COS 521: Advanced Algorithm Design
Homework 4
Due: Wed, Dec 6 (in class)

Collaboration Policy: You may collaborate with other students on these problems. Collaboration is limited to discussion of ideas only, and you should write up the solutions entirely *on your own* and *list your collaborators* as well as *cite any references* you may have used.

1. Given a graph $G(V, E)$, we would like to find a subset of vertices $S \subseteq V$ so as to maximize $\frac{|E(S)|}{|S|}$. Here $E(S)$ is the set of edges in the subgraph induced by S. Write a linear programming relaxation for this problem using a variable for every vertex and a variable for every edge. Show how the optimum solution to the problem can be obtained from the linear program.

2. Given a collection of sets and a number k, we would like to pick k sets so as to maximize the size of their union. Analyze the approximation ratio achieved by the natural greedy algorithm for this problem.

3. Consider k unit vectors $v_1, v_2, \ldots, v_k$. Prove that
$$\max_{i \neq j}\{v_i \cdot v_j\} \geq \frac{-1}{k-1}$$

4. As with linear programs, semidefinite programs have duals. The dual of the MAX CUT SDP we looked at is
$$\frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\min \sum_i \gamma_i$$
$$\text{subject to:} \qquad W + diag(\gamma) \text{ symmetric, positive semidefinite,}$$
where the matrix W is the symmetric matrix of the edge weights and the matrix $diag(\gamma)$ is the matrix with zeroes on the off-diagonal entries and γ_i as the ith entry on the diagonal. Show that the value of any feasible solution for this dual is an upper bound on the cost of any cut.

5. Given a graph $G(V, E)$, the *sparsest cut* problem is the problem of computing
$$\min_{S \subset V} \frac{|E(S, \bar{S})|}{|S||\bar{S}|}$$

Consider the following vector program with a vector v_i for every $i \in V$.

$$\begin{aligned} \min \sum_{(i,j)\in E} (v_i - v_j)^2 \\ \text{such that} \quad \sum_{i<j} (v_i - v_j)^2 = 1 \end{aligned}$$

(a) Show that the vector program is a relaxation for sparsest cut.

(b) Suppose that graph G is a d-regular graph. Show that the optimum value of the vector program is the second eigenvalue of the Laplacian of G (suitably scaled).