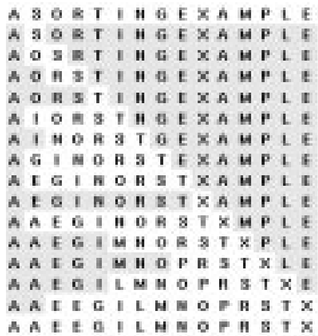
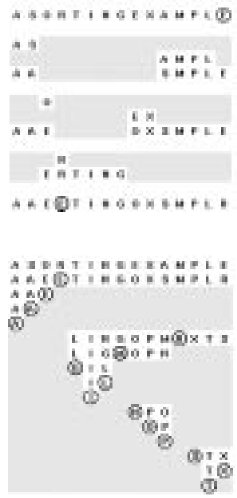
Lecture T4: Analysis of Algorithm





Overview

Lecture T3:

- What is an algorithm?
  - Turing machine.
- Is it possible, in principle, to write a program to solve any problem?
  - No. Halting problem and others are unsolvable.

This Lecture:

- For many problems, there may be several competing algorithms.
  - Which one should I use?
- Computational complexity:
  - Rigorous and useful framework for comparing algorithms and predicting performance.
- Use sorting as a case study.

Linear Growth

Grade school addition.

- Work is proportional to number of digits N.
- Linear growth: k N for some constant k.

1	1	1	0	
	1	0	1	1
+	1	1	1	0
1	1	0	0	1

N = 4

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

N = 8

2N	read operations
2N + 1	write operations
N	odd parity operations
N	majority operations

Quadratic Growth

Grade school multiplication.

- Work is proportional to square of number of digits N.
- Quadratic growth: k N^2 for some constant k.

N = 4

```
        1 0 1 1
      * 1 1 0 1
        1 0 1 1
      0 0 0 0
    1 0 1 1
  1 0 1 1
1 0 0 0 1 1 1 1
```

N = 8

```
                  1 1 0 1 0 1 0 1
                * 0 1 1 1 1 1 0 1
                1 1 0 1 0 1 0 1 0
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1 0
```

2N reads
N^2 + 2N + 1 writes
N-1 adds on N-bit integers

Why Does It Matter?

Run time (nanoseconds)		$1.3\ N^3$	$10\ N^2$	$47\ N \log_2 N$	$48\ N$
Time to solve a problem of size	1000	1.3 seconds	10 msec	0.4 msec	0.048 msec
	10,000	22 minutes	1 second	6 msec	0.48 msec
	100,000	15 days	1.7 minutes	78 msec	4.8 msec
	million	41 years	2.8 hours	0.94 seconds	48 msec
	10 million	41 millennia	1.7 weeks	11 seconds	0.48 seconds
Max size problem solved in one	second	920	10,000	1 million	21 million
	minute	3,600	77,000	49 million	1.3 billion
	hour	14,000	600,000	2.4 trillion	76 trillion
	day	41,000	2.9 million	50 trillion	1,800 trillion
N multiplied by 10, time multiplied by		1,000	100	10+	10

Orders of Magnitude

Seconds	Equivalent
1	1 second
10	10 seconds
10^2	1.7 minutes
10^3	17 minutes
10^4	2.8 hours
10^5	1.1 days
10^6	1.6 weeks
10^7	3.8 months
10^8	3.1 years
10^9	3.1 decades
10^{10}	3.1 centuries
. . .	forever
10^{21}	age of universe

Meters Per Second	Imperial Units	Example
10^{-10}	1.2 in / decade	Continental drift
10^{-8}	1 ft / year	Hair growing
10^{-6}	3.4 in / day	Glacier
10^{-4}	1.2 ft / hour	Gastro-intestinal tract
10^{-2}	2 ft / minute	Ant
1	2.2 mi / hour	Human walk
10^2	220 mi / hour	Propeller airplane
10^4	370 mi / min	Space shuttle
10^6	620 mi / sec	Earth in galactic orbit
10^8	62,000 mi / sec	1/3 speed of light

Powers of 2	2^{10}	thousand
	2^{20}	million
	2^{30}	billion

Historical Quest for Speed

Multiplication: a × b.

- Naïve: add a to itself b times. $N\ 2^N$ steps
- Grade school. N^2 steps
- Divide-and-conquer (Karatsuba, 1962). $N^{1.58}$ steps
- Ingenuity (Schönhage and Strassen, 1971).
  N log N log log N steps

N = # bits in binary representation of a, b

step = integer division

Greatest common divisor: gcd(a, b).

- Naïve: factor a and b, then find gcd(a, b). 2^N steps
- Euclid (20 BCE): gcd(a, b) = gcd(b, a mod b). N steps

Better Machines vs. Better Algorithms

New machine.

- Costs $$$ or more.
- Makes "everything" finish sooner.
- Incremental quantitative improvements (Moore's Law).
- May not help much with some problems.

New algorithm.

- Costs $ or less.
- Dramatic qualitative improvements possible! (million times faster)
- May make the difference, allowing specific problem to be solved.
- May not help much with some problems.

Impact of Better Algorithms

Example 1: N-body-simulation.

- Simulate gravitational interactions among N bodies.
  - physicists want N = # atoms in universe
- Brute force method: N^2 steps.
- Appel (1981). N log N steps, enables new research.

Example 2: Discrete Fourier Transform (DFT).

- Breaks down waveforms (sound) into periodic components.
  - foundation of signal processing
  - CD players, JPEG, analyzing astronomical data, etc.
- Grade school method: N^2 steps.
- Runge-König (1924), Cooley-Tukey (1965).
  FFT algorithm: N log N steps, enables new technology.

Case Study: Sorting

Sorting problem:

- Given N items, rearrange them so that they are in increasing order.
- Among most fundamental problems.

name
Hauser
Hong
Hsu
Hayes
Haskell
Hanley
Hornet
Hill

name
Hanley
Haskell
Hauser
Hayes
Hill
Hong
Hornet
Hsu

Case Study: Sorting

Sorting problem:

- Given N items, rearrange them so that they are in increasing order.
- Among most fundamental problems.

Insertion sort

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

Generic Item to Be Sorted

Define generic Item type to be sorted.

- Associated operations:
  - less, show, swap, rand
- Example: integers.

return 1 if a < b

swap 2 Items

ITEM.h

```
typedef int Item;

int  ITEMless(Item a, Item b);
void ITEMshow(Item a);
void ITEMswap(Item *pa, Item *pb);
int  ITEMscan(Item *pa);
```

Item Implementation

swap integers – need to use pointers

item.c

```
#include <stdio.h>
#include "ITEM.h"

int ITEMless(Item a, Item b) {
   return (a < b);
}

void ITEMswap(Item *pa, Item *pb) {
  Item t;
  t = *pa; *pa = *pb; *pb = t;
}

void ITEMshow(Item a) {
  printf("%d\n", a);
}

 void ITEMscan(Item *pa) {
  return scanf("%d", pa);
}
```


Generic Sorting Program

sort.c (see Sedgewick 6.1)

Max number of items to sort.

Read input.

Call generic sort function.

Print results.

```
#include <stdio.h>
#include <stdlib.h>
#include "Item.h"
#define N 2000000

int main(void) {
  int i, n = 0;
  Item a[N];

  while(ITEMscan(&a[n]) != EOF)
    n++;

  sort(a, 0, n-1);

  for (i = 0; i < n; i++)
     ITEMshow(a[i]);
  return 0;
}
```


Insertion Sort Function

insertionsort.c (see Sedgewick Program 6.1)

```
void insertionsort(Item a[], int left, int right) {
  int i, j;

  for (i = left + 1; i <= right; i++)
     for (j = i; j > left; j--)
        if (ITEMless(a[j], a[j-1]))
           ITEMswap(&a[j], &a[j-1]);
        else
           break;
}
```


Profiling Insertion Sort Empirically

Use lcc "profiling" capability.

- Automatically generates a file `prof.out` that has frequency counts for each instruction.
- Striking feature:
  - HUGE numbers!

Unix

```
% lcc -b insertion.c item.c
% a.out < sort1000.txt
% bprint
```

Insertion Sort prof.out

```
void insertionsort(Item a[], int left, int right) <1>{
  int i, j;
  for (<1>i = left + 1; <1000>i <= right; <999>i++)
     for (<999>j = i; <256320>j > left; <255321>j--)
        if (<256313>ITEMless(a[j], a[j-1]))
           <255321>ITEMswap(&a[j], &a[j-1]);
        else
           <992>break;
<1>}
```

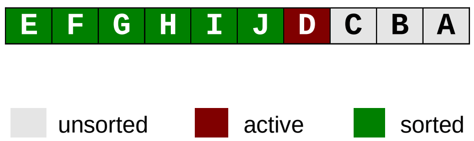
Profiling Insertion Sort Analytically

How long does insertion sort take?

- **Depends on number of elements N to sort.**
- **Depends on specific input.**
- **Depends on how long compare and exchange operation takes.**

Worst case.

- **Elements in reverse sorted order.**
  - i^{th} iteration requires i - 1 compare and exchange operations
  - total = 0 + 1 + 2 + . . . + N-1 = N (N-1) / 2



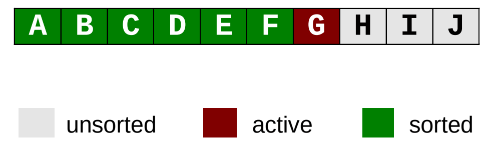
Profiling Insertion Sort Analytically

How long does insertion sort take?

- **Depends on number of elements N to sort.**
- **Depends on specific input.**
- **Depends on how long compare and exchange operation takes.**

Best case.

- **Elements in sorted order already.**
  - i^{th} iteration requires only 1 compare operation
  - total = 0 + 1 + 1 + . . . + 1 = N -1



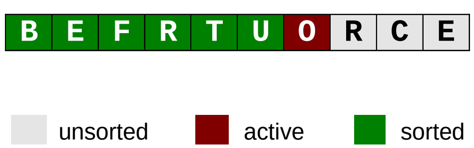
Profiling Insertion Sort Analytically

How long does insertion sort take?

- **Depends on number of elements N to sort.**
- **Depends on specific input.**
- **Depends on how long compare and exchange operation takes.**

Average case.

- **Elements are randomly ordered.**
  - i^{th} iteration requires i / 2 comparison on average
  - total = 0 + 1/2 + 2/2 + . . . + (N-1)/2 = N (N-1) / 4
  - check with profile: 249,750 vs. 256,313



Profiling Insertion Sort Analytically

How long does insertion sort take?

- **Depends on number of elements N to sort.**
- **Depends on specific input.**
- **Depends on how long compare and exchange operation takes.**

Worst case: **N (N - 1) / 2.**

Best case: **N - 1.**

Average case: **N (N – 1) / 4.**

Estimating the Running Time

Total run time:

- Sum over all instructions: frequency * cost.

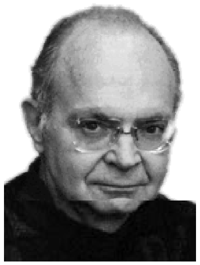
Frequency:

- Determined by algorithm and input.
- Can use `lcc -b` (or analysis) to help estimate.

Cost:

- Determined by compiler and machine.
- Could use `lcc -S` (plus manuals).

Estimating the Running Time



Donald Knuth

Easier alternative.

(i) Analyze asymptotic growth.

(ii) For medium N, run and measure time.

For large N, use (i) and (ii) to predict time.

Asymptotic growth rates.

- Estimate time as a function of input size.
  - N, $N \log N$, N^2, N^3, 2^N, $N!$
- Ignore lower order terms and leading coefficients.
  - Ex. $6N^3 + 17N^2 + 56$ is proportional to N^3

Insertion sort is quadratic. On arizona: 1 second for N = 10,000.

- How long for N = 100,000? 100 seconds (100 times as long).
- N = 1 million? 2.78 hours (another factor of 100).
- N = 1 billion? 317 years (another factor of 10^6).

Sorting Case Study: mergesort

Insertion sort (brute-force)

Mergesort (divide-and-conquer)

- Divide array into two halves.
- Sort each half separately. How do we sort half size files?
  - Any sorting algorithm will do.
  - Use mergesort recursively!

M E R G E S O R T M E

M E R G E S   O R T M E   divide

E E G M R S   E M O R T   sort

Sorting Case Study: mergesort

Insertion sort (brute-force)

Mergesort (divide-and-conquer)

- Divide array into two halves.
- Sort each half separately.
- Merge two halves to make sorted whole.
  - How do we merge efficiently?

M E R G E S O R T M E

M E R G E S   O R T M E   divide

E E G M R S   E M O R T   sort

E E E G M M O R R S T   merge

Profiling Mergesort Analytically

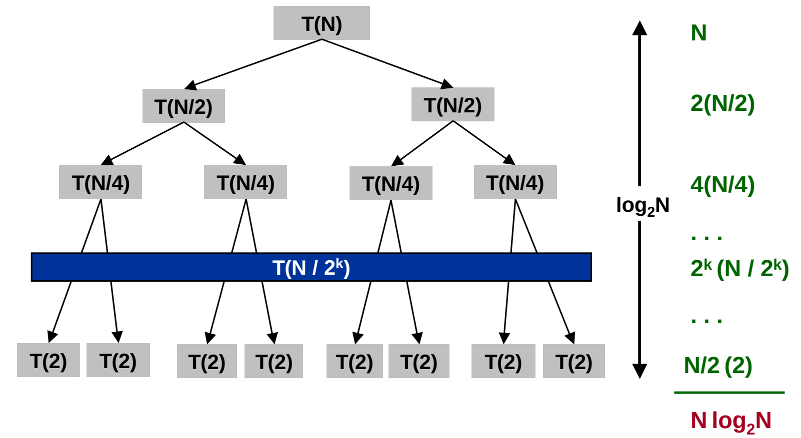
How long does mergesort take?

- Bottleneck = merging (and copying).
  - merging two files of size N/2 requires N comparisons
- T(N) = comparisons to mergesort N elements.

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ \underbrace{2T(N/2)}_{\text{sorting both halves}} + \underbrace{N}_{\text{merging}} & \text{otherwise} \end{cases}$$

Profiling Mergesort Analytically

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ \underbrace{2T(N/2)}_{\text{sorting both halves}} + \underbrace{N}_{\text{merging}} & \text{otherwise} \end{cases}$$



Profiling Mergesort Analytically

How long does mergesort take?

- Bottleneck = merging (and copying).
  - merging two files of size N/2 requires N comparisons
- N $\log_2$ N comparisons to sort ANY array of N elements.
  - even already sorted array!

How much space?

- Can't do "in-place" like insertion sort.
- Need auxiliary array of size N.

Implementing Mergesort

mergesort (see Sedgewick Program 8.3)

```
Item aux[MAXN];        uses scratch array

void mergesort(Item a[], int left, int right) {
  int mid = (right + left) / 2;
  if (right <= left)
     return;
  mergesort(a, left, mid);
  mergesort(a, mid + 1, right);
  merge(a, left, mid, right);
}
```

Implementing Mergesort

merge (see Sedgewick Program 8.2)

```
void merge(Item a[], int left, int mid, int right) {
  int i, j, k;

  for (i = mid+1; i > left; i--)
     aux[i-1] = a[i-1];
  for (j = mid; j < right; j++)
     aux[right+mid-j] = a[j+1];

  for (k = left; k <= right; k++)
     if (ITEMless(aux[i], aux[j]))
        a[k] = aux[i++];
     else
        a[k] = aux[j--];
}
```

copy to temporary array

merge two sorted sequences

Profiling Mergesort Empirically

Mergesort prof.out

```
void merge(Item a[], int left, int mid, int right) <999>{
  int i, j, k;
  for (<999>i = mid+1; <6043>i > left; <5044>i--)
     <5044>aux[i-1] = a[i-1];
  for (<999>j = mid; <5931>j < right; <4932>j++)
     <4932>aux[right+mid-j] = a[j+1];
  for (<999>k = left; <10975>k <= right; <9976>k++)
     if (<9976>ITEMless(aux[i], aux[j]))
        <4543>a[k] = aux[i++];
     else
        <5433>a[k] = aux[j--];
<999>}

void mergesort(Item a[], int left, int right) <1999>{
  int mid = <1999>(right + left) / 2;
  if (<1999>right <= left)
     return<1000>;
  <999>mergesort(a, aux, left, mid);
  <999>mergesort(a, aux, mid+1, right);
  <999>merge(a, aux, left, mid, right);
<1999>}
```

\# comparisons
Theory ~ $N \log_2 N$ = 9,966
Actual = 9,976

Striking feature: no HUGE numbers!

Quicksort

Quicksort.

- Partition array so that:
  - some partitioning element `a[m]` is in its final position
  - no larger element to the left of `m`
  - no smaller element to the right of `m`

partitioning element

Q	U	I	C	K	S	O	R	T	I	S	C	O	O	L
I	C	K	I	C	L	Q	U	S	O	R	T	S	O	O

≤ L

≥ L

partitioned array

Quicksort

Quicksort.

- Partition array so that:
  - some partitioning element `a[m]` is in its final position
  - no larger element to the left of `m`
  - no smaller element to the right of `m`
- Sort each "half" recursively.

partitioning element

Q	U	I	C	K	S	O	R	T	I	S	C	O	O	L
C	C	I	I	K	L	O	O	O	Q	R	S	S	T	U

Sort each "half."

Sorting Case Study: quicksort

Insertion sort (brute-force)
Mergesort (divide-and-conquer)
Quicksort (conquer-and-divide)

- Partition array so that:
  - some partitioning element `a[m]` is in its final position
  - no larger element to the left of `m`
  - no smaller element to the right of `m`
- Sort each "half" recursively.

quicksort.c (see Sedgewick Program 7.1)

```
void quicksort(Item a[], int left, int right) {
  int m;
  if (right > left) {
    m = partition(a, left, right);
    quicksort(a, left, m - 1);
    quicksort(a, m + 1, right);
  }
}
```


Sorting Case Study: quicksort

Insertion sort (brute-force)
Mergesort (divide-and-conquer)
Quicksort (conquer-and-divide)

- Partition array so that:
  - some partitioning element `a[m]` is in its final position
  - no larger element to the left of `m`
  - no smaller element to the right of `m`
- Sort each "half" recursively.
- How do we partition efficiently?
  - N - 1 comparisons
  - easy with auxiliary array
  - better solution: use no extra space!

Implementing Partition

partition (see Sedgewick Program 7.2)

```
int partition(Item a[], int left, int right) {
  int i = left-1;    /* left to right pointer */
  int j = right;     /* right to left pointer */
  Item p = a[right]; /* partition element     */

  while(1) {
    while (ITEMless(a[++i], p))
      ;
    while (ITEMless(p, a[--j]))
      if (j == left)
        break;

    if (i >= j)
      break;
    ITEMswap(&a[i], &a[j]);
  }

  ITEMswap(&a[i], &a[right]);
  return i;
}
```

find element on left to swap

look for element on right to swap, but don't run off end

pointers cross

swap partition element

Profiling Quicksort Empirically

Quicksort prof.out

```
void quicksort(Item a[], int left, int right) <1337>{
  int p;
  if (<1337>right <= left)
     return<669>;
  <668>p = partition(a, left, right);
  <668>quicksort(a, left, p-1);
  <668>quicksort(a, p+1, right);
<1337>}
```

Striking feature: no HUGE numbers!

Profiling Quicksort Empirically

Quicksort prof.out (cont)

```
int partition(Item a[], int left, int right) <668>{
  int i = <668>left-1, j = <668>right;
  Item swap, p = <668>a[right];

  <668>while<1678>(<1678>1) {
     while (<5708>ITEMless(a[++i], p))
        <3362>;
     while (<6664>ITEMless(p, a[--j]))
        if (<4495>j == left)
          <177>break;
     if (<2346>i >= j)
        <668>break;
     <1678>ITEMswap(&a[i], &a[j]);
  }
  <668>ITEMswap(&a[i], &a[right]);
  return <668>i;
<668>}
```

Striking feature: no HUGE numbers!

Profiling Quicksort Analytically

Intuition.

- Assume all elements unique.
- Assume we always select median as partition element.
- T(N) = # comparisons.

$$T(N) = \begin{cases} 0 & \text{if } N = 1 \\ \underbrace{2T(N/2)}_{\text{sorting both halves}} + \underbrace{N}_{\text{partitioning}} & \text{otherwise} \end{cases}$$

If N is a power of 2.
$\Rightarrow$ T(N) = N $\log_2$ N

- Analysis "almost true" if you partition on random element.

Can you find median in O(N) time?

- Yes, see COS 226/423.

Bob Tarjan, et al (1973)

Profiling Quicksort Analytically

Partition on median element.

- Proportional to N $\log_2$ N in best and worst case.

Partition on rightmost element.

- Proportional to N^2 in worst case.
- Already sorted file: takes $N^2/2 + N/2$ comparisons.

Partition on random element.

- Roughly 2 N $\log_e$ N steps.
- Choose random partition element.

Check profile.

- 2 N $\log_e$ N: 13815 vs. 12372 (5708 + 6664).
- Running time for N = 100,000 about 1.2 seconds.
- How long for N = 1 million ?
  - slightly more than 10 times (about 12 seconds)

Sorting Analysis Summary

Running time estimates:

- Home pc executes 10^8 comparisons/second.
- Supercomputer executes 10^{12} comparisons/second.

	Insertion Sort (N^2)			Quicksort (N lg N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	0.3 sec	6 min
super	instant	1 second	1.6 weeks	instant	instant	instant

Lesson: good algorithms are more powerful than supercomputers.

Design, Analysis, and Implementation of Algorithms

Algorithm.

- "Step-by-step recipe" used to solve a problem.
- Generally independent of programming language or machine on which it is to be executed.

Design.

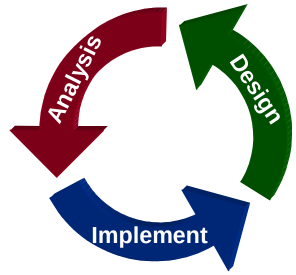
- Find a method to solve the problem.

Analysis.

- Evaluate its effectiveness and predict theoretical performance.

Implementation.

- Write actual code and test your theory.



Sorting Analysis Summary

Comparison of Different Sorting Algorithms

Attribute	insertion	quicksort	mergesort
Worst case complexity	N^2	N^2	$N \log_2 N$
Best case complexity	N	$N \log_2 N$	$N \log_2 N$
Average case complexity	N^2	$N \log_2 N$	$N \log_2 N$
Already sorted	N	N^2	$N \log_2 N$
Reverse sorted	N^2	N^2	$N \log_2 N$
Space	N	N	2 N
Stable	yes	no	yes

Sorting algorithms have different performance characteristics.

- Other choices: BST sort, bubblesort, heapsort, shellsort, selection sort, shaker sort, radix sort, distribution sort, solitaire sort, hybrid methods.
- Which one should I use?
  - Depends on application.

Computational Complexity

Framework to study efficiency of algorithms.

- Depends on machine model, average case, worst case.
- UPPER BOUND = algorithm to solve the problem.
- LOWER BOUND = proof that no algorithm can do better.
- OPTIMAL ALGORITHM: lower bound = upper bound.

Example: sorting.

- Measure costs in terms of comparisons.
- Upper bound = $N \log_2 N$ (mergesort).
  - quicksort usually faster, but mergesort never slow
- Lower bound = $N \log_2 N - N \log_2 e$ (applies to any comparison-based algorithm).
  - Why?

Computational Complexity

Caveats.

- Worst or average case may be unrealistic.
- Costs ignored in analysis may dominate.
- Machine model may be restrictive.

Complexity studies provide:

- Starting point for practical implementations.
- Indication of approaches to be avoided.

Summary

How can I evaluate the performance of a proposed algorithm?

- **Computational experiments.**
- **Complexity theory.**

What if it's not fast enough?

- **Use a faster computer.**
  - **performance improves incrementally**
- **Understand why.**
- **Discover a better algorithm.**
  - **performance can improve dramatically**
  - **not always easy / possible to develop better algorithm**