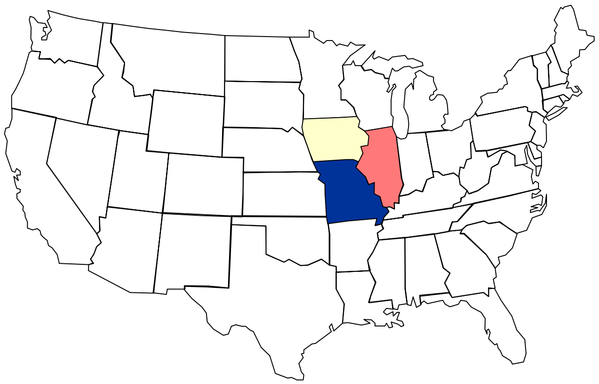
Lecture T5: NP-Completeness



Can you color each of the 48 states red, white, or blue so that no two adjacent states have the same color?

Overview

Lecture T3:

- **What is an algorithm?**
  - **Turing machine**
- **Which problems can be solved on a computer?**
  - **not the halting problem**

Lecture T4:

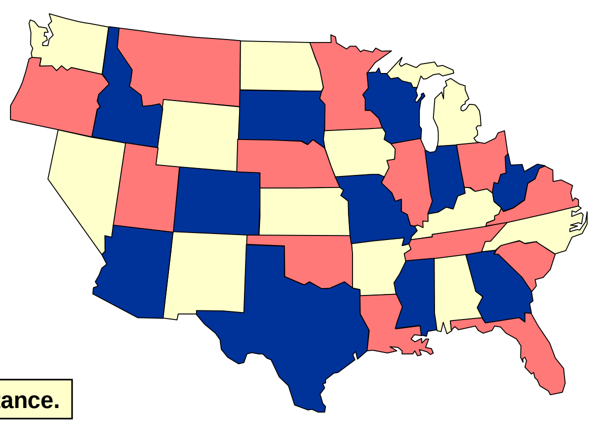
- **Which algorithms will be useful in practice?**
  - **polynomial vs. exponential algorithms**

This lecture:

- **Which problems can be solved in practice?**
  - **probably not 3-COLOR or TSP**

Some Hard Problems

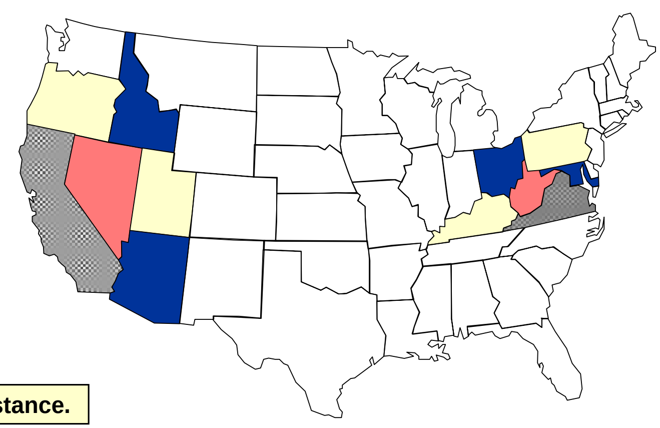
3-COLOR: Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?



YES instance.

Some Hard Problems

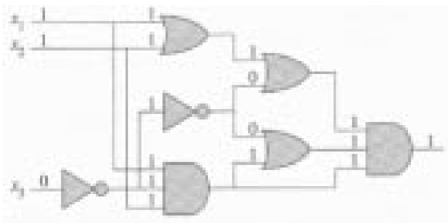
3-COLOR: Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

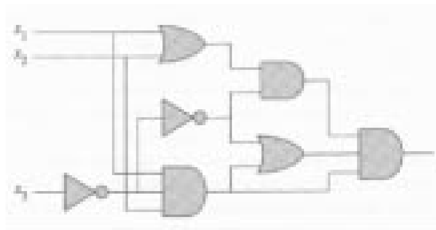


NO instance.

Some Hard Problems

CIRCUIT-SAT: Is there a way to assign inputs to a given Boolean (combinational) circuit that makes it true?





YES instance.

NO instance.

Some Hard Problems

FACTOR: Given two positive integers x and U, is there a nontrivial factor of x that is less than U?

- Factoring is at the heart of RSA encryption.

Example 1: x = 23,536,481,273, U = 110,000.

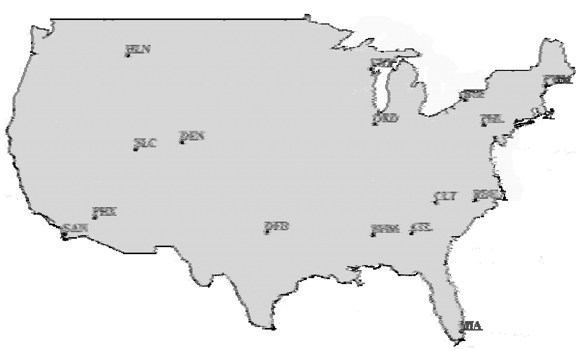
- YES: x = 224,737 $\times$ 104,729.

Example 2: x = 23,536,481,277, U = 110,000.

- NO: x is prime.

Some Hard Problems

TSP: A travelling salesperson needs to visit N cities. Is there a route of length at most D?



More Hard Problems

More hard computational problems.

- **Biology:** protein folding.
- **Chemistry:** chemical synthesis.
- **Civil engineering:** equilibrium of urban traffic flow.
- **Finance:** find minimum risk portfolio of given return.
- **Electrical engineering:** VLSI layout.
- **Medicine:** reconstructing 3-D shape from biplane angiocardiogram.
- **Operations research:** optimal resource allocation.
- **Physics:** anti-ferromagnetic Potts model.
- **Politics:** Shapley-Shubik voting power.
- **Pop culture:** Minesweeper consistency.
- **Statistics:** optimal experimental design.

Properties of Algorithms

A given problem can be solved by many different algorithms (TMs).

- Which ones are useful in practice?

A working definition: (Jack Edmonds, 1962)

- Efficient: polynomial time for ALL inputs.
  - mergesort requires N $\log_2 N$ steps
- Inefficient: "exponential time" for SOME inputs.
  - brute force TSP takes N! > 2^N steps

Robust definition has led to explosion of useful algorithms for wide spectrum of problems.

Exponential Growth

Exponential growth dwarfs technological change.

- Suppose each electron in the universe had power of today's supercomputers.
- And each works for the life of the universe in an effort to solve TSP problem using brute force N! algorithm from Lecture P6.

Some Numbers

quantity	number
Home PC instructions/second	10^9
Supercomputer instructions per second	10^{12}
Seconds per year	10^9
Age of universe in years (estimated)	10^{13}
Electrons in universe (estimated)	10^{79}

- Will not succeed for 1,000 city TSP!

$1000! \gg 10^{1000} \gg 10^{79} * 10^{13} * 10^9 * 10^{12}$

Properties of Problems

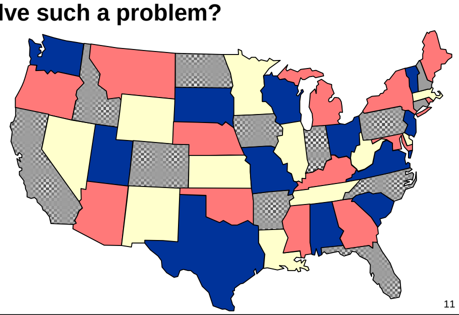
Which ALGORITHMS will be useful in practice?

- Efficient: polynomial-time for ALL inputs.
  - broad and robust definition
  - covers virtually all algorithms running on actual computers
- Inefficient: "exponential-time" for SOME inputs.

Which PROBLEMS will we be able to solve in practice?

- Those with efficient algorithms.
- How can I tell if I am trying to solve such a problem?
  - 2-COLOR: yes
  - 3-COLOR: probably no
  - 4-COLOR: yes

Theorem (Appel-Haken, 1976).
Every planar map is 4 colorable.



P

Definition of P:

- Set of all decision problems solvable in polynomial time on a deterministic Turing machine.

MULTIPLE: Is the integer y a multiple of x?

- YES: (x, y) = (17, 51).

RELPRIME: Are the integers x and y relatively prime?

- YES: (x, y) = (34, 39).

Definition important because of Strong Church-Turing thesis.

Strong Church-Turing Thesis

Strong Church-Turing thesis:

- P is the set of all decision problems solvable in polynomial time on REAL computers.

Evidence supporting thesis:

- True for all physical computers: can create deterministic TM that efficiently simulates any existing digital computer.

Possible exception?

- Quantum computers – no conventional gates.

NP

EXP: set of all decision problems solvable in exponential time on a deterministic Turing machine.

NP: does NOT mean "not polynomial."

NP: set of all decision problems with efficient certification algorithm.

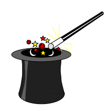
- Efficient: polynomial number of steps on deterministic TM.
- Certifier: algorithm to check whether a proposed "solution" is correct.
  - proposed solution is called CERTIFICATE (a hint)
  - technical condition: certificate must be of polynomial-size.

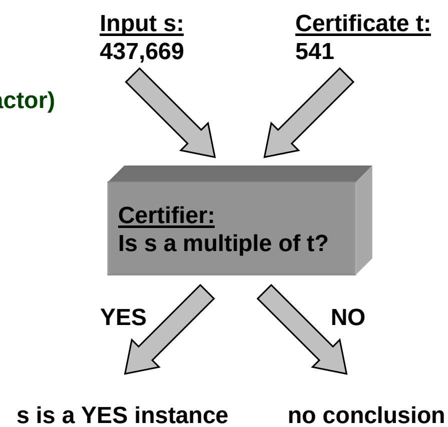
Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

Observation. s is composite $\Leftrightarrow$ there exists an integer $1 < t < s$ such that s is a multiple of t.

- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)



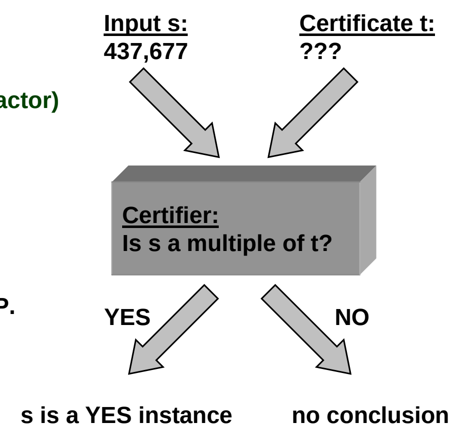


Certifiers and Certificates

COMPOSITE: Given integer s, is s composite?

Observation. s is composite $\Leftrightarrow$ there exists an integer $1 < t < s$ such that s is a multiple of t.

- YES instance: s = 437,669.
  - certificate t = 541 or 809 (a factor)
- NO instance: s = 437,677.
  - no witness can fool verifier into saying YES
- Conclusion: COMPOSITE $\in$ NP.

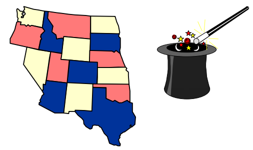


Certifiers and Certificates

3-COLOR: Given planar map, can it be colored with 3 colors?

Input s:

Certificate t:



Certifier:
1. Check that s and t describe same map.
2. Count number of distinct colors in t.
3. Check all pairs of adjacent states.

YES — s is a YES instance

NO — no conclusion

3-COLOR $\in$ NP.

NP

NP: set of decision problems with efficient certification algorithms.

NP: set of all decision problems solvable in polynomial time on a NONDETERMINISTIC Turing machine.

- Equivalent definition.
- Intuition: nondeterministic TM can guess and check all possible solutions in parallel.
- Real computer can simulate nondeterministic TM, but takes exponential time unless you get "lucky."
  - $P \subseteq NP \subseteq EXP$

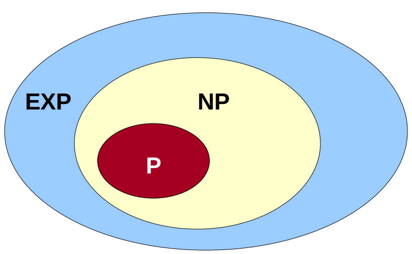
The Main Question

Does P = NP? (Edmonds, 1962)

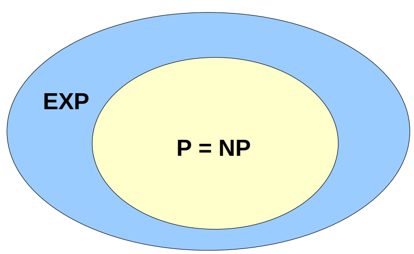
- Is the original DECISION problem as easy as CERTIFICATION?
- Does nondeterminism help you solve problems faster?

Most important open problem in computer science.

- If yes, staggering practical significance.
- Clay Foundation Millennium $1 million prize.



If $P \neq NP$



If P = NP

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as CERTIFICATION?

If yes, then:

- Efficient algorithms for 3-COLOR, TSP, FACTOR.
- Cryptography is impossible (except for one-time pads) on conventional machines.
- Modern banking system will collapse.
- Harmonial bliss.

If no, then:

- Can't hope to write efficient algorithm for TSP.
  - see NP-completeness
- But maybe efficient algorithm still exists for factoring?

The Main Question

Does P = NP?

- Is the original DECISION problem as easy as CERTIFICATION?

Probably no, since:

- Thousands of researchers have spent four decades in search of polynomial algorithms for many fundamental NP problems without success.
- Consensus opinion: P $\neq$ NP.

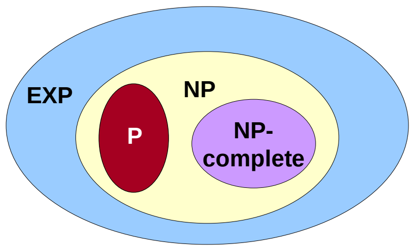
But maybe yes, since:

- No success in proving P $\neq$ NP either.

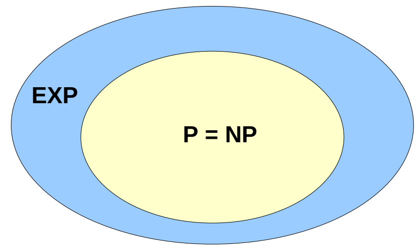
NP-Complete

Definition of NP-complete:

- A problem in NP with the property that if it can be solved efficiently, then it can be used as a subroutine to solve any other problem in NP efficiently.
- "Hardest computational problems" in NP.



If P $\neq$ NP



If P = NP

NP-Complete

Links together a huge and diverse number of fundamental problems:

- TSP, 3-COLOR, CIRCUIT-SAT, thousands more.
- Given an efficient algorithm for 3-COLOR, can efficiently solve TSP, CIRCUIT-SAT, FACTOR, etc.
- Can implement any program in 3-COLOR.

Note: FACTOR not known to be NP-complete.

Notorious complexity class.

- Only exponential algorithms known for these problems.
- Called intractable - unlikely that they can be solved given limited computing resources.

Reduction

Reduction is a general technique for showing that one problem is harder (easier) than another.

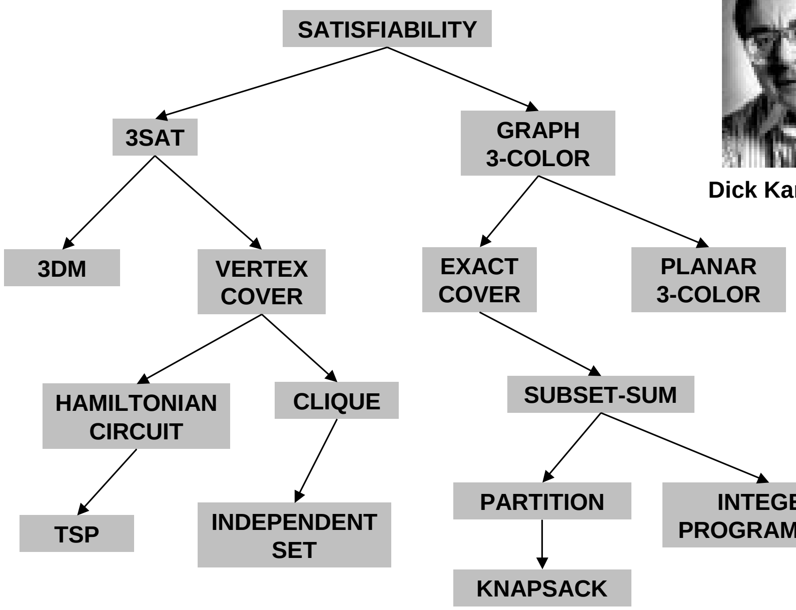
- For problems Y and X, we can often show: if Y can be solved efficiently, then so can X.
- In this case, we say X reduces to Y. (X is "easier" than Y).

Warmup: PRIMALITY reduces to FACTOR.

- Given an efficient algorithm for FACTOR(x, U), want to design an efficient algorithm for PRIMALITY(p).
  - Step 1: Compute FACTOR(p, p).
  - Step 2: If answer = YES, return NO; otherwise return YES.

- Original problem: Is p = 437,669 prime?

Reduction



Dick Karp (1972)

The "World's First" NP-Complete Problem

SAT is NP-complete. (Cook-Levin, 1960s)

Idea of proof:

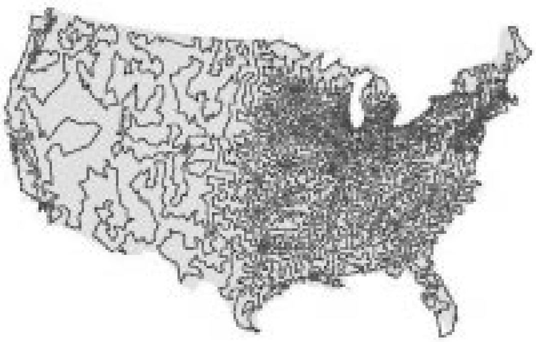
- Given problem X ∈ NP, by definition there exists nondeterministic TM M that solves X in polynomial time.
- Use Boolean variables to model which symbol occupies cell i at step t, location of read head at step t, state of finite control at step t, etc.
- Use logic gates to ensure machine makes legal moves, etc.
- SAT instance is satisfiable if and only if TM outputs YES.

Stephen Cook

Coping With NP-Completeness

Hope that worst case doesn't occur.

- Complexity theory deals with worst case behavior. The instance(s) you want to solve may be "easy."
  - TSP where all points are on a line or circle
  - 13,509 US city TSP problem solved



(Cook et. al., 1998)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

- Develop a heuristic, and hope it produces a good solution.
  - TSP assignment
  - Metropolis algorithm, simulating annealing, genetic algorithms
- Design an approximation algorithm: algorithm that is guaranteed to find a high-quality solution in polynomial time.
  - active area of research, but not always possible!
  - Euclidean TSP tour within 1% of optimal

Sanjeev Arora (1997)

Coping With NP-Completeness

Hope that worst case doesn't occur.

Change the problem.

Exploit intractability.

Keep trying to prove P = NP.

Summary

Many fundamental problems are NP-complete.

- TSP, CIRCUIT-SAT, 3-COLOR.

Theory says we probably won't be able to design efficient algorithms for NP-complete problems.

- You will likely run into these problems in your scientific life.
- If you know about NP-completeness, you can identify them and avoid wasting time.