Directed Graphs

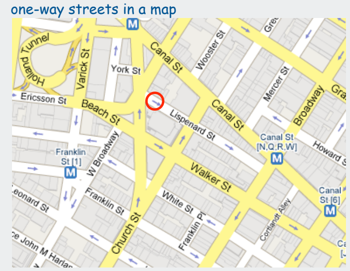
- digraph search
- transitive closure
- topological sort
- strong components

References:
Algorithms in Java, Chapter 19
http://www.cs.princeton.edu/introalgsds/52directed

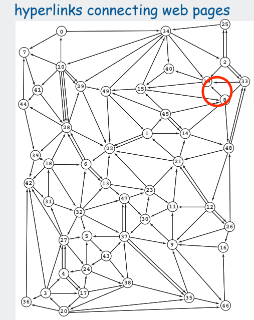
Directed graphs (digraphs)

Set of objects with oriented pairwise connections.

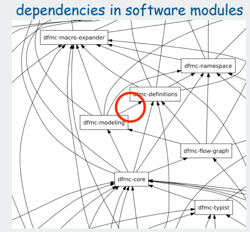
one-way streets in a map



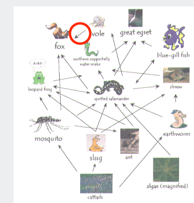
hyperlinks connecting web pages



dependencies in software modules



prey-predator relationships



Digraph applications

digraph	vertex	edge
financial	stock, currency	transaction
transportation	street intersection, airport	highway, airway route
scheduling	task	precedence constraint
WordNet	synset	hypernym
Web	web page	hyperlink
game	board position	legal move
telephone	person	placed call
food web	species	predator-prey relation
infectious disease	person	infection
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

Some digraph problems

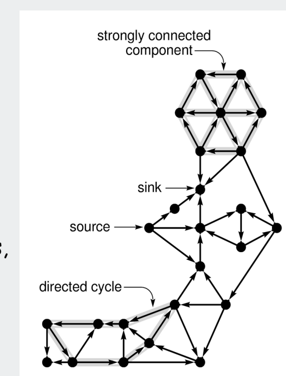
Transitive closure.
Is there a directed path from `v` to `w`?

Strong connectivity.
Are all vertices mutually reachable?

Topological sort.
Can you draw the digraph so that all edges point from left to right?

PERT/CPM.
Given a set of tasks with precedence constraints, how we can we best complete them all?

Shortest path. Find best route from `s` to `t` in a weighted digraph



PageRank. What is the importance of a web page?

Digraph representations

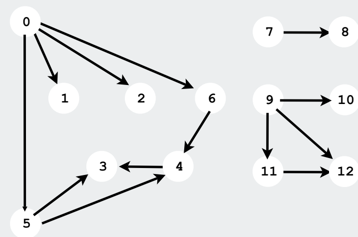
Vertices

- this lecture: use integers between 0 and V-1.
- real world: convert between names and integers with symbol table.

Edges: four easy options

- list of vertex pairs
- vertex-indexed adjacency arrays (adjacency matrix)
- vertex-indexed adjacency lists
- vertex-indexed adjacency SETs

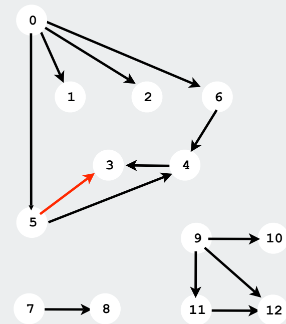
Same as undirected graph
BUT
orientation of edges is significant.



Adjacency matrix digraph representation

Maintain a two-dimensional $V \times V$ boolean array.
For each edge $v \to w$ in graph: `adj[v][w] = true`.

one entry for each edge

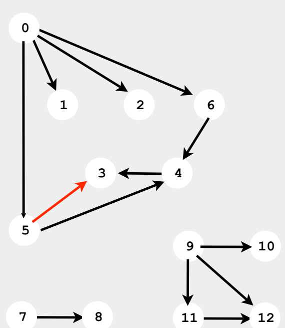


to

from	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0	0
5	0	0	0	1	1	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	0	0	0	0	0	0	0	0	0

Adjacency-list digraph representation

Maintain vertex-indexed array of lists.

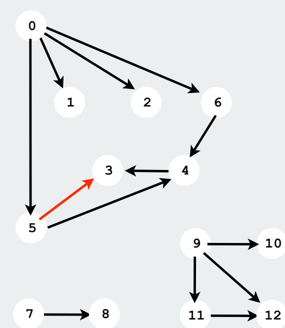


```
 0:  5 • → 2 • → 1 • → 6 •
 1:
 2:
 3:
 4:  3 •
 5:  4 • → 3 •
 6:  4 •
 7:  8 •
 8:
 9:  10 • → 11 • → 12 •
10:
11:  12 •
12:
```

one entry for each edge

Adjacency-SET digraph representation

Maintain vertex-indexed array of SETs.



```
 0:  { 1  2  5  6 }
 1:  { }
 2:  { }
 3:  { }
 4:  { 3 }
 5:  { 3  4 }
 6:  { 4 }
 7:  { 8 }
 8:  { }
 9:  { 10  11  12 }
10:  { }
11:  { 12 }
12:  { }
```

one entry for each edge

Adjacency-SET digraph representation: Java implementation

Same as `Graph`, but only insert one copy of each edge.

```
public class Digraph
{
   private int V;
   private SET<Integer>[] adj;            // adjacency SETs

   public Digraph(int V)
   {
      this.V = V;
      adj = (SET<Integer>[]) new SET[V];  // create empty V-vertex graph
      for (int v = 0; v < V; v++)
         adj[v] = new SET<Integer>();
   }

   public void addEdge(int v, int w)
   {
      adj[v].add(w);                      // add edge from v to w (Graph also has adj[w].add[v])
   }

   public Iterable<Integer> adj(int v)
   {
      return adj[v];                      // iterable SET for v's neighbors
   }
}
```


Digraph representations

Digraphs are abstract mathematical objects, BUT

- ADT implementation requires specific representation.
- Efficiency depends on matching algorithms to representations.

representation	space	edge between v and w?	iterate over edges incident to v?
list of edges	E	E	E
adjacency matrix	V^2	1	V
adjacency list	E + V	degree(v)	degree(v)
adjacency SET	E + V	log (degree(v))	degree(v)

In practice: Use adjacency SET representation

- Take advantage of proven technology
- Real-world digraphs tend to be "sparse"
  [huge number of vertices, small average vertex degree]
- Algs all based on iterating over edges incident to v.

Typical digraph application: Google's PageRank algorithm

Goal. Determine which web pages on Internet are important.
Solution. Ignore keywords and content, focus on hyperlink structure.

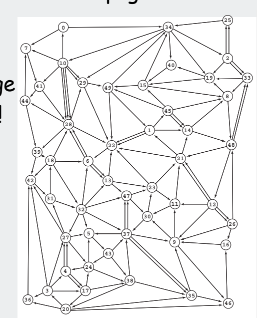
Random surfer model.

- Start at random page.
- With probability 0.85, randomly select a hyperlink to visit next; with probability 0.15, randomly select any page.
- PageRank = proportion of time random surfer spends on each page.

Solution 1: Simulate random surfer for a long time.
Solution 2: Compute ranks directly until they converge
Solution 3: Compute eigenvalues of adjacency matrix!

None feasible without sparse digraph representation

Every square matrix is a weighted digraph



▸ **digraph search**
▸ transitive closure
▸ topological sort
▸ strong components

Digraph application: program control-flow analysis

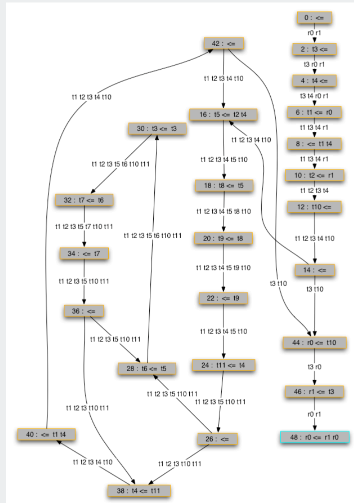
Every program is a digraph (instructions connected to possible successors)

Dead code elimination.
Find (and remove) unreachable code

can arise from compiler optimization (or bad code)

Infinite loop detection.
Determine whether exit is unreachable

can't detect all possible infinite loops (halting problem)



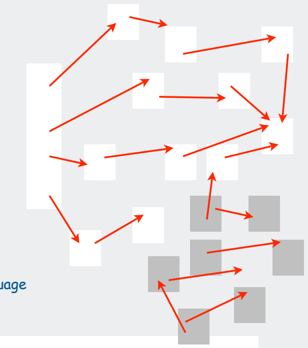
Digraph application: mark-sweep garbage collector

Every data structure is a digraph (objects connected by references)

Roots. Objects known to be directly accessible by program (e.g., stack).

Reachable objects. Objects indirectly accessible by program (starting at a root and following a chain of pointers).

easy to identify pointers in type-safe language



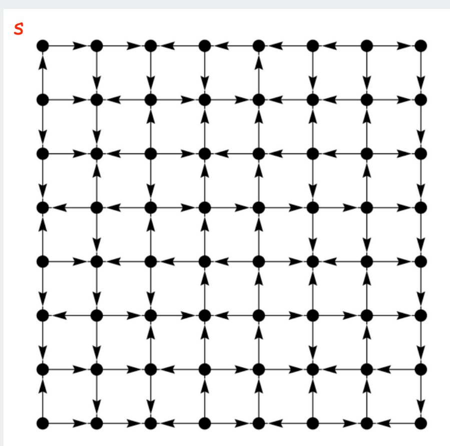
Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage, so add to free list.

Memory cost: Uses 1 extra mark bit per object, plus DFS stack.

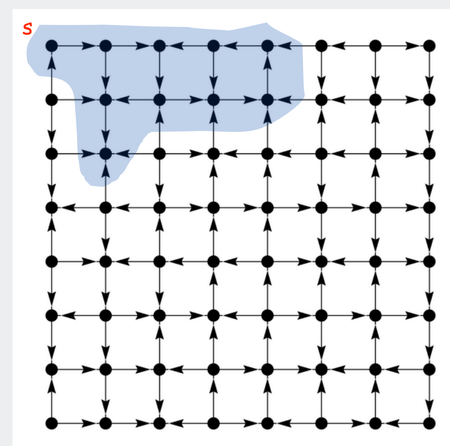
Reachability

Goal. Find all vertices reachable from s along a directed path.



Reachability

Goal. Find all vertices reachable from s along a directed path.

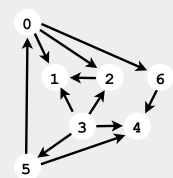


Digraph-processing challenge 1:

Problem: Mark all vertices reachable from a given vertex.

How difficult?

1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows




```
0-1
0-6
0-2
3-4
3-2
5-4
5-0
3-5
2-1
6-4
3-1
```


Depth-first search in digraphs

Same method as for undirected graphs

Every undirected graph is a digraph

- happens to have edges in both directions
- DFS is a digraph algorithm

DFS (to visit a vertex v)

Mark v as visited.

Visit all unmarked vertices w adjacent to v.

recursive

Depth-first search (single-source reachability)

Identical to undirected version (substitute `Digraph` for `Graph`).

```
public class DFSearcher
{
   private boolean[] marked;

   public DFSearcher(Digraph G, int s)
   {
      marked = new boolean[G.V()];
      dfs(G, s);
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
   }

   public boolean isReachable(int v)
   {
      return marked[v];
   }
}
```

true if connected to s

constructor marks vertices connected to s

recursive DFS does the work

client can ask whether any vertex is connected to s

Depth-first search (DFS)

DFS enables direct solution of simple digraph problems.

- ✓ Reachability.
- Cycle detection
- Topological sort
- Transitive closure.
- Is there a path from s to t ?

Basis for solving difficult digraph problems.

- Directed Euler path.
- Strong connected components.

stay tuned

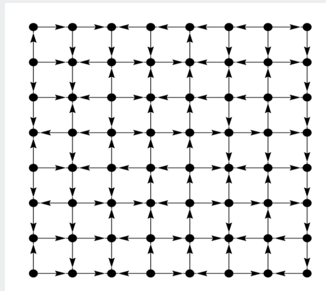
Breadth-first search in digraphs

Same method as for undirected graphs

Every undirected graph is a digraph
- happens to have edges in both directions
- BFS is a digraph algorithm

BFS (from source vertex s)

Put s onto a FIFO queue.
Repeat until the queue is empty:
- remove the least recently added vertex v
- add each of v's unvisited neighbors to the queue and mark them as visited.



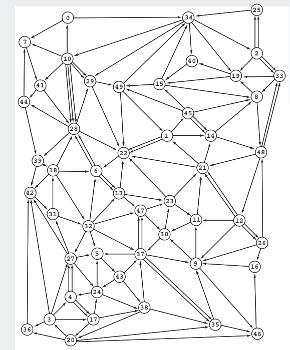
Visits vertices in increasing distance from s

Digraph BFS application: Web Crawler

The internet is a digraph

Goal. Crawl Internet, starting from some root website.
Solution. BFS with implicit graph.

BFS.
- Start at some root website (say `http://www.princeton.edu`).
- Maintain a `Queue` of websites to explore.
- Maintain a `SET` of discovered websites.
- Dequeue the next website and enqueue websites to which it links (provided you haven't done so before).



Q. Why not use DFS?
A. Internet is not fixed (some pages generate new ones when visited)

subtle point: think about it!

Web crawler: BFS-based Java implementation

```
Queue<String> q = new Queue<String>();
SET<String> visited = new SET<String>();

String s = "http://www.princeton.edu";
q.enqueue(s);
visited.add(s);

while (!q.isEmpty())
{
   String v = q.dequeue();
   System.out.println(v);
   In in = new In(v);
   String input = in.readAll();

   String regexp = "http://(\\w+\\.)*(\\w+)";
   Pattern pattern = Pattern.compile(regexp);
   Matcher matcher = pattern.matcher(input);
   while (matcher.find())
   {
      String w = matcher.group();
      if (!visited.contains(w))
      {
         visited.add(w);
         q.enqueue(w);
      }
   }
}
```

queue of sites to crawl

set of visited sites

start crawling from s

read in raw html for next site in queue

http://xxx.yyy.zzz

use regular expression to find all URLs in site

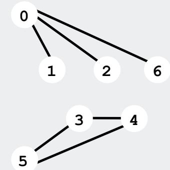
if unvisited, mark as visited and put on queue

- digraph search
- **transitive closure**
- topological sort
- strong components

Graph-processing challenge (revisited)

Problem: Is there a path from s to t ?

Goals: linear ~(V + E) preprocessing time
constant query time



0-1
0-6
0-2
4-3
5-3
5-4

How difficult?

1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible

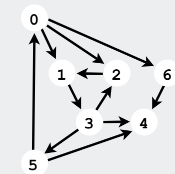
Digraph-processing challenge 2

Problem: Is there a directed path from s to t ?

Goals: linear ~(V + E) preprocessing time
constant query time

How difficult?

1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible



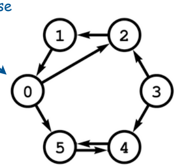
0-1
0-6
0-2
3-4
3-2
5-4
5-0
3-5
2-1
6-4
1-3

Transitive Closure

The transitive closure of G has an directed edge from v to w
if there is a directed path from v to w in G

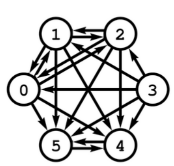
graph is usually sparse

G



	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1

Transitive closure of G



	0	1	2	3	4	5
0	1	1	1	0	1	1
1	1	1	1	0	1	1
2	1	1	1	0	1	1
3	1	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1

TC is usually dense so adjacency matrix representation is OK

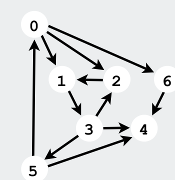
Digraph-processing challenge 2 (revised)

Problem: Is there a directed path from s to t ?

Goals: ~V^2 preprocessing time
constant query time

How difficult?

1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible

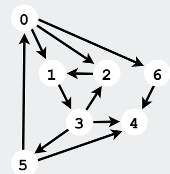


0-1
0-6
0-2
3-4
3-2
5-4
5-0
3-5
2-1
6-4
1-3

Digraph-processing challenge 2 (revised again)

Problem: Is there a directed path from s to t ?
Goals: ~VE preprocessing time (~V^3 for dense digraphs)
~V^2 space
constant query time

How difficult?
1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible



0-1
0-6
0-2
3-4
3-2
5-4
5-0
3-5
2-1
6-4
1-3

Transitive closure: Java implementation

Use an array of `DFSearcher` objects, one for each row of transitive closure

```
public class TransitiveClosure
{
   private DFSearcher[] tc;

   public TransitiveClosure(Digraph G)
   {
      tc = new DFSearcher[G.V()];
      for (int v = 0; v < G.V(); v++)
         tc[v] = new DFSearcher(G, v);
   }

   public boolean reachable(int v, int w)
   {
      return tc[v].isReachable(w);
   }
}
```

is there a directed path from v to w ?

```
public class DFSearcher
{
   private boolean[] marked;
   public DFSearcher(Digraph G, int s)
   {
      marked = new boolean[G.V()];
      dfs(G, s);
   }
   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
   }
   public boolean isReachable(int v)
   {
      return marked[v];
   }
}
```

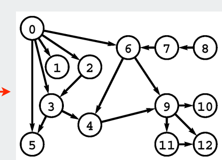

- digraph search
- transitive closure
- **topological sort**
- strong components

Digraph application: Scheduling

Scheduling. Given a set of tasks to be completed with precedence constraints, in what order should we schedule the tasks?

Graph model.
- Create a vertex **v** for each task.
- Create an edge **v→w** if task v must precede task **w**.
- Schedule tasks in topological order.

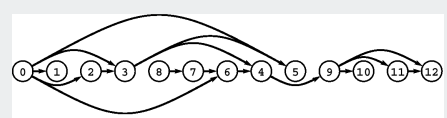
precedence constraints



tasks

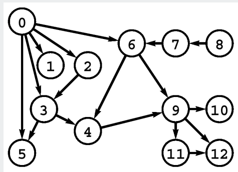
0. read programming assignment
1. download files
2. write code
3. attend precept
...
12. sleep

feasible schedule

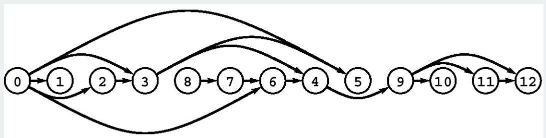


Topological Sort

DAG. Directed acyclic graph.



Topological sort. Redraw DAG so all edges point left to right.

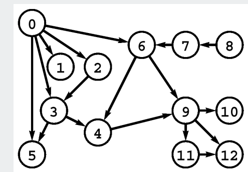


Observation. Not possible if graph has a directed cycle.

Digraph-processing challenge 3

Problem: Check that the digraph is a DAG.
If it is a DAG, do a topological sort.
Goals: linear ~(V + E) preprocessing time
provide client with vertex iterator for topological order

How difficult?
1) any CS126 student could do it
2) need to be a typical diligent CS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible




```
0-1
0-6
0-2
0-5
2-3
4-9
6-4
6-9
7-6
8-7
9-10
9-11
9-12
11-12
```


Topological sort in a DAG: Java implementation

```
public class TopologicalSorter
{
   private int count;
   private boolean[] marked;
   private int[] ts;

   public TopologicalSorter(Digraph G)
   {
      marked = new boolean[G.V()];
      ts = new int[G.V()];
      count = G.V();
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) tsort(G, v);
   }

   private void tsort(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) tsort(G, w);
      ts[--count] = v;
   }
}
```

standard DFS
with 5
extra lines of code

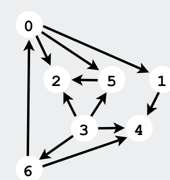
add iterator that returns
ts[0], ts[1], ts[2]...

Seems easy? Missed by experts for a few decades

Topological sort of a dag: trace

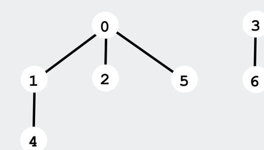
"visit" means "call tsort()" and "leave" means "return from tsort()"

	marked[]	ts[]
visit 0:	1 0 0 0 0 0 0	0 0 0 0 0 0 0
visit 1:	1 1 0 0 0 0 0	0 0 0 0 0 0 0
visit 4:	1 1 0 0 1 0 0	0 0 0 0 0 0 0
leave 4:	1 1 0 0 1 0 0	0 0 0 0 0 0 4
leave 1:	1 1 0 0 1 0 0	0 0 0 0 0 1 4
visit 2:	1 1 1 0 1 0 0	0 0 0 0 0 1 4
leave 2:	1 1 1 0 1 0 0	0 0 0 0 2 1 4
visit 5:	1 1 1 0 1 1 0	0 0 0 0 2 1 4
check 2:	1 1 1 0 1 1 0	0 0 0 0 2 1 4
leave 5:	1 1 1 0 1 1 0	0 0 0 5 2 1 4
leave 0:	1 1 1 0 1 1 0	0 0 0 5 2 1 4
check 1:	1 1 1 0 1 1 0	0 0 0 5 2 1 4
check 2:	1 1 1 0 1 1 0	0 0 0 5 2 1 4
visit 3:	1 1 1 1 1 1 0	0 0 0 5 2 1 4
check 2:	1 1 1 1 1 1 0	0 0 0 5 2 1 4
check 4:	1 1 1 1 1 1 0	0 0 0 5 2 1 4
check 5:	1 1 1 1 1 1 0	0 0 0 5 2 1 4
visit 6:	1 1 1 1 1 1 1	0 0 0 5 2 1 4
leave 6:	1 1 1 1 1 1 1	0 6 0 5 2 1 4
leave 3:	1 1 1 1 1 1 1	3 6 0 5 2 1 4
check 4:	1 1 1 1 1 1 0	3 6 0 5 2 1 4
check 5:	1 1 1 1 1 1 0	3 6 0 5 2 1 4
check 6:	1 1 1 1 1 1 0	3 6 0 5 2 1 4



adj SETs

```
0: 1 2 5
1: 4
2:
3: 2 4 5 6
4:
5: 2
6: 0 4
```



3 6 0 5 2 1 4

Topological sort in a DAG: correctness proof

Invariant:

`tsort(G, v)` visits all vertices reachable from `v` with a directed path

```
public class TopologicalSorter
{
   private int count;
   private boolean[] marked;
   private int[] ts;

   public TopologicalSorter(Digraph G)
   {
      marked = new boolean[G.V()];
      ts = new int[G.V()];
      count = G.V();
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) tsort(G, v);
   }

   private void tsort(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) tsort(G, w);
      ts[--count] = v;
   }
}
```

Proof by induction:

- `w` marked: vertices reachable from `w` are already visited
- `w` not marked: call `tsort(G, w)` to visit the vertices reachable from `w`

Therefore, algorithm is correct in placing `v` before all vertices visited during call to `tsort(G, v)` just before returning.

Q. How to tell whether the digraph has a cycle (is not a DAG)?
A. Use `TopologicalSorter` (exercise)

Topological sort applications.

- Causalities.
- Compilation units.
- Class inheritance.
- Course prerequisites.
- Deadlocking detection.
- Temporal dependencies.
- Pipeline of computing jobs.
- Check for symbolic link loop.
- Evaluate formula in spreadsheet.
- Program Evaluation and Review Technique / Critical Path Method

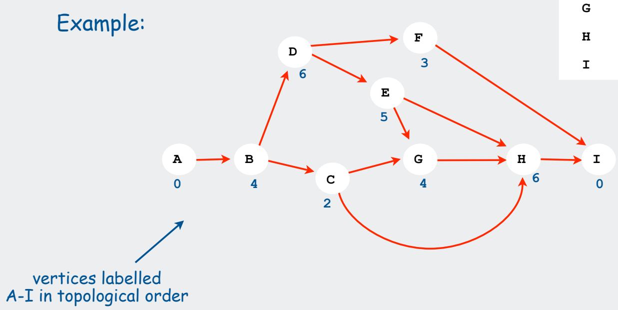
Topological sort application (weighted DAG)

Precedence scheduling

- Task `v` takes `time[v]` units of time.
- Can work on jobs in parallel.
- Precedence constraints:
- must finish task `v` before beginning task `w`.
- Goal: finish each task as soon as possible

index	task	time	prereq
A	begin	0	-
B	framing	4	A
C	roofing	2	B
D	siding	6	B
E	windows	5	D
F	plumbing	3	D
G	electricity	4	C, E
H	paint	6	C, E
I	finish	0	F, H

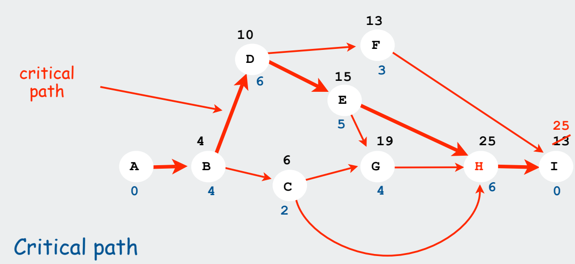
Example:



Program Evaluation and Review Technique / Critical Path Method

PERT/CPM algorithm.

- compute topological order of vertices.
- initialize `fin[v] = 0` for all vertices `v`.
- consider vertices `v` in topologically sorted order.
  - for each edge `v→w`, set `fin[w]= max(fin[w], fin[v] + time[w])`



Critical path

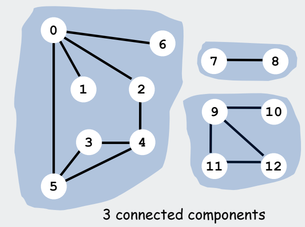
- remember vertex that set value.
- work backwards from sink

- digraph search
- transitive closure
- topological sort
- **strong components**

Strong connectivity in digraphs

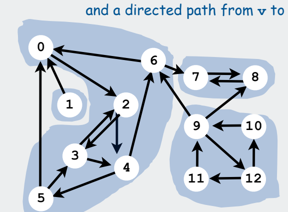
Analog to connectivity in undirected graphs

In a `Graph`, `u` and `v` are connected when there is a path from `u` to `v`



3 connected components
(sets of mutually connected vertices)

In a `Digraph`, `u` and `v` are strongly connected when there is a directed path from `u` to `v` and a directed path from `v` to `u`



4 strongly connected components
(sets of mutually strongly connected vertices)

Connectivity table (easy to compute with DFS)

	0	1	2	3	4	5	6	7	8	9	10	11	12
cc	0	0	0	0	0	0	0	1	1	2	2	2	2

```
public int connected(int v, int w)
{  return cc[v] == cc[w];  }
```

constant-time client connectivity query

Strong connectivity table (how to compute?)

	0	1	2	3	4	5	6	7	8	9	10	11	12
sc	2	1	2	2	2	2	3	3	0	0	0	0	0

```
public int connected(int v, int w)
{  return cc[v] == cc[w];  }
```

constant-time client strong connectivity query

Digraph-processing challenge 4

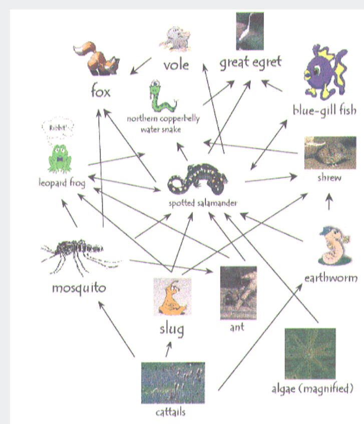
Problem: Is there a directed cycle containing s and t ?
Equivalent: Are there directed paths from s to t and from t to s?
Equivalent: Are s and t strongly connected?

Goals: linear (V + E) preprocessing time (like for undirected graphs)
constant query time

How difficult?
1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible

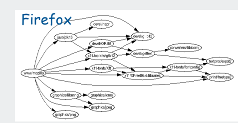
Typical strong components applications

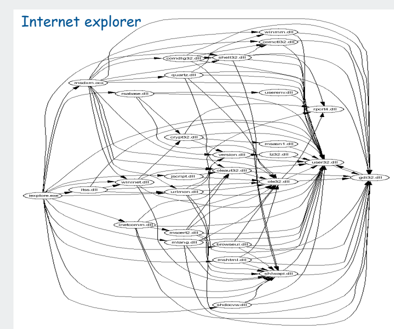
Ecological food web



Strong component: subset with common energy flow
- source in kernel DAG: needs outside energy?
- sink in kernel DAG: heading for growth?

Software module dependency digraphs





Strong component: subset of mutually interacting modules
- approach 1: package strong components together
- approach 2: use to improve design!

Strong components algorithms: brief history

1960s: Core OR problem
- widely studied
- some practical algorithms
- complexity not understood

1972: Linear-time DFS algorithm (Tarjan)
- classic algorithm
- level of difficulty: CS226++
- demonstrated broad applicability and importance of DFS

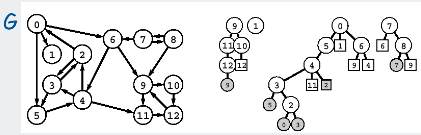
1980s: Easy two-pass linear-time algorithm (Kosaraju)
- forgot notes for teaching algorithms class
- developed algorithm in order to teach it!
- later found in Russian scientific literature (1972)

1990s: More easy linear-time algorithms (Gabow, Mehlhorn)
- Gabow: fixed old OR algorithm
- Mehlhorn: needed one-pass algorithm for LEDA

Kosaraju's algorithm

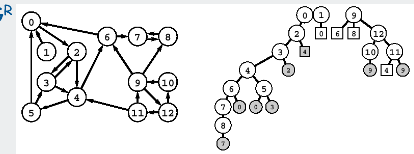
Simple (but mysterious) algorithm for computing strong components
- Run DFS on G^R and compute postorder.
- Run DFS on G, considering vertices in reverse postorder
- [has to be seen to be believed: follow example in book]

G



	0	1	2	3	4	5	6	7	8	9	10	11	12
post	8	7	6	5	4	3	2	0	1	11	10	12	9

G^R



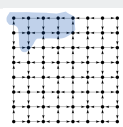
	0	1	2	3	4	5	6	7	8	9	10	11	12
sc	2	1	2	2	2	2	2	3	3	0	0	0	0

Theorem. Trees in second DFS are strong components. (!)

Proof. [stay tuned in COS 423]

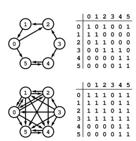
Digraph-processing summary: Algorithms of the day

Single-source reachability



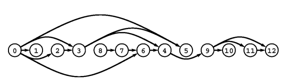
DFS

transitive closure



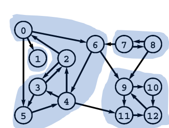
DFS from each vertex

topological sort (DAG)



DFS

strong components



Kosaraju DFS (twice)