Graph Theory: Matchings and Hall's Theorem

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Definition 1 *A* matching M *in a graph* $G(V, E)$ *is a subset of the edge set* E *such that no two edges in* M *are incident on the same vertex, i.e. if* $\{w, x\}, \{y, z\} \in M$*, then the vertices* w, x, y, z *are distinct.*

The size of a matching M is the number of edges in M. For a graph $G(V, E)$, a matching of maximum size is called a *maximum* matching.

Definition 2 *If* M *is a matching in a graph* G*, a vertex* v *is said to be* M-saturated *if there is an edge in* M *incident on* v*. Vertex* v *is said to be* M*-unsaturated if there is no edge in* M *incident on* v*.*

If $G(V_1, V_2, E)$ is a bipartite graph than a matching M of G that saturates all the vertices in V_1 is called a *complete* matching (also called a *perfect* matching).

When does a bipartite graph have a complete matching ? Given a graph, if we wanted to prove that the graph has a complete matching, we can simply give the edges in the matching. On the other hand, how do you prove that a graph has no complete matching ? In this note, we state and prove Hall's theorem which gives a necessary and sufficient condition for the existence of a complete matching in a bipartite graph.

Before we state Hall's theorem, we will need some definitions and preliminaries.

Definition 3 *Given a matching* M *in graph* G*, an* M*-alternating path (cycle) is a path (cycle) in* G *whose edges are alternately in* M *and outside of* M *(i.e. if an edge of the path is in* M*, the next edge is outside* M *and vice versa). An* M*-alternating path whose end vertices are* M*-unsaturated is called an* M*-augmenting path.*

Lemma 1 *If* M *is a maximum matching in a graph* $G(V, E)$*, there can be no* M*-augmenting paths in* G*.*

Proof: Assume, for contradiction, that there exists an M-augmenting path P. Then we can flip the edges of P to obtain a new matching by removing the edges of $P \cap M$ and adding the edges of $P \cap \bar{M}$. More formally, we set $M' = M \cup (P \cap \bar{M}) \setminus (P \cap M)$. It is easy to verify that M' is indeed a valid matching in G. Further, $|M'| = |M| + 1$. This contradicts the fact that M is a maximum matching. ∎

Given a bipartite graph $G(V_1, V_2, E)$, and a subset of vertices $S \subseteq V_1$, the neighborhood $N(S)$ is the subset of vertices of V_2 that are adjacent to some vertex in S, i.e.

$$N(S) = \{v \in V_2 : \exists u \in S, (u, v) \in E\}$$

If a bipartite graph has a complete matching saturating V_1, it is easy to see that $|S| \leq |N(S)|$ for every subset $S \subseteq V_1$. Amazingly, this condition is also sufficient to guarantee the existence of a complete matching.

Theorem 1 (Hall's Theorem) *Let $G(V_1, V_2, E)$ be a bipartite graph with $|V_1| \leq |V_2|$. Then G has a complete matching saturating every vertex of V_1 iff $|S| \leq |N(S)|$ for every subset $S \subseteq V_1$.*

Proof: First we prove that the condition of the theorem is necessary. If G has a complete matching M and S is any subset of V_1, every vertex in S is matched by M into a different vertex in $N(S)$, so that $|S| \leq |N(S)|$.

Now we prove that the condition is sufficient. Suppose that $|S| \leq |N(S)|$ for every subset $S \subseteq V_1$. Assume for contradiction that G has no complete matching. Let M be a maximum matching, i.e. a matching that saturates the maximum number of vertices in V_1. Since M is not complete, there exists an M-unsaturated vertex s in V_1. Let Z be the set of vertices of G reachable from s by M-alternating paths. Since M is a maximum matching, there are no M-augmenting paths among these (by Lemma 1). Let $S = Z \cap V_1$ and $T = Z \cap V_2$. Then, every vertex of T is matched under M to some vertex of $S - \{s\}$ and every vertex of $S - \{s\}$ is matched under M to some vertex of T. Thus $|T| = |S| - 1$. Also, $T = N(S)$. Thus S is a subset of V_1 such that $|N(S)| = |S| - 1$, giving a contradiction. This proves the reverse direction. ■