Subdivision Surfaces

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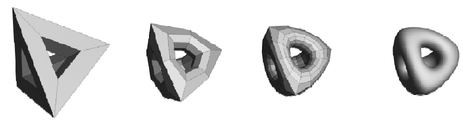
Acknowledgments: Denis Zorin, Peter Schröder, Szymon Rusinkiewicz

Subdivision Surfaces

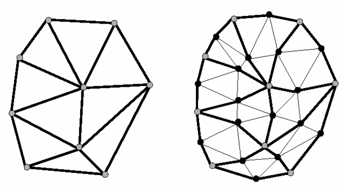
- Coarse mesh & subdivision rule
  - Smooth surface = limit of sequence of refinements



[Zorin & Schröder]

Key Questions

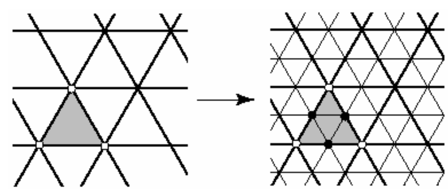
- How to refine mesh?
- Where to place new vertices?
  - Provable properties about limit surface



[Zorin & Schröder]

Loop Subdivision Scheme

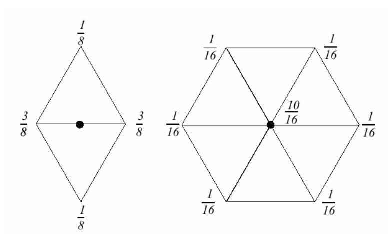
- How refine mesh?
  - Refine each triangle into 4 triangles by splitting each edge and connecting new vertices



[Zorin & Schröder]

Loop Subdivision Scheme

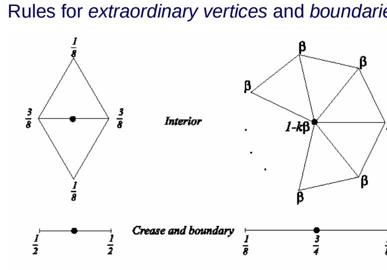
- Where to place new vertices?
  - Choose locations for new vertices as weighted average of original vertices in local neighborhood



[Zorin & Schröder]

Loop Subdivision Scheme

- Where to place new vertices?
  - Rules for *extraordinary vertices* and *boundaries*:



[Zorin & Schröder]

Loop Subdivision Scheme

- Choose β by analyzing continuity of limit surface
  - Original Loop

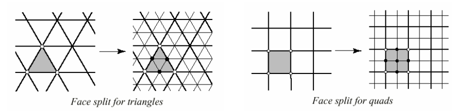
$$\beta = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2\right)$$

  - Warren

$$\beta = \begin{cases} \frac{3}{8n} & n > 3 \\ \frac{3}{16} & n = 3 \end{cases}$$

A Variety of Subdivision Schemes

- Triangles vs. Quads
- Interpolating vs. approximating



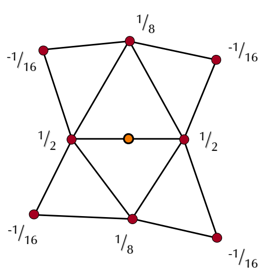
Face split		
	Triangular meshes	*Quad. meshes*
Approximating	Loop (C^2)	Catmull-Clark (C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)

Vertex split
Doo-Sabin, Midedge (C^1)
Biquartic (C^2)

[Zorin & Schröder]

Butterfly Subdivision

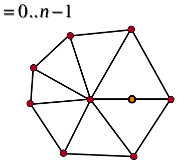
- Interpolating subdivision: larger neighborhood



Modified Butterfly Subdivision

- Need special weights near extraordinary vertices
  - For $n = 3$, weights are $5/12$, $-1/12$, $-1/12$
  - For n = 4, weights are 3/8, 0, -1/8, 0
  - For n ≥ 5, weights are

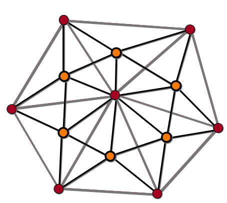
$$\frac{1}{n}\left(\frac{1}{4} + \cos\frac{2\pi j}{n} + \frac{1}{2}\cos\frac{4\pi j}{n}\right),\ j = 0..n-1$$



Weight of extraordinary vertex = 1 - Σ other weights

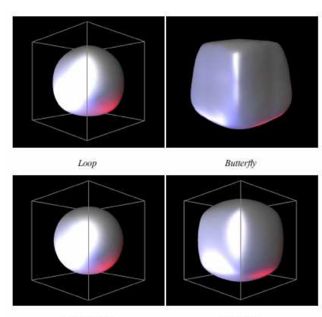
More Exotic Methods

- Kobbelt's subdivision:



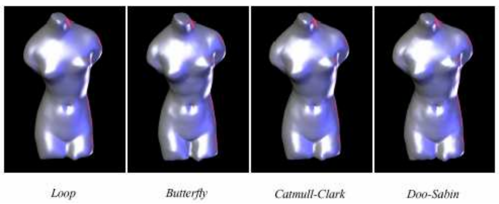
Number of faces triples per iteration: gives finer control over polygon count

Subdivision Schemes



[Zorin & Schröder]

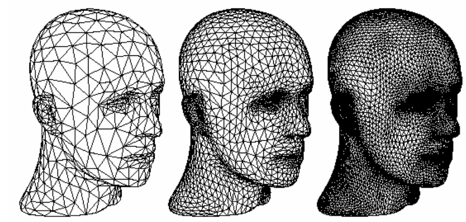
Subdivision Schemes



[Zorin & Schröder]

Analyzing Subdivision Schemes

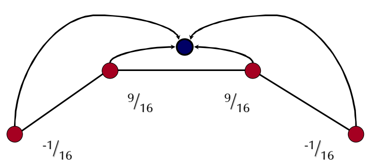
- Limit surface has provable smoothness properties



[Zorin & Schröder]

Analyzing Subdivision Schemes

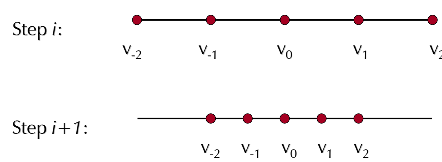
- Start with curves: 4-point interpolating scheme



(old points left where they are)

4-Point Scheme

- What is the support?



So, 5 new points depend on 5 old points

Subdivision Matrix

- How are vertices in neighborhood refined? (with vertex renumbering like in last slide)

$$\begin{pmatrix} v_{-2}^{(i+1)} \\ v_{-1}^{(i+1)} \\ v_{0}^{(i+1)} \\ v_{1}^{(i+1)} \\ v_{2}^{(i+1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_{-2}^{(i)} \\ v_{-1}^{(i)} \\ v_{0}^{(i)} \\ v_{1}^{(i)} \\ v_{2}^{(i)} \end{pmatrix}$$

Subdivision Matrix

- How are vertices in neighborhood refined? (with vertex renumbering like in last slide)

$$\vec{\mathbf{V}}^{(i+1)} = \mathbf{S}\vec{\mathbf{V}}^{(i)}$$

After *n* rounds: $\vec{\mathbf{V}}^{(n)} = \mathbf{S}^n\vec{\mathbf{V}}^{(0)}$

Convergence Criterion

$$\vec{\mathbf{V}}^{(n)} = \mathbf{S}^n \vec{\mathbf{V}}^{(0)}$$

Expand in eigenvectors of **S**:

$$\mathbf{S} = \sum_{i=0}^{4} \lambda_i \mathbf{e}_i$$

$$\vec{\mathbf{V}}^{(0)} = \sum_{i=0}^{4} a_i \mathbf{e}_i$$

$$\vec{\mathbf{V}}^{(n)} = \sum_{i=0}^{4} a_i \lambda_i^n \mathbf{e}_i$$

Criterion I: $|\lambda_i| \le 1$

Convergence Criterion

- What if all eigenvalues of **S** are < 1?
  - All points converge to 0 with repeated subdivision

Criterion II: $\lambda_0 = 1$

Translation Invariance

- For any translation t, want:

$$\begin{pmatrix} v_{-2}^{(i+1)} + t \\ v_{-1}^{(i+1)} + t \\ v_{0}^{(i+1)} + t \\ v_{1}^{(i+1)} + t \\ v_{2}^{(i+1)} + t \end{pmatrix} = \mathbf{S} \begin{pmatrix} v_{-2}^{(i)} + t \\ v_{-1}^{(i)} + t \\ v_{0}^{(i)} + t \\ v_{1}^{(i)} + t \\ v_{2}^{(i)} + t \end{pmatrix}$$

$$\vec{\mathbf{V}}^{(i+1)} + t\vec{\mathbf{1}} = \mathbf{S}\left(\vec{\mathbf{V}}^{(i)} + t\vec{\mathbf{1}}\right)$$

$$\vec{\mathbf{1}} = \mathbf{S}\vec{\mathbf{1}}$$

Criterion III: $\mathbf{e}_0 = 1$, all other $|\lambda_i| < 1$

Smoothness Criterion

- Plug back in:

$$\vec{\mathbf{V}}^{(n)} = a_0 \mathbf{e}_0 + \sum_{i=1}^{4} a_i \lambda_i^n \mathbf{e}_i$$

- Dominated by largest λi
- Case 1: |λ1| > |λ2|

$$\vec{\mathbf{V}}^{(n)} = a_0 \mathbf{e}_0 + a_0 \lambda_1^n \mathbf{e}_1 + (small)$$

  - Group of 5 points gets shorter
  - All points approach multiples of e1 → on a straight line
  - Smooth!

Smoothness Criterion

- Case 2: |λ1| = |λ2|
  - Points can be anywhere in space spanned by e1, e2
  - No longer have smoothness guarantee

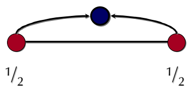
Criterion IV: Smooth iff $\lambda_0 = 1 > |\lambda_1| > |\lambda_i|$

Continuity and Smoothness

- So, what about 4-point scheme?
  - Eigenvalues = 1, $^1/_2$, $^1/_4$, $^1/_4$, $^1/_8$
  - $\mathbf{e}_0 = \mathbf{1}$
  - Stable ü
  - Translation invariant ü
  - Smooth ü

2-Point Scheme

- In contrast, consider 2-point interpolating scheme



  - Support = 3
  - Subdivision matrix = $\begin{pmatrix} ½ & ½ & 0 \\ 0 & 1 & 0 \\ 0 & ½ & ½ \end{pmatrix}$

Continuity of 2-Point Scheme

- Analysis
  - Eigenvalues = 1, $^1/_2$, $^1/_2$
  - $\mathbf{e}_0$ = **1**
  - Stable ü
  - Translation invariant ü
  - Smooth X
    - » *Not* smooth; in fact, this is piecewise linear

For Surfaces…

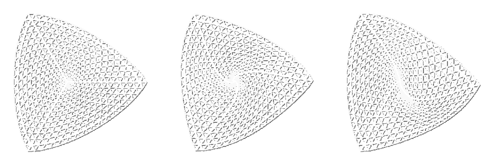
- Similar analysis: determine support, construct subdivision matrix, find eigenstuff
  - Caveat 1: separate analysis for each vertex valence
  - Caveat 2: consider more than 1 subdominant eigenvalue

Reif's smoothness condition: $\lambda_0 = 1 > |\lambda_1| \geq |\lambda_2| > |\lambda_i|$

- Points lie in subspace spanned by e1 and e2
  - If |λ1|≠|λ2|, neighborhood stretched when subdivided, but remains 2-manifold

Fun with Subdivision Methods

- Behavior of surfaces depends on eigenvalues



Real Complex Degenerate

[Zorin & Schröder]

Summary

- Advantages:
  - Simple method for describing complex, smooth surfaces
  - Relatively easy to implement
  - Arbitrary topology
  - Local support
  - Guaranteed continuity
  - Multiresolution
- Difficulties:
  - Intuitive specification
  - Parameterization
  - Intersections

[Pixar]