PRINCETON UNIVERSITY F'02 COS 597D: A THEORIST'S TOOLKIT

Homework 4

Out: *November 11* Due: *November 25*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince me. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Estimate the mixing time of the random walk on the *lollipop* graph, which consists of a complete graph on n nodes and a path of length n attached to one of those nodes.

§2 Give an efficient algorithm for the following task, which is used for dimension reduction in many *clustering* algorithms. We are given n vectors $x_1, x_2, \ldots x_n \in \Re^n$ and a number k, and we desire the k dimensional subspace S of $\Re^n$ which minimizes the sum of the squared lengths of the projections of $x_1, x_2, \ldots, x_n$ to S. You may assume that eigenvalues and eigenvectors can be efficiently computed.

§3 In this question you will prove that random walks on constant degree expanders mix very rapidly: for any subset of vertices C that is fairly large, the probability that a random walk of length l avoids C is $\exp(-l)$. Let $G = (V, E)$ be an unweighted undirected graph and A be its adjacency matrix.

(a) Show that the number of walks of length l is $g^T A^l g$ where g is the all-1 vector.

(b) Suppose now that G is d-regular and has n vertices. Suppose that each of the eigenvalues of A except the largest (which is d) has magnitude at most λ. Let C be a subset of cn vertices and let A' be the adjacency matrix of the induced graph on $V \setminus C$. Show that every eigenvalue of A' is at most $(1-c)d + c\lambda$ in magnitude.

(c) Conclude that if $\lambda < 0.9d$ (i.e., G is an expander) and $c = 1/2$ then the probability that a random walk of length l in G (starting at a randomly chosen vertex) avoids C is at most $\exp(-l)$.