Combining both:

$$n_1 = 2[\sum_{k=1}^{\infty} n_k] - [n_1 + 2\sum_{k=2}^{\infty} n_k] \geq (2\beta - d)|S|.$$

This is the number of nodes in R that send acceptances. Any node in S_i can receive at most d acceptances, so the number that drop out is at least n_1/d. Thus $|S_{i+1}| \leq |S_i| - n_1/d$ and the claim follows.

REMARK 1 This simple algorithm only scratches the surface of what is possible. One can improve the algorithm to run in $O(\log N)$ time, and furthermore, route calls in a *nonblocking* fashion. This means that callers can make calls and hang up any number of times and in any (adversarially determined) order, but still every unused input can call any unused output and the call is placed within $O(\log N)$ steps using local control. The main idea in proving the nonblocking is to treat busy nodes in the circuit —those currently used by other paths— as faulty, and to show that the remaining graph/circuit still has high expansion. See the paper by Arora, Leighton, Maggs and an improvement by Pippenger that requires expansion much less than $d/2$.

2 Spectral properties of graphs and expanders

2.1 Basic facts from linear algebra

We begin by stating several definitions and results from linear algebra:
Let $M \in \Re^{n \times n}$ be a square symmetric matrix of n rows and columns.

DEFINITION 2.1 *An **eigenvalue** of M is a scalar $\lambda \in \Re$ such that exists a vector $x \in \Re^n$ for which $M \cdot x = \lambda \cdot x$. The vector x is called the **eigenvector** corresponding to the eigenvalue λ. (The multiset of eigenvalues is called the* spectrum*.)*

Facts about eigenvalues and eigenvectors of symmetric matrices over $\Re$:

1. M has n real eigenvalues denoted $\lambda_1 \leq ... \leq \lambda_n$. The eigenvectors associated with these eigenvalues form an orthogonal basis for the vector space $\Re^n$ (for any two such vectors the inner product is zero and all vectors are linear independent).

2. The smallest eigenvalue satisfies:

$$\lambda_1 = \min_{x \in R^n, x \neq 0} \frac{x^T M x}{x^T x}$$

   Denote the eigenvector corresponding to λ_i as x_i. Denote the vector space of all vectors in R^n that are orthogonal to x_1 as: $W(x_1) := \Re^n \setminus span\{x_1\}$. Then the second smallest eigenvalues satisfies:

$$\lambda_2 = \min_{x \in W(x_1)} \frac{x^T M x}{x^T x}$$

   If we denote $W(x_1, ..., x_{k-1}) := \Re^n \setminus span\{x_1, ..., x_{k-1}\}$, then the k'th smallest eigenvalue is:

$$\lambda_k = \min_{x \in W(x_1, ..., x_{k-1})} \frac{x^T M x}{x^T x}$$

   This characterization of the spectrum is called the *Courant Fisher Theorem*.

3. Denote by $Spec(M)$ the spectrum of matrix M, that is the multi-set of its eigenvalues. Then for a block diagonal matrix M, that is, a matrix of the form:

$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

The following holds: $Spec(M) = Spec(A) \cup Spec(B)$

4. Eigenvalues of a matrix can be computed in polynomial time. (Eigenvalues are the roots of the characteristic polynomial of a matrix).

5. The Interlacing theorem:
A matrix B is denoted a **principal minor** of matrix M if it can be obtained from M by deleting $k < n$ columns and k rows.
Let $A \in \Re^{(n-1)\times(n-1)}$ be a principal minor of the matrix M. Let:

$$Spec(A) = \{\mu_1 \leq ... \leq \mu_{n-1}\}$$

Then:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq ... \leq \mu_{n-1} \leq \lambda_n$$

2.2 Matrices of Graphs

The most common matrix associated with graphs in literature is the **adjacency matrix**. For a graph $G = (V, E)$, the adjacency matrix $A = A_G$ is defined as:

$$A_{i,j} = \begin{cases} 1 & (i,j) \in E \\ \\ 0 & \text{otherwise} \end{cases}$$

Another common matrix is the **Laplacian** of a graph, denoted $\mathcal{L}_G = \mathcal{L}$, and defined as:

$$\mathcal{L}_{i,j} = \begin{cases} 1 & i = j \\ \\ 0 & i \neq j \text{ and } (i,j) \notin E \\ \\ -\frac{1}{\sqrt{d_i d_j}} & i \neq j \text{ and } (i,j) \in E \end{cases}$$

(where d_i is the degree of the node $v_i \in V$)
Notice that if the graph G is d-regular, then its matrices satisfy $A_G = d(I - \mathcal{L}_G)$. Denote by $\{\lambda_i\}$ the eigenvalues of A and by $\{\mu_i\}$ the eigenvalues of $\mathcal{L}$. Then the previous relation implies: $\lambda_i = d(1 - \mu_i)$.
Fact: for any graph, the laplacian is a positive semi-definite matrix, that is, for any vector $y \in R^n$:

$$\forall y \in R^n \ . \ y^T \mathcal{L} y \geq 0$$

(or equivalently, all eigenvalues are nonnegative).

2.2.1 Example - The cycle

Consider a cycle on n nodes. The laplacian is:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -\frac{1}{2} \\ \vdots & & \ddots & \vdots \\ 0 & -\frac{1}{2} & \cdots & 1 \end{bmatrix}$$

This matrix has 1's on the diagonal, and 0's or $-\frac{1}{2}$ elsewhere, depending on whether the indices correspond to an edge. Since a cycle is 2-regular, in each row and column there are exactly two entries with $-\frac{1}{2}$.

Claim 2.2 *The all ones vector $\vec{1}$ is an eigenvector of the laplacian of the n-node cycle, corresponding to eigenvalue 0.*

Proof: Observe that since every row has exactly two $-\frac{1}{2}$ then:

$$\mathcal{L} \cdot \vec{1} = 0 = 0 \cdot \vec{1}$$

□ In fact, we can characterize all eigenvalues of the cycle:

Claim 2.3 *The eigenvalues of the laplacian of the n-node cycle are:*

$$\{1 - \cos\frac{2\pi k}{n}\ ,\ k = 0, 1, ..., n-1\}$$

Proof: Observe that if the vertices are named consecutively, then each vertex i is connected to $i-1, i+1 \mod n$. Therefore, a value λ is an eigenvalue with an eigenvector $\vec{x}$ if and only if for every index of $\vec{x}$:

$$x_i - \frac{1}{2}(x_{i-1} + x_{i+1}) = \lambda \cdot x_i$$

(where sums are modulo n)
It remain to show the eigenvectors. For eigenvalue $\lambda_k = 1 - \cos\frac{2\pi k}{n}$ we associate the eigenvector $\vec{x^k}$ with coordinates:

$$x_i^k = \cos\frac{2\pi i k}{n}$$

And indeed (recall the identity $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$):

$$\begin{aligned} x_i^k - \frac{1}{2}(x_{i-1}^k + x_{i+1}^k) &= \cos\frac{2\pi i k}{n} - \frac{1}{2}\left(\cos\frac{2\pi(i+1)k}{n} + \cos\frac{2\pi(i-1)k}{n}\right) \\ &= \cos\frac{2\pi i k}{n} - \frac{1}{2}\left(2\cos\frac{2\pi i k}{n}\cos\frac{2\pi k}{n}\right) \\ &= \cos\frac{2\pi i k}{n}\left(1 - \cos\frac{2\pi k}{n}\right) \\ &= x_i^k \cdot \lambda_k \end{aligned}$$

□

2.3 Expansion and spectral properties

In this section we state the connection between expansion of a graph (defined below) and the eigenvalues of its characteristic matrices.

Definition 2.4 *Let $G = (V, E)$ be a graph. For any subset of vertices $S \subseteq V$ define its **volume** to be:*

$$Vol(S) := \sum_{i \in S} d_i$$

For a subset of vertices $S \subseteq V$ denote by $E(S, \overline{S})$ the set of edges crossing the cut defined by S. Using the above definition we can define edge expansion of a graph:

Definition 2.5 *The **Cheeger constant** of G is:*

$$h_G := \min_{S \subseteq V} \frac{|E(S, \overline{S})|}{\min\{Vol(S), Vol(\overline{S})\}} = \min_{S \subseteq V, Vol(S) \leq |E|} \frac{|E(S, \overline{S})|}{Vol(S)}.$$

(N.B. This number is closely related to the conductance *of the graph.)*

The vertex expansion is defined as:

Definition 2.6 *A graph $G = (V, E)$ is a c**-expander** if for every subset of vertices of cardinality less then half of the vertices, the number of vertices in the set of neighbors is at least c times the original set. Formally:*

$$\forall S \subseteq V, |S| < \frac{1}{2}|V| \ . \ |\Gamma(S)| \geq c \cdot |S|$$

Denote by c_G the maximal constant c for which the graph G is a c-expander.

Observe that for a d-regular graph G (or if d denotes the average degree in *any* graph):

$$h_G \leq c_G \leq d \cdot h_G$$

We now arrive at the main theorem of the lesson, describing the connection between eigenvalues of the Laplacian and the edge expansion:

Theorem 2.7 (Cheeger-Alon) *If λ is the second smallest eigenvalue of the Laplacian of the graph G, then:*

$$\frac{h_G^2}{2} \leq \lambda \leq 2h_G$$

Proof:[first part] We first prove that $\lambda \leq 2h_G$.
We begin with a small claim regarding the smallest eigenvalue:

Claim 2.8 *G has an eigenvalue 0, corresponding to the eigenvector $\vec{w}$ with coordinates:*

$$w_i = \sqrt{d_i}$$

PROOF: By straightforward calculation. Denote by $\vec{b}$ the vector:

$$\vec{b} := \mathcal{L} \cdot w$$

The i'th coordinate of $\vec{b}$ is:

$$b_i = w_i - \sum_{j \in \Gamma(i)} \frac{w_j}{\sqrt{d_i d_j}} = \sqrt{d_i} - \sum_{j \in \Gamma(i)} \frac{1}{\sqrt{d_i}} = 0$$

Hence $\vec{b}$ is the zero vector, and thus 0 is an eigenvalue corresponding to the above eigenvector. □

Using the previous claim, we can write an explicit expression for the second smallest eigenvalue:

$$\lambda = \min_{x:\Sigma_i \sqrt{d_i} x_i = 0} \frac{x^T \mathcal{L} x}{x^T x} \tag{2.1}$$

Using the identity: $z^T M x = \sum_{i,j} z_i M_{i,j} x_j$:

$$= \min_{x:\Sigma_i \sqrt{d_i} x_i = 0} \frac{1}{\sum_i x_i^2} \left[\sum_i \left(x_i^2 - \sum_{j \in \Gamma(i)} \frac{x_i x_j}{\sqrt{d_i d_j}} \right) \right] \tag{2.2}$$

Now substitute $y_i := \frac{x_i}{\sqrt{d_i}}$:

$$= \min_{\Sigma_i \sqrt{d_i} x_i = 0} \frac{\left(\sum_i d_i y_i^2 - \sum_{j \in \Gamma(i)} y_i y_j \right)}{\sum_i d_i y_i^2} \tag{2.3}$$

$$= \min_{\Sigma_i \sqrt{d_i} x_i = 0} \frac{\sum_{(i,j) \in E} (y_i - y_j)^2}{\sum_i d_i y_i^2} \tag{2.4}$$

(Aside: This characterization of the second eigenvalue is worth keeping in mind.)

Now let $S \subseteq V$ so that $Vol(S) \leq |E|$ (note that $Vol(V) = 2|E|$). Fix $\vec{a}$ to be with coordinates:

$$a_i = \begin{cases} \frac{1}{Vol(S)} & i \in S \\ -\frac{1}{Vol(\overline{S})} & i \notin S \end{cases}$$

Notice that $\vec{a}$ is legal as:

$$\sum_i d_i a_i = \sum_{i \in S} \frac{d_i}{Vol(S)} - \sum_{i \notin S} \frac{d_i}{Vol(\overline{S})} = \frac{Vol(S)}{Vol(S)} - \frac{Vol(\overline{S})}{Vol(\overline{S})} = 0$$

Now, according to the last expression obtained for λ we get:

$$\begin{aligned}
\lambda &\leq \frac{\sum_{(i,j)\in E}(a_i-a_j)^2}{\sum_i d_i a_i^2} \\
&= \frac{\left(\frac{1}{Vol(S)}+\frac{1}{Vol(\overline{S})}\right)^2 \cdot E(S,\overline{S})}{\sum_{i\in S}\frac{d_i}{Vol(S)^2}+\sum_{i\notin S}\frac{d_i}{Vol(\overline{S})^2}} \\
&= \left(\frac{1}{Vol(S)}+\frac{1}{Vol(\overline{S})}\right)\cdot E(S,\overline{S}) \\
&\leq \frac{2}{Vol(S)}\cdot E(S,\overline{S})
\end{aligned}$$

And since this holds for any $S \subseteq V$, it specifically holds for the set that minimizes the quantity in Cheeger's constant, and we get:

$$\lambda \leq 2h_G$$

□

Before we proceed to the more difficult part of the theorem, we recall the Cauchy-Schwartz inequality:
Let $a_1,...,a_n \in R$; $b_1,...,b_n \in R$, then:

$$\sum_{i=1}^{n} a_i b_i \leq (\sum_i a_i^2)^{\frac{1}{2}} \cdot (\sum_i b_i^2)^{\frac{1}{2}}$$

PROOF:[second part] Let $\vec{y}$ be the vector so that:

$$\lambda = \frac{\sum_{(i,j)\in E}(y_i-y_j)^2}{\sum_i d_i y_i^2}$$

Define two vectors $\vec{u},\vec{v}$ with coordinates:

$$u_i = \begin{cases} -y_i & y_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$v_i = \begin{cases} y_i & y_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Observe that:

$$(y_i-y_j)^2 \geq (u_i-u_j)^2+(v_i-v_j)^2$$

And hence:

$$\lambda \geq \frac{\sum_{(i,j)\in E}[(u_i-u_j)^2+(v_i-v_j)^2]}{\sum_i d_i(u_i+v_i)^2}$$

Since $\frac{a+b}{c+d} \geq \min\{\frac{a}{c}, \frac{b}{d}\}$ it suffices to show that:

$$\frac{\sum_{(i,j)\in E}(u_i - u_j)^2}{\sum_i d_i u_i^2} \geq \frac{h_G^2}{2}$$

Now comes the mysterious part (at least to Sanjeev): multiply and divide by the same quantity.

$$\begin{aligned}
\frac{\sum_{(i,j)\in E}(u_i - u_j)^2}{\sum_i d_i u_i^2} &= \frac{\sum_{(i,j)\in E}(u_i - u_j)^2}{\sum_i d_i u_i^2} \times \frac{\sum_{(i,j)\in E}(u_i + u_j)^2}{\sum_{(i,j)\in E}(u_i + u_j)^2} \\
&\geq \frac{[\sum_{(i,j)\in E}(u_i - u_j)^2][\sum_{(i,j)\in E}(u_i + u_j)^2]}{\sum_i d_i u_i^2 \cdot 2\sum_{(i,j)\in E}(u_i^2 + u_j^2)}
\end{aligned}$$

Where the last inequality comes from $(a+b)^2 \leq 2(a^2+b^2)$. Now using Cauchy-Schwartz:

$$\begin{aligned}
&\geq \frac{[\sum_{(i,j)\in E}(u_i - u_j)(u_i + u_j)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= \frac{[\sum_{(i,j)\in E}(u_i^2 - u_j^2)]^2}{2(\sum_i d_i u_i^2)^2}
\end{aligned}$$

Now denote by $S_k = \{v_1, ..., v_k\} \subseteq V$ the set of the first k vertices. Denote by C_k the size of the cut induced by S_k:

$$C_k := |E(S_k, \overline{S_k})|$$

Now, since $u_i^2 - u_j^2 = u_i^2 - u_{i+1}^2 + u_{i+1}^2 - u_{i+1}^2 ... + u_{j-1}^2 - u_i^2$, we can write:

$$\sum_{(i,j)\in E}(u_i^2 - u_j^2) = \sum_k (u_k^2 - u_{k+1}^2) \cdot C_k$$

And thus, returning to the chain of inequalities we get:

$$\begin{aligned}
\frac{\sum_{(i,j)\in E}(u_i - u_j)^2}{\sum_i d_i u_i^2} &\geq \frac{[\sum_{(i,j)\in E}(u_i^2 - u_j^2)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot C_k]^2}{2(\sum_i d_i u_i^2)^2}
\end{aligned}$$

According to the definition of h_G we know that $C_k \geq h_G \cdot (\sum_{i\leq k} d_i)$ (as h_G is the minimum of a set of expressions containing these). Hence:

$$\begin{aligned}
&\geq \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot h_G \cdot (\sum_{i\leq k} d_i)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= h_G^2 \cdot \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot (\sum_{i\leq k} d_i)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= h_G^2 \cdot \frac{(\sum_k d_k u_k^2)^2}{2(\sum_i d_i u_i^2)^2} \\
&= \frac{h_G^2}{2}
\end{aligned}$$

And this concludes the second part of the theorem. $\square$

Remark 2 Note that we proved the stronger result that actually gives us a method to find a cut $(S, \overline{S})$ such that $\frac{E(S,\overline{S})}{\min Vol(S), Vol(\overline{S})} \leq \sqrt{2\lambda}$. Namely, take the eigenvector $(y_1, y_2, \ldots, y_n)$ corresponding to λ, and check all the n cuts of the type $S_k = \{i : x_i \leq x_k\}$.