, 0);
   }

   // See next slide for prim() implementation.
}
```

marks vertices in MST

distance to MST

`pred[v]` is edge attaching v to MST

Lazy implementation of Prim's algorithm

```
private void prim(WeightedGraph G, int s)
{
   dist[s] = 0.0;
   marked[s] = true;

   MinPQplus<Double, Integer> pq;
   pq = new MinPQplus<Double, Integer>();
   pq.put(dist[s], s);

   while (!pq.isEmpty())
   {
      int v = pq.delMin();
      if (marked[v]) continue;
      marked[v] = true;
      for (Edge e : G.adj(v))
      {
         int w = e.other(v);
         if (!marked[w] && (dist[w] > e.weight()))
         {
            dist[w] = e.weight();
            pred[w] = e;
            pq.insert(dist[w], w);
         }
      }
   }
}
```

key-value PQ

get next vertex

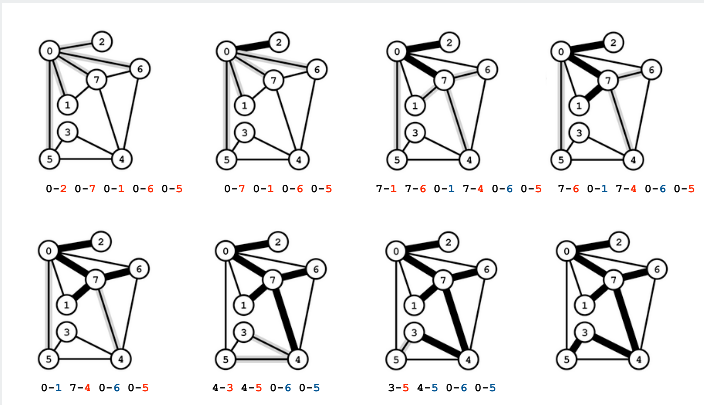
ignore if already in MST

add to PQ any vertices brought closer to S by v

Prim's algorithm (lazy) example

Priority queue key is distance (edge weight); value is vertex

Lazy version leaves obsolete entries in the PQ
therefore may have multiple entries with same value



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3-5	0.18
4-5	0.40
4-6	0.51
4-7	0.46
6-7	0.25

red: pq value (vertex)
blue: obsolete pq value (vertex)

Eager implementation of Prim's algorithm

Use indexed priority queue that supports:

- `contains(v)`: is there a key associated with value `v`?
- `decreaseKey(key, v)`: decrease the key associated with `v` to `key`.

Implementation. More complicated than `MinPQ`, see text.

Main benefit: reduces PQ size guarantee from E to V.

- Not important for the huge sparse graphs found in practice.
- PQ size is far smaller in practice.
- Widely used, but practical utility is debatable.

Removing the distinct edge weight assumption

Simplifying assumption. All edge weights w_e are distinct.

Approach 1: introduce tie-breaking rule for `compare()`.

```
public int compare(Edge e, Edge f)
{
   if (e.weight < f.weight) return -1;
   if (e.weight > f.weight) return +1;
   if (e.v < f.v) return -1;
   if (e.v > f.v) return +1;
   if (e.w < f.w) return -1;
   if (e.w > f.w) return +1;
   return 0;
}
```

Approach 2: Prim and Kruskal still find MST if equal weights!
(only our proof of correctness fails)

- weighted graph API
- cycles and cuts
- Kruskal's algorithm
- Prim's algorithm
- **advanced topics**

Advanced MST theorems: does an algorithm with a linear-time guarantee exist?

Year	Worst Case	Discovered By
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

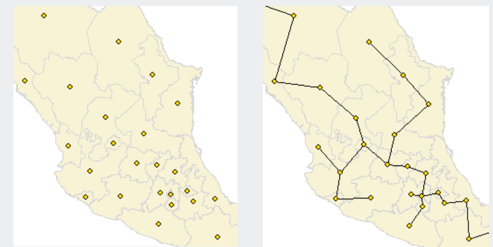
deterministic comparison-based MST algorithms

Year	Problem	Time	Discovered By
1976	planar MST	E	Cheriton-Tarjan
1992	MST verification	E	Dixon-Rauch-Tarjan
1995	randomized MST	E	Karger-Klein-Tarjan

related problems

Euclidean MST

Euclidean MST. Given N points in the plane, find MST connecting them.
(distances between point pairs are Euclidean distances)

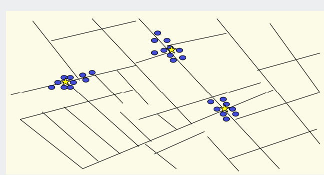


Brute force. Compute $\sim N^2/2$ distances and run Prim's algorithm.
Ingenuity. Exploit geometry and do it in O(N log N)
[stay tuned for geometric algorithms]

Scientific application: clustering

k-clustering. Divide a set of objects classify into k coherent groups.
distance function. Numeric value specifying "closeness" of two objects.

Fundamental problem.
Divide into clusters so that points in different clusters are far apart.



outbreak of cholera deaths in London in 1850s
Reference: Nina Mishra, HP Labs

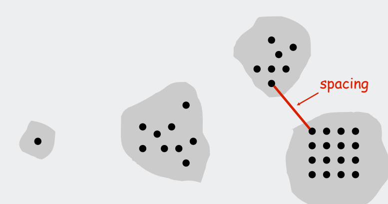
Applications.
- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

k-clustering of maximum spacing

k-clustering. Divide a set of objects classify into k coherent groups.
distance function. Numeric value specifying "closeness" of two objects.

Spacing. Min distance between any pair of points in different clusters.

k-clustering of maximum spacing.
Given an integer k, find a k-clustering such that spacing is maximized.



k = 4

Single-link clustering algorithm

"Well-known" algorithm for single-link clustering:

- Form V clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat until there are exactly k clusters.

Observation. This procedure is precisely Kruskal's algorithm (stop when there are k connected components).

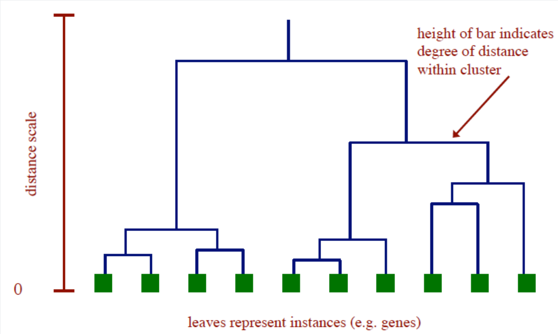
Proposition. Kruskal's algorithm finds a k-clustering of maximum spacing.

Alternate algorithm. Run Prim and delete k-1 edges of largest weight.

Clustering application: dendrograms

Dendrogram.
Scientific visualization of hypothetical sequence of evolutionary events.

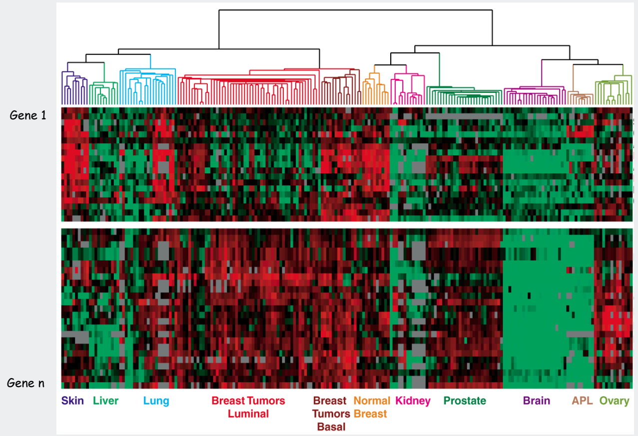
- Leaves = genes.
- Internal nodes = hypothetical ancestors.



Reference: http://www.biostat.wisc.edu/bmi576/fall-2003/lecture13.pdf

Dendrogram of cancers in human

Tumors in similar tissues cluster together.



Reference: Botstein & Brown group

gene expressed
gene not expressed