Type reconstruction example

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Claim. The following program can be well typed.

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fun g(f) is fun h(x) is f (f x) end end (fun j(x) is x end)
```

Definitions.

$$\begin{array}{rcl} H & \equiv & \texttt{fun}\, h\,\texttt{(}x\texttt{:}\alpha_1\texttt{):}\alpha_2\,\texttt{is}\, f(f(x))\,\texttt{end} \\ G & \equiv & \texttt{fun}\, g\,\texttt{(}f\texttt{:}\alpha_3\texttt{):}\alpha_4\,\texttt{is}\, H\,\texttt{end} \\ J & \equiv & \texttt{fun}\, j\,\texttt{(}x\texttt{:}\alpha_5\texttt{):}\alpha_6\,\texttt{is}\, x\,\texttt{end} \end{array}$$

Theorem. $\vdash (GJ)^- \rightsquigarrow GJ : \tau$, for some τ.

Derivation using rules 11.19–11.26.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{g, f : \alpha_3, h, x \vdash f \rightsquigarrow f : \alpha_3[\top]} \quad \dfrac{\dfrac{}{g, f : \alpha_3, h, x \vdash f \rightsquigarrow f : \alpha_3[\top]} \quad \dfrac{}{g, f, h, x : \alpha_1 \vdash x \rightsquigarrow x : \alpha_1[\top]}}{g, f : \alpha_3, h, x : \alpha_1 \vdash fx \rightsquigarrow fx : \alpha_8[\phi_3]}}{g : \alpha_3 \to \alpha_4, f : \alpha_3, h : \alpha_1 \to \alpha_2, x : \alpha_1 \vdash f(fx) \rightsquigarrow f(fx) : \alpha_9[\phi_4]}}{g : \alpha_3 \to \alpha_4, f : \alpha_3 \vdash H^- \rightsquigarrow H : \alpha_1 \to \alpha_2[\phi_5]}}{\vdash G^- \rightsquigarrow G : \alpha_3 \to \alpha_4[\phi_6]}$$

$$\dfrac{\begin{array}{c}\vdots \\ \vdash G^- \rightsquigarrow G : \alpha_3 \to \alpha_4[\phi_6]\end{array} \quad \dfrac{\dfrac{}{j : \alpha_5 \to \alpha_6, x : \alpha_5 \vdash x \rightsquigarrow x : \alpha_5[\top]}}{\vdash J^- \rightsquigarrow J : \alpha_5 \to \alpha_6[\phi_2]}}{\vdash (GJ)^- \rightsquigarrow GJ : \alpha_7[\phi_7]}$$

where

$$\begin{array}{rcl} \phi_2 & = & \alpha_5 = \alpha_6 \\ \phi_3 & = & \alpha_3 = \alpha_1 \to \alpha_8 \\ \phi_4 & = & \phi_3 \wedge \alpha_3 = \alpha_8 \to \alpha_9 \\ \phi_5 & = & \phi_4 \wedge \alpha_9 = \alpha_2 \\ \phi_6 & = & \phi_5 \wedge \alpha_1 \to \alpha_2 = \alpha_4 \\ \phi_7 & = & \phi_6 \wedge \phi_2 \wedge \alpha_3 \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7 \end{array}$$

Solution of the constraints using rules 11.27–11.28

$$
\begin{aligned}
&(\{\},\quad \alpha_3 = \alpha_1 \to \alpha_8 \land \alpha_3 = \alpha_8 \to \alpha_9 \land \alpha_9 = \alpha_2 \land \alpha_1 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land \alpha_3 \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_3 = \alpha_1 \to \alpha_8\},\quad \alpha_1 \to \alpha_8 = \alpha_8 \to \alpha_9 \land \alpha_9 = \alpha_2 \land \alpha_1 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_1 \to \alpha_8) \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_3 = \alpha_1 \to \alpha_8\},\quad \alpha_1 = \alpha_8 \land \alpha_8 = \alpha_9 \land \alpha_9 = \alpha_2 \land \alpha_1 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_1 \to \alpha_8) \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_1 = \alpha_8, \alpha_3 = \alpha_8 \to \alpha_8\},\quad \alpha_8 = \alpha_9 \land \alpha_9 = \alpha_2 \land \alpha_8 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_8 \to \alpha_8) \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_8 = \alpha_9, \alpha_1 = \alpha_9, \alpha_3 = \alpha_9 \to \alpha_9\},\quad \alpha_9 = \alpha_2 \land \alpha_9 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_9 \to \alpha_9) \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_9 = \alpha_2, \alpha_8 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 = \alpha_2 \to \alpha_2\},\quad \alpha_2 \to \alpha_2 = \alpha_4 \land \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_2 \to \alpha_2) \to \alpha_4 = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_4 = \alpha_2 \to \alpha_2, \alpha_9 = \alpha_2, \alpha_8 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 = \alpha_2 \to \alpha_2\},\quad \alpha_5 = \alpha_6 \\
&\qquad\qquad \land (\alpha_2 \to \alpha_2) \to (\alpha_2 \to \alpha_2) = (\alpha_5 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_5 = \alpha_6, \alpha_4 = \alpha_2 \to \alpha_2, \alpha_9 = \alpha_2, \alpha_8 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 = \alpha_2 \to \alpha_2\}, \\
&\qquad\qquad (\alpha_2 \to \alpha_2) \to (\alpha_2 \to \alpha_2) = (\alpha_6 \to \alpha_6) \to \alpha_7) \\
&(\{\alpha_5 = \alpha_6, \alpha_4 = \alpha_2 \to \alpha_2, \alpha_9 = \alpha_2, \alpha_8 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 = \alpha_2 \to \alpha_2\}, \\
&\qquad\qquad (\alpha_2 \to \alpha_2) = (\alpha_6 \to \alpha_6) \land (\alpha_2 \to \alpha_2) = \alpha_7) \\
&(\{\alpha_5 = \alpha_6, \alpha_4 = \alpha_2 \to \alpha_2, \alpha_9 = \alpha_2, \alpha_8 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 = \alpha_2 \to \alpha_2\}, \\
&\qquad\qquad \alpha_2 = \alpha_6 \land \alpha_2 = \alpha_6 \land (\alpha_2 \to \alpha_2) = \alpha_7) \\
&(\{\alpha_2 = \alpha_6, \alpha_5 = \alpha_6, \alpha_4 = \alpha_6 \to \alpha_6, \alpha_9 = \alpha_6, \alpha_8 = \alpha_6, \alpha_1 = \alpha_6, \alpha_3 = \alpha_6 \to \alpha_6\}, \\
&\qquad\qquad \alpha_6 = \alpha_6 \land (\alpha_6 \to \alpha_6) = \alpha_7) \\
&(\{\alpha_2 = \alpha_6, \alpha_5 = \alpha_6, \alpha_4 = \alpha_6 \to \alpha_6, \alpha_9 = \alpha_6, \alpha_8 = \alpha_6, \alpha_1 = \alpha_6, \alpha_3 = \alpha_6 \to \alpha_6\},\quad (\alpha_6 \to \alpha_6) = \alpha_7) \\
&(\{\alpha_7 = \alpha_6 \to \alpha_6, \alpha_2 = \alpha_6, \alpha_5 = \alpha_6, \alpha_4 = \alpha_6 \to \alpha_6, \alpha_9 = \alpha_6, \alpha_8 = \alpha_6, \alpha_1 = \alpha_6, \alpha_3 = \alpha_6 \to \alpha_6\},\quad \top)
\end{aligned}
$$

If we want to, we can apply this substition to the type-inference proof, yielding a proof using rules 4.1–4.12:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{g, f : (\alpha_6 \to \alpha_6), h, x \vdash f : (\alpha_6 \to \alpha_6)} \quad \dfrac{\dfrac{}{g, f : (\alpha_6 \to \alpha_6), h, x \vdash f : (\alpha_6 \to \alpha_6)} \quad \dfrac{}{g, f, h, x : \alpha_6 \vdash x : \alpha_6}}{g, f : (\alpha_6 \to \alpha_6), h, x : \alpha_6 \vdash fx : \alpha_6}}{g : (\alpha_6 \to \alpha_6) \to (\alpha_6 \to \alpha_6), f : (\alpha_6 \to \alpha_6), h : \alpha_6 \to \alpha_6, x : \alpha_6 \vdash f(fx) : \alpha_9}}{g : (\alpha_6 \to \alpha_6) \to (\alpha_6 \to \alpha_6), f : (\alpha_6 \to \alpha_6) \vdash H : \alpha_6 \to \alpha_6}}{\vdash G : (\alpha_6 \to \alpha_6) \to (\alpha_6 \to \alpha_6)}
$$

$$
\dfrac{\begin{array}{c}\vdots \\ \vdash G : (\alpha_6 \to \alpha_6) \to (\alpha_6 \to \alpha_6)\end{array} \quad \dfrac{\dfrac{}{j : \alpha_6 \to \alpha_6, x : \alpha_6 \vdash x : \alpha_6}}{\vdash J : \alpha_6 \to \alpha_6}}{\vdash GJ : \alpha_6 \to \alpha_6}
$$

But this isn't a proof, because α_6 isn't a type. What we have discovered is that this expression is polymorphic; it can have many types. Since MinML is not a polymorphic language, we can make this into a real proof of static semantics by choosing some arbitrary type, for example `int`:

$$
\begin{array}{c}\vdots \\ \vdash GJ : \texttt{int} \to \texttt{int}\end{array}
$$