PRINCETON UNIVERSITY F'02 COS 597D: A THEORIST'S TOOLKIT

Homework 1

Out: *September 16* Due: *September 23*

You can collaborate with your classmates, but be sure to list your collaborators with your answer. If you get help from a published source (book, paper etc.), cite that. Also, limit your answers to one page or less —you just need to give enough detail to convince me. If you suspect a problem is open, just say so and give reasons for your suspicion.

§1 For every integer $k \geq 1$ show that any graph with more than k^k nodes has either a clique or an independent set of size at least k. (*Moral:* A graph cannot be completely disordered: there is always some local order in it.) Can you prove the same statement for graphs with fewer than k^k nodes?

§2 A bipartite graph $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$ is said to be an (α, β) *expander* if every subset $S \subseteq V_1$ of size at most $\alpha |V_1|$ has at least $\beta |S|$ neighbors in V_2. Show that for every $\alpha, \beta > 0$ satisfying $\alpha\beta < 1$ there is an integer d such that a d-regular expander exists for every large enough $|V_1|$.

§3 A graph is said to be *nontransitive* if there is no triple of vertices i, j, k such that all of $\{i, j\}, \{j, k\}, \{k, i\}$ are edges. Show that in a nontransitive d-regular graph, there is an independent set of size at least $n \log d/8d$. (Hint: Suppose we try to pick an independent set S randomly from all independent sets in the graph. For any vertex v, suppose you have picked the portion of S except for v and the neighbors of v. How would you pick the rest of S?)