PRINCETON UNIVERSITY F'02 COS 597D: A THEORIST'S TOOLKIT

Homework 3

Out: *October 21* Due: *November 4*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince me. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 Show that $\operatorname{rank} A$ for an $n \times n$ matrix A is the least k such that A can be expressed as the sum of k rank 1 matrices. (This characterization of rank is often useful.)

§2 Compute all eigenvalues and eigenvectors of the Laplacian of the boolean hypercube on $n = 2^k$ nodes.

§3 Suppose $\lambda_1 (= d) \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the Laplacian of a connected d-regular graph G. Then show that $\alpha(G) \leq -\frac{n(\lambda_n}{d-\lambda_n}$.

§4 Let G be an n-vertex connected graph. Let λ_2 be the second largest eigenvalue of its adjacency matrix and x be the corresponding eigenvector. Then show that the subgraph induced on $S = \{i : x_i \geq 0\}$ is connected.