PRINCETON UNIVERSITY F'02 COS 597D: A THEORIST'S TOOLKIT

Lecture 2: The Duality Theorem

Lecturer: *Sanjeev Arora* Scribe: *Satyen Kale*

1 Linear Programming and Farkas' Lemma

A Linear Program involves optimizing a linear cost function with respect to *linear* inequality constraints. They are useful for algorithm design as well as a tool in mathematical proofs. The typical program looks as follows.

GIVEN: vectors $\mathbf{c}, \mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m \in \mathbf{R}^n$, and real numbers $b_1, b_2, \ldots b_m$.
OBJECTIVE: find $\mathbf{X} \in \mathbf{R}^n$ to minimize $\mathbf{c} \cdot \mathbf{X}$, subject to:

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{X} & \geq & b_1 \\ \mathbf{a}_2 \cdot \mathbf{X} & \geq & b_2 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{X} & \geq & b_m \\ \mathbf{X} & \geq & 0 \end{array} \tag{1}$$

The notation $\mathbf{X} > \mathbf{Y}$ simply means that $\mathbf{X}$ is componentwise larger than $\mathbf{Y}$. Now we represent the system in (1) more compactly using matrix notation. Let

$$A = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then the Linear Program (LP for short) can be rewritten as:

$$\begin{array}{rl} \min & \mathbf{c}^T\mathbf{X} : \\ A\mathbf{X} & \geq \mathbf{b} \\ \mathbf{X} & \geq 0 \end{array} \tag{2}$$

This form is general enough to represent any possible linear program. For instance, if the linear program involves a linear equality $\mathbf{a} \cdot \mathbf{X} = b$ then we can replace it by two inequalities

$$\mathbf{a} \cdot \mathbf{X} \geq b \quad \text{and} \quad -\mathbf{a} \cdot \mathbf{X} \geq -b.$$

If the variable X_i is unconstrained, then we can replace each occurence by $X_i^+ - X_i^-$ where X_i^+, X_i^- are two new non-negative variables.

The set of conditions in an LP may not be satisfiable, however. Farkas' Lemma tells us when this happens.

LEMMA 1
Farkas' Lemma. *The set of linear inequalities (1) is infeasible if and only if using positive linear combinations of the inequalities it is possible to derive* $-1 \geq 0$*, i.e. there exist* $\lambda_1, \lambda_2, \ldots \lambda_m \geq 0$ *such that*

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i < 0 \quad \textit{and} \quad \sum_{i=1}^{m} \lambda_i b_i > 0.$$

2 The Duality Theorem

With every LP we can associate another LP called its *dual*. The original LP is called the *primal*. If the primal has n variables and m constraints, then the dual has m variables and n constraints.

$$\begin{array}{rl|rl} \multicolumn{2}{c|}{\underline{\text{Primal}}} & \multicolumn{2}{c}{\underline{\text{Dual}}} \\ \min & \mathbf{c}^T\mathbf{X}: & \max & \mathbf{Y}^T\mathbf{b}: \\ A\mathbf{X} & \geq \mathbf{b} & \mathbf{Y}^T A & \leq \mathbf{c^T} \\ \mathbf{X} & \geq 0 & \mathbf{Y} & \geq 0 \end{array} \tag{3}$$

(Aside: if the primal contains an equality constraint instead of inequality then the corresponding dual variable is unconstrained.)

It is an easy exercise that the dual of the dual is just the primal.

THEOREM 2
The Duality Theorem. *If both the Primal and the Dual of an LP are feasible, then the two optima coincide.*

PROOF: The proof involves two parts:

1. Primal optimum $\geq$ Dual optimum.
   This is the easy part. Suppose $\mathbf{X}^*, \mathbf{Y}^*$ are the respective optima. This implies that

   $$A\mathbf{X}^* \geq \mathbf{b}.$$

   Now, since $\mathbf{Y}^* \geq 0$, the product $\mathbf{Y}^* A\mathbf{X}^*$ is a non-negative linear combination of the rows of $A\mathbf{X}^*$, so the inequality

   $$\mathbf{Y}^{*T} A\mathbf{X}^* \geq \mathbf{Y}^{*T}\mathbf{b}$$

   holds. Again, since $\mathbf{X}^* \geq 0$ and $\mathbf{c}^T \geq \mathbf{Y}^{*T} A$, the inequality

   $$\mathbf{c}^T\mathbf{X}^* \geq (\mathbf{Y}^{*T} A)\mathbf{X}^* \geq \mathbf{Y}^{*T}\mathbf{b}$$

   holds, which completes the proof of this part.

2. Dual optimum $\geq$ Primal optimuml.
   Let k be the optimum value of the primal. Since the primal is a minimization problem, the following set of linear inequalities is infeasible for any $\epsilon > 0$:

   $$\begin{array}{rl} -\mathbf{c}^T\mathbf{X} & \geq -(k-\epsilon) \\ A\mathbf{X} & \geq \mathbf{b} \end{array} \tag{4}$$

Here, ϵ is a small positive quantity. Therefore, by Farkas' Lemma, there exist $\lambda_0, \lambda_1, \ldots \lambda_m \geq 0$ such that

$$-\lambda_0 \mathbf{c} + \sum_{i=1}^{m} \lambda_i \mathbf{a}_i < 0 \tag{5}$$

$$-\lambda_0 (k - \epsilon) + \sum_{i=1}^{m} \lambda_i b_i > 0. \tag{6}$$

Note that $\lambda_0 > 0$ omitting the first inequality in (4) leaves a feasible system by assumption about the primal. Thus, consider the vector

$$\Lambda = (\frac{\lambda_1}{\lambda_0}, \ldots \frac{\lambda_m}{\lambda_0})^T.$$

The inequality (5) implies that $\Lambda^T A \leq \mathbf{c}^T$. So Λ is a feasible solution to the Dual. The inequality (6) implies that $\Lambda^T \mathbf{b} > (k - \epsilon)$, and since the Dual is a maximization problem, this implies that the Dual optimal is bigger than $k - \epsilon$. Letting ϵ go to zero, we get that the Dual optimal $\geq k =$ Primal optimal. Thus, this part is proved, too. Hence the Duality Theorem is proved.

□

Sanjeev's thoughts on this business: (1) Usually textbooks bundle the case of infeasible systems into the statement of the Duality theorem. He feels that this muddies the issue. Usually all applications of LPs fall into two cases: (a) We either know (for trivial reasons) that the system is feasible, and are only interested in the value of the optimum or (b) We do not know if the system is feasible and that is precisely what we want to determine. Then it is best to just use Farkas' Lemma. (2) The proof of the Duality theorem is interesting. The first part shows that for any dual feasible solution $\mathbf{Y}$ the various Y_i's can be used to obtain a *weighted* sum of primal inequalities, and thus obtain a lowerbound on the primal. The second part shows that this method of taking weighted sums of inequalities is *sufficient* to obtain the best possible lowerbound on the primal: there is no need to do anything fancier (e.g., taking products of inequalities or some such thing).

3 Example: Max Flow Min Cut theorem in graphs

The input is a directed graph $G(V, E)$ with one source s and one sink t. Each edge e has a capacity c_e. The flow on any edge must be less than its capacity, and at any node apart from s and t, flow must be conserved: total incoming flow must equal total outgoing flow. We wish to maximize the flow we can send from s to t. The maximum flow problem can be formulated as a Linear Program as follows:

Let $\mathcal{P}$ denote the set of all (directed) paths from s to t. Then the max flow problem

becomes:

$$\max \sum_{P \in \mathcal{P}} f_P : \tag{7}$$

$$\forall P \in \mathcal{P} : f_P \geq 0 \tag{8}$$

$$\forall e \in E : \sum_{P : e \in P} f_P \leq c_e \tag{9}$$

Going over to the dual, we get:

$$\min \sum_{e \in E} c_e y_e : \tag{10}$$

$$\forall e \in E : y_e \geq 0 \tag{11}$$

$$\forall P \in \mathcal{P} : \sum_{e \in P} y_e \geq 1 \tag{12}$$

Notice that the dual in fact represents the Fractional min $s - t$ cut problem: think of each edge e being picked up to a fraction y_e. The constraints say that a total weight of 1 must be picked on each path. Thus the usual min cut problem simply involves $0 - 1$ solutions to the y_e's in the dual.

EXERCISE 1 Prove that the optimum solution does have $y_e \in \{0, 1\}$.

Thus, the max- st-flow = (capacity of) min-cut.