COS 341 – Discrete Math
Office Hours

• Currently, my office hours are on Friday, from 2:30 to 3:30.
Office Hours

• Currently, my office hours are on Friday, from 2:30 to 3:30.
• Nobody seems to care.
• Change office hours? Tuesday, 8 PM to 9 PM.
Homework 8

- Due on Wednesday at the beginning of class.
- No collaboration!

Question 3:
- “Never crosses itself” is the key.

Question 4:
- Assume $n > 4$ (the theorem is not true for $n=4$).
- For some values of $n > 4$, the bound may not be an integer. It doesn’t matter (the number of crossings will be strictly greater than that).
From last class

• Jordan curve theorem:
  – Any Jordan curve divides the plane into two parts, the *interior* and the *exterior*.

• $K_5$ is not planar.

• $K_{3,3}$ is not planar.
2-Connected Graphs

- Recall that a graph is 2-connected if it has at least 3 vertices, and by deleting any single vertex we obtain a connected graph.

- We also know the following:
  - A graph $G$ is 2-connected if and only if it can be created from a triangle ($K_3$) by a sequence of edge subdivisions and edge insertions.
Faces and Cycles

• Theorem:
  – Let $G$ be a 2-vertex-connected planar graph. Then every face in any planar drawing of $G$ is a region of some cycle of $G$. 
Faces and Cycles

• Theorem:
  
  – Let $G$ be a 2-vertex-connected planar graph. Then every face in any planar drawing of $G$ is a region of some cycle of $G$.  

(We do need it to be 2-vertex-connected.)
Faces and Cycles

• Proof: by induction on $n$ (number of vertices)
  – Base case: $n = 3$
    • only 2-connected graph is the triangle
    • one cycle, two regions: OK.
  – Hypothesis: assume true for $n = n_0 - 1$, with $n_0 > 3$.
  – Let’s prove it is true for $n = n_0$.
    • 2-connected graph $G$ with at least 4 vertices.
Faces and Cycles

– Take a planar 2-connected graph $G$ with $n > 3$ vertices.
– Can be built from a triangle by a sequence of edge insertions and subdivisions.
– One of these must be true:
  (a) There is an edge $e$ such that $G' = G - e$ is 2-connected.
  (b) There is a graph $G' = (V', E')$ and there is an edge $e'$ in $E'$ such that the subdivision of $e'$ creates $G$.
– In either case, $G'$ is a smaller 2-connected graph.
  • By the inductive hypothesis, every face in any planar drawing of $G'$ is a region of some cycle of $G'$. 
Faces and Cycles

– Case (a): there is an edge $e$ such that $G' = G - e$ is 2-connected.

• Let $e = \{v, w\}$.
• There is a face $F$ in $G'$ corresponding to a cycle that contains both $v$ and $w$.
  $$v - \alpha_1 - w - \alpha_2 - v$$  
  ($\alpha_1$ and $\alpha_2$ are arcs in the cycle)
• The arc corresponding to $e$ divides $F$ into two faces, each corresponding to a different cycle.
  $$v - \alpha_1 - w - \alpha(e) - v$$
  $$v - e - w - \alpha_2 - v$$
Faces and Cycles

– Case (b): there is a graph $G' = (V, E')$ with an edge $e'$ in $E'$ such that the subdivision of $e'$ creates $G$.

  • Each face of $G'$ is a region of some cycle $G'$.
  • Subdividing $e'$ amounts to drawing a vertex inside the edge.
  • This extends the length of the cycles $e'$ participates in, but doesn’t change the property.
Combinatorial Characterization
Combinatorial Characterization

• Every subgraph of a planar graph must be planar:
  – cannot contain $K_5$
  – cannot contain $K_{3,3}$
Combinatorial Characterization

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• More generally: no subgraph of a planar graph can be a subdivision of a non-planar graph.
  – cannot contain a subdivision of $K_5$
  – cannot contain a subdivision of $K_{3,3}$
Combinatorial Characterization

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• More generally: no subgraph of a planar graph can be a subdivision of a non-planar graph.
  – cannot contain a subdivision of $K_5$
  – cannot contain a subdivision of $K_{3,3}$

• Is that enough?
Combinatorial Characterization

• Kuratowski’s theorem:
  – *A graph $G$ is planar if and only if it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or to a subdivision of $K_5$.*

• We can test if a graph is planar without actually drawing it:
  – we just have to verify if there are violating subgraphs.
  – (There are faster ways of testing planarity, though.)
Euler’s Formula

• Theorem:
  
  – Let $G = (V, E)$ be a connected planar graph, and let $f$ be the number of faces of any planar drawing of $G$. Then

  \[ |V| - |E| + f = 2. \]

• The number of faces does not depend on the (planar) drawing, just on the graph itself.
Euler’s Formula

• Proof by induction on $|E|$.
  – Base case: $|E| = 0$ (single vertex, single face):
    $$|V| - |E| + f = 1 - 0 + 1 = 2.$$
  – $|E| > 0$ and $G$ does not contain a cycle (it’s a tree):
    $$|V| - |E| + f = |V| - (|V| - 1) + 1 = 2.$$
  – $|E| > 0$ and $G = (V, E)$ contains a cycle:
    • Some edge $e$ belongs to a cycle; remove it.
    • The resulting graph $G’$ obeys the formula: $|V’| - |E’| + f’ = 2$
      – Clearly, $|V’| = |V|$ and $|E’| = |E| - 1$.
      – $e$ was adjacent to two faces (by Jordan) that become one: $f’ = f - 1$
        $$|V’| - |E’| + f’ = 2$$
        $$|V| - (|E| - 1) + (f - 1) = |V| - |E| + f = 2.$$
Regular Polytopes

• 3-dimensional convex bodies;
• finite number of faces;
• faces are congruent copies of the same regular polygon;
• same number of faces meet at each vertex;
• also known as *Platonic Solids*. 
Regular Polytopes

- Tetrahedron: 4 faces
- Hexahedron (a.k.a. cube): 6 faces
- Octahedron: 8 faces
- Dodecahedron: 12 faces
- Icosahedron: 20 faces

- Are there more?

[images from mathworld.wolfram.com]
Regular Polytopes

• Every convex polytope can be converted to a planar graph:
  – Find a sphere such that:
    • center of sphere inside polytope;
    • sphere contains the whole polytope.
  – Project the polytope onto the sphere:
    • we get a graph of the surface of a sphere;
    • that graph can be converted to a planar graph with a stereographic projection.
  – Vertices, faces, and edges of the polytope become vertices, faces, and edges of a planar graph.
Regular Polytopes

- Tetrahedron
Regular Polytopes

- Cube
Regular Polytopes

- Octahedron
Regular Polytopes

• Dodecahedron
Regular Polytopes

• Icosahedron
Regular Polytopes

• Parameters of a regular convex polytope:
  – $k$: number of sides in each polygon (face)
  – $d$: number of faces that meet at each vertex
  – $n$: vertices
  – $m$: edges
  – $f$: faces

• Looking at the vertices:
  – Every edge appears in exactly two vertices:
    \[ dn = 2m \]
    \[ n = \frac{2m}{d} \]
Regular Polytopes

- Parameters of a regular convex polytopes:
  - $k$: number of sides in each face
  - $d$: number of faces that meet at each vertex
  - $n$: vertices
  - $m$: edges
  - $f$: faces

- Looking at the faces:
  - Every edge appears in exactly two faces:
    \[
    kf = 2m \\
    f = \frac{2m}{k}
    \]
Regular Polytopes

• Parameters of a regular convex polytopes:
  – $k$: number of sides in each face
  – $d$: number of faces that meet at each vertex
  – $n$: vertices
  – $m$: edges
  – $f$: faces

• Looking at the whole graph:
  – It is planar, so we can apply Euler’s formula:
    \[ n - m + f = 2 \]
Regular Polytopes

• So we have:

\[ f = \frac{2m}{k} \]
\[ n = \frac{2m}{d} \]
\[ n - m + f = 2 \]

• Substituting \( n \) and \( f \) in the third equation:

\[ n - m + f = 2 \]
\[ \frac{2m}{d} - m + \frac{2m}{k} = 2 \]

(dividing by \( 2m \) and rearranging...)

\[ \frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} \]
Regular Polytopes

• So every regular polytope must obey \( \frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} \)

• In particular,

\[
\frac{1}{d} + \frac{1}{k} > \frac{1}{2}
\]

• If both \( d \geq 4 \) and \( k \geq 4 \), we would have:

\[
\frac{1}{d} + \frac{1}{k} \leq \frac{1}{2}
\]

• Se either \( d=3 \) or \( k=3 \) (or both).
Regular Polytopes

• Assume \( d = 3 \):

\[
\frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} \\
\frac{1}{3} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} \\
\frac{1}{k} - \frac{1}{6} = \frac{1}{m}
\]

• The right-hand side is positive, so \( k < 6 \).
• \( k = \{3, 4, 5\} \)
Regular Polytopes

- Assume $k=3$:

  \[
  \frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} \\
  \frac{1}{d} + \frac{1}{3} = \frac{1}{2} + \frac{1}{m} \\
  \frac{1}{d} - \frac{1}{6} = \frac{1}{m}
  \]

- The right-hand side is positive, so $d < 6$.
- $d = \{3, 4, 5\}$
# Regular Polytopes

- So the only possibilities are:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$k$</th>
<th>$n$</th>
<th>$m$</th>
<th>$f$</th>
<th>Polytope</th>
</tr>
</thead>
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<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>tetrahedron</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>cube</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
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<td>30</td>
<td>12</td>
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<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>octahedron</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>icosahedron</td>
</tr>
</tbody>
</table>
Number of Edges

• Theorem:
  – Let $G = (V,E)$ be a planar graph with at least 3 vertices. Then $|E| \leq 3|V| - 6$.
  – If the graph is maximal (no edge can be added without violating planarity), the equality holds: $|E| = 3|V| - 6$.

• It suffices to prove the second statement; if the graph is not maximal, we can always add edges until it becomes one.
Number of Edges

• Lemma:
  – *Every maximal planar graph* $G$ *is a triangulation* (every face is a triangle).

• Proof: we show that if $G$ is not a triangulation, it is always possible to add an edge without violating planarity.
  – Three cases to consider:
    • $G$ is disconnected.
    • If $G$ is connected but not 2-connected.
    • $G$ is 2-connected.
Number of Edges

- Case 1: $G$ is not connected:
  - An edge can be added between two components.
Number of Edges

• Case 1: $G$ is not connected:
  – An edge can be added between two components.

\[ V_1 \quad V_2 \quad V_3 \quad V_4 \]
Number of Edges

- **Case 2:** $G$ is connected, but not 2-connected:
  - There is a vertex $v$ whose removal disconnects $G$.
  - Let $V_1, V_2, ..., V_k$ be the resulting components ($k > 2$).
  - An edge can be added between components associated with edges drawn next to each other around $v$. 

![Diagram of connected components](image-url)
Number of Edges

- Case 2: $G$ is connected, but not 2-connected:
  - There is a vertex $v$ whose removal disconnects $G$.
  - Let $V_1, V_2, ..., V_k$ be the resulting components ($k > 2$).
  - An edge can be added between components associated with edges drawn next to each other around $v$. 

![Diagram showing $V_1$, $V_2$, $V_3$, $V_4$, and vertex $v$.]
Number of Edges

• Case 3: $G$ is 2-connected.
  – Every face is bounded by a cycle.
  – Take any face with 4 or more edges:
Number of Edges

- **Case 3:** $G$ is 2-connected.
  - Every face is bounded by a cycle.
  - Take any face with 4 or more edges:

    ![Diagram of a graph with vertices $v_1, v_2, v_3, v_4$ and edges connecting them]

    - If $v_1$ and $v_3$ are not connected, you can add an edge between them.
Case 3: $G$ is 2-connected.
- Every face is bounded by a cycle.
- Take any face with 4 or more edges:
  - If $v_1$ and $v_3$ are connected, $v_2$ and $v_4$ can’t be.
  - So you can add an edge between $v_2$ and $v_4$. 
Number of Edges

- So every maximal planar graph is a triangulation.
  - Because every face is a triangle and every edge is incident to exactly two faces, we have:
    \[ 3f = 2|E| \]
    \[ f = 2|E|/3. \]
  - Using this value in Euler’s formula:
    \[ |V| - |E| + f = 2 \]
    \[ |V| - |E| + 2|E|/3 = 2 \]
    \[ |V| - |E|/3 = 2 \]
    \[ |E| = 3|V| - 6. \]
  - Corollary: there exists a vertex of degree at most 5.
Triangle-Free Planar Graphs

• Theorem:
  – Let $G=(V,E)$ be a planar graph with no triangles (i.e., without $K_3$ as a subgraph) and at least 3 vertices. Then $|E| \leq 2|V| - 4$.

• Proof (similar to the previous one)
  – Consider a maximal triangle-free planar graph $G$;
    • we can always add edges until it becomes one.
  – $G$ is clearly connected.
Triangle-Free Planar Graphs

– Assume $G$ is connected, but not 2-connected.
– There is a vertex $v$ whose removal disconnects $G$.
– Let $V_1, V_2, ..., V_k$ be the resulting components ($k > 2$).
  • Edges can be added between these components without violating planarity.
  • But we could create a triangle if we joined vertices that are adjacent to $v$.
– If every $V_i$ is a single vertex, then $G$ is a tree:
  $|E| = |V| - 1$
  $|E| = |V| + 3 - 4$
  $|E| \leq |V| + |V| - 4$ (because $G$ has at least three vertices)
  $|E| \leq 2|V| - 4$ (the inequality holds)
Triangle-Free Planar Graphs

– Now consider the case in which component $V_1$ has at least two vertices.

– Consider a face $F$ having both a vertex of $V_1$ and a vertex of some other $V_i$ on its boundary.

– $V_1$ must have at least one edge $\{v_1, v_2\}$ on the boundary of $F$.

– We can’t have both $v_1$ and $v_2$ connected to $v$ (or these vertices would constitute a triangle).

– So an edge can be added between one of these vertices and a vertex in $V_i$.
  
  • $G$ is not maximal – a contradiction.
  
  • Maximal triangle-free planar graphs must be 2-connected.
Triangle-Free Planar Graphs

- $G$ is a 2-connected, maximal triangle-free planar graph.
- 2-connected:
  - every face is a region of a cycle.
- Triangle-free:
  - every cycle has at least 4 edges.
- Counting edges from faces: $2 |E| \geq 4f \Rightarrow f \leq |E|/2$
- From Euler’s formula:
  $$|V| - |E| + f = 2$$
  $$2 - |V| + |E| = f \leq |E|/2$$
  $$|E| \leq 2|V| - 4.$$ 
- Corollary: there exists a vertex of degree at most 3.
Scores of Planar Graphs

• Theorem:
  
  – Let $G=(V,E)$ be a 2-connected planar graph with at least 3 vertices. Define:
    
    • $n_i$: number of vertices of degree $i$;
    
    • $f_i$: number of faces (in some fixed drawing of $G$) bounded by cycles of length $i$.

  Then we have

  $$
  \sum_{i \geq 1} (6-i) n_i = 12 + 2 \sum_{j \geq 3} (j-3) f_j.
  $$

  

Scores of Planar Graphs

• Why is this relevant?
• We can rewrite

\[ \sum_{i \geq 1} (6 - i)n_i = 12 + 2 \sum_{j \geq 3} (j - 3)f_j. \]

as

\[ 5n_1 + 4n_2 + 3n_3 + 2n_2 + n_1 + (...) = 12 + (...). \]

– The first “(...)” contains only negative terms.
– The second “(...)” contains only positive terms.
– So \( 5n_1 + 4n_2 + 3n_3 + 2n_2 + n_1 \geq 12. \)
– Among other things, this means that there are at least 3 vertices of degree at most 5 in every planar graph.
Scores of Planar Graphs

• Proof of the theorem:
  – Obvious facts:
    \[ f = \sum_j f_j \quad \text{and} \quad |V| = \sum_i n_i \]
  – From Euler’s formula:
    \[ |V| - |E| + f = 2 \]
    \[ \sum_i n_i - |E| + \sum_j f_j = 2 \]
    \[ 2|E| = \sum_i 2n_i + \sum_j 2f_j - 4 \]
Scores of Planar Graphs

• Proof of the theorem:
  – From previous slide:  \[ 2 |E| = \sum_i 2n_i + \sum_j 2f_j - 4 \]
  – Counting edges from the faces:
    \[ \sum_j (j \cdot f_j) = 2 |E| = \sum_i 2n_i + \sum_j 2f_j - 4 \]
    \[ \sum_j (j \cdot f_j) - \sum_j 2f_j + 4 = \sum_i 2n_i \]
    \[ \sum_j (j - 2) f_j + 4 = \sum_i 2n_i \]
  – Counting edges from the vertices:
    \[ \sum_i (i \cdot n_i) = 2 |E| = \sum_i 2n_i + \sum_j 2f_j - 4 \]
    \[ \sum_j 2f_j = \sum_i (i \cdot n_i) - \sum_i 2n_i + 4 \]
    \[ \sum_j 2f_j = \sum_i n_i (i - 2) + 4 \]
Scores of Planar Graphs

• Proof of the theorem:
  – From the previous slide:
    \[ \sum_j (j - 2) f_j + 4 = \sum_i 2n_i \quad (\times 2) \]
    \[ \sum_j (2j \cdot f_j - 4f_j) + 8 = \sum_i 4n_i \quad (i) \]

    \[ \sum_j 2f_j = \sum_i n_i(i - 2) + 4 \quad (\times (-1)) \]
    \[ \sum_j (-2)f_j = \sum_i (2n_i - i \cdot n_i) - 4 \quad (ii) \]
  – Adding (i) and (ii), we get the final expression:
    \[ \sum_j (2j \cdot f_j - 4f_j - 2f_j) + 8 = \sum_i (4n_i + 2n_i - i \cdot n_i) - 4 \]
    \[ 2 \sum_j (j - 3)f_j + 12 = \sum_i (6 - i)n_i \]